Automorphic Forms - Home Assignment 5

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The goal of this exercise is to classify the infinitesimal equivalence classes of representations of $GL_2(\mathbb{R})$, similarly to what was done in class for $SL_2(\mathbb{R})$. Our general strategy is as follows: after doing some geometry, we will explicitly define representations of $GL_2(\mathbb{R})$ that will be called *the principal series representations.* We will show out that all irreducible Harish-Chandra modules of $GL_2(\mathbb{R})$ can be embedded in the principal series, and give a list of all non-zero morphisms between these representations.

Question 1

We begin with some geometry. Let $G = GL_2(\mathbb{R})$, $G^+ = \{g \in G \mid \det(g) > 0\}$, $K = O_2(\mathbb{R})$, and:

$$\begin{aligned} P &= \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \right\}, \\ P^+ &= \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \mid a > 0, d > 0 \right\}, \\ U &= \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in G \right\}, \\ Z &= \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \in G \right\}, \\ Z^+ &= \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \in G \mid r > 0 \right\}. \end{aligned}$$

(a) Prove the Iwasawa decomposition: any element $g \in G$ may be written uniquely as $g = p^+k$, where $p^+ \in P^+$ and $k \in K$.

(b) Conclude that $Z\backslash G/K$ is isomorphic to the upper half plane H^+.

(c) Show that $Z^+ \cdot U\backslash G$ is the union of two punctured planes $\mathbb{R}^2 - \{0\}$.

Question 2

Let us define the principal series representation for $G = GL_2(\mathbb{R})$. Let $\chi_1, \chi_2 : \mathbb{R}^\times \to \mathbb{C}^\times$ be two multiplicative characters. Let the *trivially extended* character $\chi : P \to \mathbb{C}^\times$ be

$$\chi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \left|\frac{a}{d}\right|^{1/2} \chi_1(a)\chi_2(d).$$

(The normalization $\left|\frac{a}{d}\right|^{1/2}$ might seem a bit odd now, but as we will see it makes our life much simpler later on).

Now, define the principal series represntation $(\pi, V(\chi_1, \chi_2))$ to be the space

$$V(\chi_1, \chi_2) = \{f : G \to \mathbb{C} \mid f \text{ is smooth, } f(pg) = \chi(p)f(g) \quad \forall p \in P\},$$

with action given by right translation, that is, $(\pi(h) \cdot f)(g) = f(gh)$.

The goal of this question is to determine the structure of this representation, and its relation to the principal series of $SL_2(\mathbb{R})$.

(a) Show that any $f \in V(\chi_1, \chi_2)$ is completely determined by its values on the circle $SO(2,\mathbb{R}) \subseteq K$.

(b) Conclude that $V(\chi_1, \chi_2)$ has a basis given by Fourier series, that is, it is generated as a Banach space by the functions

$$f_k\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right) = \left|\frac{a}{d}\right|^{1/2} \chi_1(a)\chi_2(d)\exp(ik\theta),$$

for all k such that $(-1)^k = \chi_1(-1)\chi_2(-1)$. Show that these functions are a base for its $(\mathfrak{g}, K)$-module, and show that it is weakly admissible.

(c) Since we already have a basis for the $(\mathfrak{g}, K)$-module corresponding to $V(\chi_1, \chi_2)$, let us explicitly write down the actions of $\mathfrak{g}$ and K on this basis.

For any $\alpha \in \mathfrak{g}$, let

$$D_\alpha f(g) = \frac{\partial}{\partial t} f(g \exp(\alpha t))\bigg|_{t=0}$$

be the action of the Lie algebra. Denote (note that these matrices are in a different basis than the ones defined in class for $SL_2(\mathbb{R})$):

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{g}$$
$$H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g}$$
$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{g}$$
$$Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g}.$$

Show that (letting $\chi_1(r) = |r|^{s_1}\,\mathrm{sign}(r)^{\varepsilon_1}$, $\chi_2(r) = |r|^{s_2}\,\mathrm{sign}(r)^{\varepsilon_2}$):

$$\begin{aligned}
\pi\left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right) f_k &= \exp(ik\theta) f_k \\
\pi\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right) f_k &= (-1)^{\varepsilon_1} f_{-k} \\
D_Z f_k &= (s_1 + s_2) f_k \\
D_H f_k &= ik f_k \\
D_E f_k := (D_X - iD_Y) f_k &= (s_1 - s_2 + 1 + k) f_{k+2} \\
D_F f_k := (D_X + iD_Y) f_k &= (s_1 - s_2 + 1 - k) f_{k-2}.
\end{aligned}$$

Conclude that $V(\chi_1, \chi_2)$ is admissible.

(d) Show that if we let $\chi_2 = 1$, and

$$v_k(r\cos\theta, r\sin\theta) = f_k\left(\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right),$$

then the function v_k is an element of the principal series representation V_{χ_1} of $SL_2(\mathbb{R})$ defined in class, and in fact this defines a morphism from the restriction of the principal series representation $V(\chi_1, 1)$ to $SL_2(\mathbb{R})$ to the principal series representation V_{χ_1}.

