Problem assignment 5.

Algebra B3 – Commutative Algebra

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Let A be a commutative algebra, $X = SpecA$ (in most cases you can assume for simplicity that A is Noetherian).

For every point $x \in X$ denote by $\mathfrak{p}_x$ the corresponding prime ideal in A and by A_x the localized algebra $A_{\mathfrak{p}_x}$. For an A-module M we denote my M_x the localized module $M_x = A_x \otimes_A M$.

We denote by $k(x)$ the residue field $k(x) = A_x/A_x\mathfrak{p}_x$. For an A-module M we denote by $M(x)$ its fiber over the point x - this is the $k(x)$-vector space $M(x) := M_x/\mathfrak{p}_x M_x = k(X) \otimes_A M$.

Note that any element $f \in A$ defines a "function" $x \mapsto f(x) \in k(X)$ and any element $m \in M$ defines a "section" $x \mapsto m(x) \in M(x)$.

Definition. For every finitely generated A-module M we define $SuppM = V(Ann(M)) \subset X = SpecA$.

[P] 1. Show that the following conditions are equivalent.

(a) $x \in SuppM$

(b) $M_x \neq 0$

(c) M has a quotient isomorphic to $A/\mathfrak{p}_x$.

(d) M has a subquotient isomorphic to $A/\mathfrak{p}_x$

[P] 2. Let M be a finitely generated A-module. Suppose $SuppM \subset F \bigcup F'$, where F, F' are closed subsets of X.

Show that there exists a submodule $L \subset M$ such that L has support inside F and $N = M/L$ has support inside F'.

Show that if F and F' do not intersect, then $M \simeq L \bigoplus N$.

[P] 3. Let M be a finitely generated module over a Noetherian algebra A. Show that if M is not zero then it contains a submodule isomorphic to $A/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset A$. Deduce from this that M has a finite filtration by A-modules $0 = M_0 \subset M_1 \subset ... \subset M_n = M$ with quotients isomorphic to $A/\mathfrak{p}_i$.

Hint. Consider ideals $Ann(m)$ for all elements $m \in M$ and show that this set of ideals has a maximal element.

Definition. For any open subset $U \subset X$ we consider the algebra of all sections $\phi : x \mapsto \phi(x) \in A_x$.

We say that a section ϕ is **regular** at a point $x \in U$ if there exists a basic neighborhood X_f of x in U on which the section ϕ is defined by some element $b \in A_f$. We say that the section ϕ is regular if it is regular at all points of U. The algebra of such regular sections we denote $\mathcal{O}(U)$.

∇ **4.** Show that $\mathcal{O}(X) = A$. Deduce that for any $f \in A$ we have $\mathcal{O}(X_f) = A_f$.

Remark. Collection of algebras of sections $\mathcal{O}(U)$ is called the **structure sheaf** of X. This construction plays central role in algebraic geometry since it allows to glue affine schemes $SpecA$ into larger schemes.

Definition. (i) Let X be a Noetherian space. Define **dimension** of X to be supremum of lengths d of strict chains of irreducible closed subsets $F_0 \subsetneq F_1 \subsetneq ... \subsetneq F_d \subset X$.

(ii) Given a point $a \in X$ we define the local dimension $\dim_a X$ to be the minimum of $\dim U$ for all open neighborhoods U of a.

[P] 5. (i) Let X be a Noetherian topological space, $Y \subset X$ a locally closed subset. Denote by $cl(Y)$ the closure of Y in X and define the **boundary** ∂Y of the set Y by $\partial Y := cl(Y) \setminus Y$.

Show that $\dim cl(Y) = \dim Y$ and $\dim \partial Y \leq \dim Y - 1$.

(ii) Let A be a Noetherian algebra and $X = SpecA$. Show that $\dim_a X = \dim A_{\mathfrak{p}_a}$

(iii) Let X be an irreducible affine algebraic variety. Show that for any closed point $a \in X$ the local dimension $\dim_a X$ equals $\dim X$.

Remark. Analogous statement does not hold for arbitrary Noetherian spaces.

(iv) Let X be an irreducible affine algebraic variety over a field k. Denote by $k(X)$ the field of rational functions on X. Show that the dimension $\dim X$ equals to the transcendence degree of the field $k(X)$ over k.

6. (i) Let $p : X \to Y$ be a continuous map of Noetherian spaces. Suppose that p is closed and epimorphic. Show that $\dim X \geq \dim Y$.

(ii) Let $\nu : A \to B$ be a finite morphism of Noetherian algebras and $p : X = SpecB \to Y = SpecA$ the induced map. Show that $\dim X \leq \dim Y$.

[P] 7. Let $p : A \to B$ be a morphism of Noetherian algebras and $p : X = SpecB \to Y = SpecA$ the induced map. Fix a point $y \in Y$ and consider the field $L = k(y)$.

Describe the natural morphism $i : Z = SpecL \to Y$. Describe the fibered product $X \times_Y Z$ in the category of affine schemes - this is the fiber of the morphism p over the point y.

Explicitly describe the corresponding topological space.

[P] 8. Let S be a commutative A algebra. Let u, v be elements of S integral over A, i.e. they satisfy monic polynomials $P, Q \in A[t]$. Explicitly $P = t^n + \sum_{i<n} a_i t^i$ and $Q = t^m + \sum_{j<m} b_j t^j$.

Show that there exists a number N and system of N polynomials F_k in $\mathbf{Z}[x_1, ..., x_n, y_1, ..., y_m]$ such that the element $w = u^2 + v^3 + 3uv \in S$ satisfies the monic polynomial equation $t^N + \sum_{k<N} c_k t^k$ in which the coefficients c_k are given by $c_k := F_k(a_1, ...a_n.b_1.,,,.b_m)$.

Give an a priory bound on the degree N of this equation.

[P] 9. Let A be a Noetherian algebra and M a finitely generated A-module. Show that there exists a bound n such that for every nilpotent endomorphism T of M we have $T^n = 0$.

Definition. Let $\mathcal{A}$ be an abelian category. An object $P \in Ob(\mathcal{A})$ is called **projective** if the functor $H_P : N \mapsto Hom(P, N)$ on the category $\mathcal{A}$ is exact.

We will analyze this notion in case of the category of modules $\mathcal{A} = \mathcal{M}(A)$.

[P] 10. (i) Show that in any abelian category an object P is projective iff for any epimorphic morphism $p : R \to P$ there exists a section $s : P \to R$, so that $ps = Id_P$. Explain that in this case $R \simeq P \bigoplus Ker(p)$

(ii) Show that any free A-module F is projective. Show that an A-module P is projective iff it can be realized as a direct summand of a free module.

(iii) Show that any projective A-module is flat. Give example of a flat A-module that is not projective.