Problem assignment 4.

Algebra B3 – Commutative Algebra

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In problems we fix an algebraically closed field k. You can use NSS (Nullstellensatz) in solutions of the problems, but please specify when you use NSS.

1. Let V be a linear space over k and $T : V \to V$ its endomorphism. Let us assume that the field k is uncountable and that $dim_k(V)$ is countable.

(i) Show that if $V \neq 0$ then the operator T has non-empty spectrum $Sp(T)$ (here $sp(T) = \{a \in A|$ operator $T - a$ is not invertible $\}$).

[P] (ii) Show that $Sp(T) = 0$ iff the operator T is locally nilpotent.

2. Let $(X, \mathcal{P})$ be an affine algebraic variety. Given a function $f \in \mathcal{P}$ we denote by X_f the basic open subset $X_f := \{x \in X | f(x) \neq 0\}$.

[P] (i) Consider the algebra B of functions on X_f generated by restrictions of functions in $\mathcal{P}$ to X_f and by the function $1/f$. Show that this algebra is isomorphic to the algebra $\mathcal{P}_f$ and that $(X_f, \mathcal{P}_f)$ is an affine algebraic variety.

(ii) Let U be an open subset of X. Define the notion of a **regular** function on U. We denote the algebra of regular functions on U by $\mathcal{O}(U)$. Prove the following

Statement (Serre's lemma). Every regular function on an affine algebraic variety X is polynomial.

[P] (iii) Using Serre's lemma compute the algebras of regular functions for the following open subsets of $X = \mathbf{A}^n$

$U = X$; $U = X \setminus H_i$, where $H_i = V(x_i)$; $U = X \setminus \bigcup H_i$; $U = X \setminus 0$.

(iv) Let X be a quadratic cone in $\mathbf{A}^n$ defined by a non-degenerate quadratic form Q. Compute algebras of regular functions on X and on $U = X \setminus 0$.

3. Let K be an infinite field. Consider the algebra $C = C_n(K) := K[x_1, ..., x_n]$. Show that it can be interpreted as the algebra of polynomial functions on the affine space $\mathbf{A}^n(K) := \{(a_a, ..., a_n)|\ a_i \in K\}$.

(i) Let $f \in C$ be a non-zero polynomial of degree d. Show that there exists a linear change of coordinates $y_i = \sum b_{ij} x_j$ such that in coordinates y the polynomial f is monic of degree d with respect to coordinate y_n.

Deduce that the algebra C/Cf is finite over the subalgebra $B = k[y_1, ..., y_{n-1}]$.

(ii) Prove **Noether's Normalization Lemma.**

Let M be a non-zero finitely generated $\mathcal{P}_n$-module. Then there exist a number d and a system of linear forms $y_i = \sum a_{ij} x_j,\ 1 \leq i \leq d$ such that

(α) module M is finitely generated over the algebra $B = K[y_1, ..., y_d]$

(β) Annihilator of M in B is 0.

In fact given one such system show that almost every system of d linear forms y_i has these properties.

∇ **Remarks.** (i). Try to formulate and prove the analogue of Noether's Normalization Lemma for the case of a finite field K.

[P] (iii) Show that for an algebraically closed field K Noether's Normalization lemma above implies Nullstellensatz.

∇ **4.** Fix an infinite field K and consider the algebra $C = K[x_1, ..., x_n]$ for some $n > 0$. Let M be a non-zero finitely generated C-module. Prove a version of the Nullstellensatz following the steps bellow.

((i) Show that there exists a non-zero polynomial $f \in C$ such that the quotient module M/fM is non-zero.

(ii) Show that there exists a non-zero linear form $y = \sum a_j x_j$ that satisfies the following condition:

(*) There exists a monic polynomial $Q \in K[t]$ such that the quotient module $M/Q(y)M$ is non-zero.

(iii) Show that any non-zero linear form y on $\mathbf{A}^n$ satisfies the property (*) above.

(iv) Show that there exists an ideal $J \subset C$ of finite codimension such that the quotient module $N = M/JM$ is non-zero.

(v) Show that for any maximal ideal $\mathfrak{m} \subset C$ the quotient field $L = C/\mathfrak{m}$ is a finite extension of the field K.

(vi) Show that all these statements hold for the case of a finite field K.

5. Let A be a finitely generated k-algebra (k is algebraically closed). Consider the set $X = M(A) := Mor_{k-alg}(A, k)$.

Show that we can identify X with the set of maximal ideals of A.

(i) Define Zariski topology on X. For every subset $Z \subset X$ consider the ideal $J_Z \subset A$ of functions that vanish on Z. For any ideal $J \subset A$ denote by $V(J)$ the set of common zeroes $a \in X$ of all elements of J. Show that

If Z is a subset of X then $V(J_Z)$ coincides with the closure of Z in Zariski topology.

If I is an ideal of A then the ideal $J_{V(I)}$ coincides with the radical of I.

[P] (ii) Let $\nu : B \to A$ be a finite morphism of finitely generated k-algebras. Show that the corresponding map of sets $\nu^* : M(A) \to M(B)$ is closed and has finite fibers.

∇ **6.** Prove the following arithmetic version of Nullstellensatz

Theorem. Let A be a non-zero finitely generated $\mathbf{Z}$-algebra. Then it has a morphism $\nu : A \to F$ where F is some finite field.

Definition. Let X be a topological space. The topological space X is called **irreducible** if it is not empty and it can not be written as a union of two proper closed subsets $F_1, F_2 \subsetneqq X$

A subset $Z \subset X$ is called **irreducible** if it is irreducible in induced topology.

7. (i) Show that a non-empty space X is irreducible iff it satisfies the following condition:

(*) Every non-empty open subset $U \subset X$ is dense in X.

(ii) Show that a subset $Z \subset X$ is irreducible iff its closure is irreducible.

(iii) Let $\nu : X \to Y$ be a continuous map of topological spaces. Show that if a subset $Z \subset X$ is irreducible then its image $\nu(Z)$ is irreducible.

8. Let A be an algebra, $X = SpecA$ its spectrum.

Define Zariski topology on X in terms of closed subsets V_J and in terms of basic open subsets X_f. Check that they give the same topology.

(i) Let $f \in A$. Show that the basic open subset X_f is empty iff f is nilpotent. Show that $X_f = X$ iff f is a unit.

More generally, show that $X_f = X_h$ iff $Rad(f) = Rad(h)$.

(ii) Show that there axists a natural order reversing bijection between radical ideals of A and closed substs of X. Show that under this bijection prime ideals bijectively correspond to closed irreducible subsets and maximal ideals bijectively correspond to closed points of X.

Deduce that a point x is contained in the closure of a point y iff $\mathfrak{p}_x \supset \mathfrak{p}_y$.

(iii) Show that if $x, y \in X$ are distinct points then there exists an open subset U that contains one of them but not another.

[P] (iv) Show that the space X is quasi-compact in Zariski topology.

Show that an open subset $U \subset X$ is quasi-compact iff it is a union of a finite number of basic open subsets.

[P] (v) Show that a closed subset $F \subset X$ is irreducible iff it is a closure of some point $x \in X$.