Automorphic Forms - Home Assignment 6

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The primary topic of this exerxise is the adeles and their properties.
Define the *ring of adeles* to be the the subring $\mathbb{A}_\mathbb{Q} \subseteq \mathbb{R} \times \prod \mathbb{Q}_p$, defined by:

$$\mathbb{A}_\mathbb{Q} = \{(a_\infty, a_2, a_3, a_5, a_7, a_{11}, \dots) \in \mathbb{R} \times \prod \mathbb{Q}_p \mid a_p \in \mathbb{Z}_p \text{ for almost all } p\}.$$

It is sometimes convenient to simply write $\prod \mathbb{Q}_v$ instead of $\mathbb{R} \times \prod \mathbb{Q}_p$, where v is either a prime or infinity. It is customary to refer to v as a *place*, and say that $v = \infty$ is the *infinite place*, and $v = p$ for any prime p is a *finite place*. Finally, we let $\mathbb{Q}_\infty = \mathbb{R}$.

We denote by $\mathbb{A}_{\text{fin}}$ the subring of inite adeles $\mathbb{A}_{\text{fin}} = \{(0, a_2, a_3, a_5, \dots) \in \mathbb{A}_\mathbb{Q}\}$.

Question 1

Define the topology on $\mathbb{A}_\mathbb{Q}$ to be the minimal topology containing all open sets of the form

$$U = U_\infty \times \prod U_p = \prod U_v,$$

where $U_p = \mathbb{Z}_p$ for almost all primes p.

(a) Prove that addition and multiplication in $\mathbb{A}_\mathbb{Q}$ are continuous with respect to this topology.

(b) Show that the set $\hat{\mathbb{Z}} = \prod \mathbb{Z}_p \subseteq \mathbb{A}_{\text{fin}}$ is compact with the subspace topology inherited from $\mathbb{A}_\mathbb{Q}$.

(c) Conclude that $[0,1] \times \hat{\mathbb{Z}} \subseteq \mathbb{A}_\mathbb{Q}$ is also compact.

(d) Define the embedding $i_{\text{diag}} : \mathbb{Q} \to \mathbb{A}_\mathbb{Q}$ by $i_{\text{diag}}(q) = (q, q, q, q, \dots)$ for all $q \in \mathbb{Q}$. That is, $i_{\text{diag}}(q)_v = q$ for all places v. Show that the set $[0,1) \times \prod \mathbb{Z}_p$ is a fundamental domain for the action of Q on $\mathbb{A}_\mathbb{Q}$ via addition, and conclude that $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ is compact.

(e) Show that $U = (-1/2, 1/2) \times \prod \mathbb{Z}_p$ is a neighborhood of the origin $0 \in \mathbb{A}_\mathbb{Q}$ such that 0 is the only rational point in it. Conclude that $\mathbb{Q}$ is discrete in $\mathbb{A}_\mathbb{Q}$.

(f) Use the Chinese remainder theorem to show that $\mathbb{Q}$ is dense in $\mathbb{A}_{\text{fin}}$, with respect to the embedding $i_{\text{fin}} : \mathbb{Q} \to \mathbb{A}_{\text{fin}}$ given by $i_{\text{fin}}(q) = (0, q, q, q, \dots)$ for all $q \in \mathbb{Q}$. That is, $i_{\text{fin}}(q)_\infty = 0$, $i_{\text{fin}}(q)_p = q$ for all fininte primes p.

(g) Show that $\mathbb{R} + \mathbb{Q}$ is dense in $\mathbb{A}_\mathbb{Q}$. This is known as strong approximation.

Definition 1. Given any function $f : \mathbb{A}_\mathbb{Q} \to \mathbb{C}$, we will say that f is *smooth* if for every $x \in \mathbb{A}_\mathbb{Q}$, there is an open neighborhood $U = \prod U_v$ of x such that $f(u) = \hat{f}(u_\infty)$ for all $u \in U$ and some smooth function $\hat{f} : U_\infty \to \mathbb{C}$. Note that since $U_\infty \subseteq \mathbb{R}$, the notion of smoothness for $\hat{f}$ is already defined. Essentially, this means that locally, $f(x)$ is constant in all of its p-adic parameters x_p and smooth in x_∞.

Question 2

In this question we give a somewhat concrete definition of the isomorphism $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \cong \varprojlim \mathbb{R}/N\mathbb{Z}$. We show how to explicitly give an isomorphism

$$\varinjlim \mathrm{Func}^{\infty}(\mathbb{R}/N\mathbb{Z}) \to \mathrm{Func}^{\infty}(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q})$$

called *the adelic lift*, and give as an example the *adelic exponent* (we let $\mathrm{Func}^{\infty}(X)$ denote the set of smooth functions $f : X \to \mathbb{C}$).

