Problem assignment 7.

Algebra B3 – Commutative Algebra

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Jordan Hoelder Theory and length function.

Let D be an algebra and $\mathcal{A} = \mathcal{M}(D)$ the category of left D-modules.

Definition. A D-module S is called **simple** (or irreducible) if $S \neq 0$ and any submodule of S is either 0 or S.

1. Show that if S is simple then any nonzero morphism $\nu : M \to S$ is epimorphic and any nonzero morphism $\nu : S \to N$ is monomorphism.

We will consider filtrations of a D-module M by D-submodules; filtration F is a collection of submodules $0 = M_0 \subset M_1 \subset ... \subset M_l = M$. We will say that the filtration F is strict if all inclusions are strict.

If we have such filtration we say that the module M is **glued** from D-modules $Q_i = M_i/M_{i-1}$.

Note that the strict filtration F is maximal, i.e. it can not be included in a larger strict filtration, iff all the quotient modules Q_i are simple.

Remark. Note that Atiyah and MacDonalds work with decreasing filtrations – this leads to the same notions up to small change of notations; they call a strict filtration a **chain**; a maximal chain they call a **composition series.**

Definition. We say that a D-module M has **finite length** if it can be glued from a finite number of simple objects S_i, $i = 1, ..., l$. In other words M has a finite composition series.

Theorem (Jordan-Hoelder). Suppose that D-module M is glued from simple modules $S_i, i = 1, ..., l$. Then

(i) Any strict filtration $\Phi : \ 0 = N_0 \subsetneqq ... \subsetneqq N_r = M$ of the module M by D-submodules has length $r \leq l$.

(ii) The filtration Φ is maximal iff it has length l. In this case the quotient modules $Q_i = N_i/N_{i-1}$ are simple and the collection of modules Q_i up to permutation of indexes and up to isomorphism of modules coincides with the collection S_i.

[P] 2. Prove JH theorem.

Hint. Consider projection $p : M \to L = M/S$, where $S = S_1$. Consider the image $p(\Phi)$ of the filtration Φ. Using induction show that if the length of Φ is $\geq l$ then the filtration $p(\Phi)$ is not strict, i.e. for some i we have $p(N_i) = p(N_{i+1})$. Show that this implies that $N_{i+1} = N_i \bigoplus S$.

JH-theorem implies that the subcategory $\mathcal{A}^{fl} \subset \mathcal{A}$ consisting of modules of finite length is an abelian subcategory of $\mathcal{A}$ closed with respect to subquotients and extensions. On this subcategory we have a well defined **length function** $M \mapsto l(M)$ (length of the module M). This function satisfies the following properties:

(i) It is additive, i.e. for every short exact sequence $0 \to L \to M \to N \to 0$ we have $l(M) = l(L) + l(N)$.

(ii) It is strictly positive, i.e $l(M) \ngeq 0$ and $l(M) = 0$ iff $M = 0$.

[P] 3. Prove that a D-module M has finite length iff it satisfies both descending and ascending chain conditions.

One can formulate another, slightly more general, version of JH theorem. Let us say that two D-modules M, N are Jordan - Hoelder equivalent (notation $M \sim_{JH} N$) if they have filtrations Φ and Ψ of the same finite length such that the sequences of their quotients are the same up to isomorphism and permutation of indexes.

Theorem GJH. The relation $\sim_{JH}$ on the set of isomorphism classes of D-modules is an equivalence relation.

∇ **4.** Prove theorem GJH. Show why Jordan-Hoelder theorem above is a special case of this theorem.

Remark. All the results of Jordan-Hoelder theory hold for an arbitrary abelian category $\mathcal{A}$.

Definition. A (commutative) algebra A is called **Artinian** if any finitely generated A-module has finite length.

5. Let A be a commutative algebra.
[P] (i) Show that the following conditions are equivalent.
(a) A is Artinian.
(b) Free A-module A has finite length.
(c) The algebra A is Noetherian of dimension 0.
(*) (ii) Show that conditions a,b,c above are equivalent to the following condition
(d) Free A-module A satisfies descending chain condition.

[P] 6. Let R be an Artinian algebra. Consider the graded algebra $A = R[x_1, ..., x_n]$ (grading with respect to degree in x's).
Let $M = \bigoplus M^k$ be a finitely generated graded A module. Show that the sequence of numbers $a(k) = l(M^k)$ is ep (eventually polynomial).

[P] 7. Let A be a Noetherian local ring, $\mathfrak{m}$ the maximal ideal of A and $k = A/\mathfrak{m}$ the residue field. Fix an ideal $\mathfrak{a} \subset \mathfrak{m}$.

(i) Show that the following conditions are equivalent

(a) $\mathfrak{a}$ is a defining ideal, i.e. it contains the ideal $\mathfrak{m}^k$ for some k.

(b) The algebra $A/\mathfrak{a}$ is Artinian.

(ii) Fix a defining ideal $\mathfrak{a} \subset A$. Consider a finitely generated A-module and fix an $\mathfrak{a}$-filtration Φ of the module M.

Show that if the filtration Φ is good then the sequence of numbers $b(k) := l(M/\Phi^k(M))$ is ep.

(iii) For any finitely generated A-module M define its Hilbert dimension $d(M)$. Show that for a short exact sequence $0 \to L \to M \to N \to 0$ we have $d(M) = \max(d(L), d(N))$.