Remark 1. We note that the main consequence of switching from $SL_2(\mathbb{R})$ to $GL_2(\mathbb{R})$ is the existence of the element $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, which enables us to refect the weights, sending elements of weight k to elements of weight $-k$. Relating this to phenomenon that we have encountered before, it can be seen that the action of this element on Maass forms is precisely the reflection sending $f(x+iy)$ to $f(-x+iy)$ (which is why we encountered that extra symmetry). Also, in the previous exercise when we took the derivative of an odd Maass form, we remarked that there was some subtlety in the choice of operator. The choice that we presented (the Maass raising operator) can in fact be seen to correspond to the operator D_E above. In a similar fashion, the operator D_F is sometimes called *the Maass lowering operator*, as it lowers the weight.

Question 3

The goal of this question is to determine the when the principal series representation is reducible, and what kinds of isomorphisms exist between principal series representations.

(a) Use the above explicit action to show that $V(\chi_1, \chi_2)$ is reducible iff there is some integer $k \neq 1$ such that $(-1)^k = \chi_1(-1)\chi_2(-1)$ and $s_1 - s_2 = k - 1$.

(b) If k is an integer as above, show that $V(\chi_1, \chi_2)$ decomposes into two irreducible subquotients (in contrast to the three we had for $SL_2(\mathbb{R})$), one of which is finite dimensional and the other infinite dimensional. We let $D_k^{\pm}(\chi)$ be the infinite dimensional irreducible subquotient of $V(\chi|\cdot|^{\frac{k-1}{2}} \operatorname{sign}(\cdot)^k, \chi|\cdot|^{-\frac{k-1}{2}})$.

We also define $D_1^{\pm}(\chi) = V(\chi \operatorname{sign}(\cdot), \chi)$, for completeness. The representations $D_k^{\pm}(\chi)$ for $k \neq 1$ are sometimes called *essentially discrete series* representations, while $D_1^{\pm}(\chi)$ is sometimes called *limit of essentially discrete series.*

(c) Use the explicit action above to show that whenever there is no integer k such that $(-1)^k = \chi_1(-1)\chi_2(-1)$ and $s_1 - s_2 = k - 1$ (that is, whenever $V(\chi_1, \chi_2)$ has no essentially discrete or limit of essentially discrete as a subquotient) then, as $(\mathfrak{g}, K)$-modules,
$$V(\chi_1, \chi_2)^{K\text{-finite}} \cong V(\chi_2, \chi_1)^{K\text{-finite}}.$$

Remark 2. Note that this isomorphism can be made into an isomorphism of the entire smooth part (for $\operatorname{Re}(s_1 - s_2)$ sufficiently large) by
$$f(g) \mapsto \int_{\mathbb{R}} f\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) du,$$
which has meromorphic continuation to any value of $s_1 - s_2$, with poles whenever $V(\chi_1, \chi_2)$ is reducible.

(d) Next, show that for any χ and k, we have $D_k^{\pm}(\chi)^{K\text{-finite}} \cong D_{2-k}^{\pm}(\chi \operatorname{sign}(\cdot)^k)^{\ K\text{-finite}}$ (that is, although $V(\chi_1, \chi_2)$ might not be isomorphic to $V(\chi_2, \chi_1)$ when it is not irreducible, they still have the same irreducible subquotient).

(e) Show that for any χ and k, we have
$$D_k^{\pm}(\chi)^{K\text{-finite}} \cong D_k^{\pm}(\chi \operatorname{sign}(\cdot))^{K\text{-finite}} \cong D_{2-k}^{\pm}(\chi)^{K\text{-finite}}.$$

Keep in mind that $V(\chi_1, \chi_2)^{K\text{-finite}} \ncong V(\chi_1 \operatorname{sign}(\cdot), \chi_2 \operatorname{sign}(\cdot))^{K\text{-finite}}$ when it is irreduible.

Remark 3. It can be seen that essentially discrete series representations and limit of essentially discrete series representations correspond to *holomorphic* modular forms, while irreducible principal series representations correspond to Maass forms.

Question 4

The goal of this question is to finish the classification of irreducible Harish-Chandra modules for $GL_2(\mathbb{R})$.

(a) Show that the differential operators D_Z and $\Delta = \frac{1}{2}D_E D_F + \frac{1}{2}D_F D_E + D_H^2$ lie in the center of the universal enveloping algebra $U(\mathfrak{g})$ (that is, they commute with all differential operators D_α with $\alpha \in \mathfrak{g}$) and in fact that they commute with the action of K as well.

(b) Conclude that D_Z and Δ act on arbitrary irreducible $(\mathfrak{g}, K)$-modules by scalars (hint: use Schur's lemma). Compute their values on the principal series representation $V(\chi_1, \chi_2)$.

(c) Show that the isomorphisms found in question **??** are, up to scalar, all of the nonzero morphisms between irreducible representations $D_k^\pm(\chi)$ (with k possibly 1) and the irreducible of the principal serires representations $V(\chi_1, \chi_2)$.

(d) Show that any admissible irreducible $(\mathfrak{g}, K)$-module is isomorphic to either an irreducible principal series representation $V(\chi_1, \chi_2)$, an essentially discrete series representation $D_k^\pm(\chi)$, or a limit of essentially discrete series representation $D_1^\pm(\chi)$.

Remark 4. We note that up to a scalar, the operator Δ above is the Laplacian we introduced on Maass forms. When added with an appropriate multiple of D_Z^2, it is also known as the *Casimir operator*.