(a) Use the Chinese remainder theorem to define a canonical morphism

$$(\frac{1}{p^m}\mathbb{Z})/(p^n\mathbb{Z}) \to (\frac{1}{p^m}\mathbb{Z})/(N' \cdot p^n\mathbb{Z}),$$

where p is a finite prime and N' is coprime to p. Conclude that there are morphisms

$$\frac{1}{p^m}\mathbb{Z}_p \to \mathbb{R}/N\mathbb{Z}$$

for all natural N. Show that these morphisms are in fact a compatible system of morphisms as N ranges over all natural numbers, and deduce that they induce a map

$$\frac{1}{p^m}\mathbb{Z}_p \to \varprojlim \mathbb{R}/N\mathbb{Z}.$$

Show that these maps too form a compatible system of morphisms as m varies, and conclude that this induces a morphism

$$\pi_p : \mathbb{Q}_p \to \varprojlim \mathbb{R}/N\mathbb{Z}.$$

(b) Show that the canonical map

$$\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/N\mathbb{Z} \xrightarrow{i} \varprojlim \mathbb{R}/N\mathbb{Z}$$

induced by the inclusion $\mathbb{Z} \subseteq \mathbb{R}$ gives a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_p & \longrightarrow & \mathbb{Q}_p \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi_p} \\ \hat{\mathbb{Z}} = \prod \mathbb{Z}_p & \xrightarrow{i} & \varprojlim \mathbb{R}/N\mathbb{Z}. \end{array}$$

(c) Show that the diagram

$$\begin{array}{ccc} \hat{\mathbb{Z}} \oplus \bigoplus \mathbb{Q}_p & & \\ \big\downarrow & \searrow^{i \oplus \bigoplus \pi_p} & \\ \mathbb{A}_{\mathrm{fin}} & \overset{\pi_{\mathrm{fin}}}{\dashrightarrow} & \varprojlim \mathbb{R}/N\mathbb{Z} \end{array}$$

can be completed to a commutative diagram as indicated.

(d) Consider the morphism $\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \mathbb{A}_{\mathrm{fin}} \xrightarrow{\pi} \varprojlim \mathbb{R}/N\mathbb{Z}$ given by

$$\pi = \pi_{\infty} \times (-\pi_{\mathrm{fin}}),$$

where $\pi_{\infty} : \mathbb{R} \to \varprojlim \mathbb{R}/N\mathbb{Z}$ is the projection. Show that π induces an isomorphism $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \xrightarrow{\sim} \varprojlim \mathbb{R}/N\mathbb{Z}$.

(e) Given a smooth periodic function $f \in \varinjlim \mathrm{Func}^\infty(\mathbb{R}/N\mathbb{Z})$ of period N, define its *adelic lift* $\tilde{f} \in \mathrm{Func}^\infty(\mathbb{A}_\mathbb{Q}/\mathbb{Q})$ as follows. First, let $N = \prod p^{a_p}$ be the prime decomposition of N, with $a_p = 0$ for almost all p. Consider the compact set $K(N) = \prod p^{a_p}\mathbb{Z}_p = N\hat{\mathbb{Z}} \subseteq \mathbb{A}_{\mathrm{fin}}$. For any $x \in \mathbb{A}_\mathbb{Q}$, use strong approximation to find $r \in \mathbb{R}$, $q \in \mathbb{Q}$ such that $r + q \in x + K(N)$ (note that $K(N)$ is not open in $\mathbb{A}_\mathbb{Q}$, but $\mathbb{R} + K(N)$ is open). Define

$$\tilde{f}(x) = f(r).$$

Check that $\tilde{f}$ is well defined, smooth and invariant to translation by $\mathbb{Q}$ (that is, $\tilde{f} \in \mathrm{Func}^\infty(\mathbb{A}_\mathbb{Q}/\mathbb{Q})$).

Remark 2. In the adelic language, a function invariant to translation by $\mathbb{Q}$ is often called *periodic.*

(f) Show that the map $f \mapsto \tilde{f}$ has an inverse given by

$$f(r) = \tilde{f}((r, 0, 0, 0, \dots)) = \tilde{f} \circ i_\infty(r),$$

where $i_\infty : \mathbb{R} \to \mathbb{A}_\mathbb{Q}$ is the embedding at infinity.

(g) As an example, consider the function $e(r) = \exp(2\pi i r) \in \mathrm{Func}^\infty(\mathbb{R}/\mathbb{Z})$. Explicitly compute the function $\tilde{e}$ to show that

- $\tilde{e}(k) = 1$ for all $k \in K(1) = \hat{\mathbb{Z}} = \prod \mathbb{Z}_p$.
- $\tilde{e}$ is multiplicative, that is, $\tilde{e}(x + y) = \tilde{e}(x) \cdot \tilde{e}(y)$ for all $x, y \in \mathbb{A}_\mathbb{Q}$.
- There are unique functions $e_v : \mathbb{Q}_v \to \mathbb{C}$ for every place v such that

$$\tilde{e}(x) = \prod e_v(x_v) = e_\infty(x_\infty) \cdot e_2(x_2) \cdot e_3(x_3) \cdot e_5(x_5) \cdots,$$

  for all $x \in \mathbb{A}_\mathbb{Q}$, where $x = (x_\infty, x_2, x_3, x_5, \dots)$.
- $e_\infty(x_\infty)$ is smooth, and the functions e_p are locally constant for all finite primes p, and all such functions are multiplicative.
- Finally, if $x_\infty \in \mathbb{R}$ then

$$e_\infty(x_\infty) = \exp(2\pi i x_\infty).$$

  Also, if $x_p \in \mathbb{Q}_p$, then

$$e_p(x_p) = \exp(-2\pi i \{x_p\}),$$

  where $\{x_p\}$ is the fractional part of x_p (that is, $\{x_p\}$ is some rational number such that $\{x_p\} \equiv x_p \pmod{\mathbb{Z}_p}$).

Question 3

We now define another important object of study, the group of *ideles*,

$$\mathbb{A}_\mathbb{Q}^\times = GL_1(\mathbb{A}_\mathbb{Q}) = \\ \left\{(x_\infty, x_2, x_3, x_5, x_7 \dots) \mid x_\infty \in \mathbb{R}^\times,\ x_p \in \mathbb{Q}_p^\times \text{ for all } p, \text{ and } x_p \in \mathbb{Z}_p^\times \text{ for almost all } p\right\}.$$

We would now like to define a topology on $\mathbb{A}_\mathbb{Q}^\times$. However, the naive topology given by the subspace topology induced on the ideles from the adeles via the embedding $i : \mathbb{A}_\mathbb{Q}^\times \to \mathbb{A}_\mathbb{Q}$ is, as we will see, inadequate. Let us refer to this topology by $\mathcal{T}_{\mathrm{naive}}$.

As we will see below, the correct topology on $\mathbb{A}_\mathbb{Q}^\times$ is the topology induced from the embedding $(i, i^{-1}) : \mathbb{A}_\mathbb{Q}^\times \to \mathbb{A}_\mathbb{Q} \times \mathbb{A}_\mathbb{Q}$, where $i^{-1}(x) = i(x)^{-1}$ is the inverse of i, and the topology on $\mathbb{A}_\mathbb{Q} \times \mathbb{A}_\mathbb{Q}$ is the usual product topology. Let us refer to this topology on $\mathbb{A}_\mathbb{Q}^\times$ as $\mathcal{T}$.

(a) Show that the topology $\mathcal{T}$ is generated by all sets of the form: $U = \prod U_v = U_\infty \times \prod U_p$, where U_v is open in $\mathbb{Q}_v^\times$ for all places v, and $U_p = \mathbb{Z}_p^\times$ for almost all p.

(b) Show that $\mathcal{T}_{\text{naive}} \subsetneq \mathcal{T}$.

(c) Show that $\{-1, 1\} \times \prod \mathbb{Z}_p^\times$ is a maximal compact subgroup of $\mathbb{A}_\mathbb{Q}^\times$.

(d) Show that $(0, \infty) \times \prod \mathbb{Z}_p^\times$ is a fundamental domain for the multiplicative action of $\mathbb{Q}^\times$ on the ideles via the embedding $i_{\text{diag}} : \mathbb{Q}^\times \to \mathbb{A}_\mathbb{Q}^\times$ (verify that the image of $\mathbb{Q}^\times$ is indeed in $\mathbb{A}_\mathbb{Q}^\times$!).

(e) Show that $\mathbb{R} \cdot \mathbb{Q}$ is not dense in the ideles (hint: show that if $i_{\text{fin}}(q) \in \prod \mathbb{Z}_p^\times$, then $q = \pm 1$). Therefore, strong approximation does not hold on the ideles.

Question 4

In a similar manner to our treatment of $\mathbb{A}_\mathbb{Q}^\times = GL_1(\mathbb{A}_\mathbb{Q})$, let us consider $GL_2(\mathbb{A}_\mathbb{Q})$ with the correct topology, that is, the one generated by all sets of the form $U = \prod U_v = U_\infty \times \prod U_p$, where U_v is open in $GL_2(\mathbb{Q}_v)$ for all places v, and $U_p = GL_2(\mathbb{Z}_p)$ for almost all p.

(a) Show that $K = O_2(\mathbb{R}) \times \prod GL_2(\mathbb{Z}_p)$ is a maximal compact subgroup of $GL_2(\mathbb{A}_\mathbb{Q})$.

(b) Let D_∞ be a fundamental domain for $GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{R})$. Show that $D_\infty \times \prod GL_2(\mathbb{Z}_p)$ is a fundamental domain for $GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_\mathbb{Q})$.

(c) Show that $SL_2(\mathbb{R}) \cdot SL_2(\mathbb{Q})$ is dense in $SL_2(\mathbb{A}_\mathbb{Q})$ (hint: show that it can approximate elements of the form $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ for $u \in \mathbb{A}_\mathbb{Q}$, and that such elements generate $SL_2(\mathbb{A}_\mathbb{Q})$).

(d) Show that

$$GL_2(\mathbb{R}) \cdot GL_2(\mathbb{Q})$$

is not dense in $GL_2(\mathbb{A}_\mathbb{Q})$, but that

$$GL_2(\mathbb{R}) \cdot GL_2(\mathbb{Q}) \cdot Z(GL_2(\mathbb{A}_\mathbb{Q}))$$

is dense in $GL_2(\mathbb{A}_\mathbb{Q})$.