CHAPTER IV

Cohomology of Groups

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1. Definition of Cohomology 94
2. The Standard Complex 96
3. Homology 97
4. Change of Groups 98
5. The Restriction–Inflation Sequence 100
6. The Tate Groups 101
7. Cup-products 105
8. Cyclic Groups; Herbrand Quotient 108
9. Cohomological Triviality 111
10. Tate's Theorem 113

1. Definition of Cohomology

Let G be a group, $\Lambda = \mathbf{Z}[G]$ its integral group ring. A (left) G-module is the same thing as a (left) Λ-module. By G-module we shall always mean *left* G-module. Note that if A is a left G-module we can define a right G-module structure on A by putting $a.g = g^{-1}.a$.

If A, B are G-modules, the group of all abelian group homomorphisms $A \to B$ is denoted by $\operatorname{Hom}(A, B)$, and the group of all G-module homomorphisms by $\operatorname{Hom}_G(A, B)$. $\operatorname{Hom}(A, B)$ has a G-module structure defined as follows: if $\varphi \in \operatorname{Hom}(A, B)$, $g.\varphi$ is the mapping $a \mapsto g.\varphi(g^{-1}a)$ $(a \in A)$.

For any G-module A, the subset of elements of A invariant under the action of G is denoted by A^G. A^G is an abelian group which depends functorially on A. It is the largest submodule of A on which G acts trivially. If A, B are G-modules then

$$\operatorname{Hom}_G(A, B) = (\operatorname{Hom}(A, B))^G; \tag{1.1}$$

in particular,

$$\operatorname{Hom}_G(\mathbf{Z}, A) = (\operatorname{Hom}(\mathbf{Z}, A))^G \cong A^G,$$

regarding $\mathbf{Z}$ as a G-module on which G acts trivially. Since the functor Hom is left-exact, it follows that A^G is a left-exact covariant functor of A, i.e. if

$$0 \to A \to B \to C \to 0 \tag{1.2}$$

is an exact sequence of G-modules, then

$$0 \to A^G \to B^G \to C^G$$

is an exact sequence of abelian groups.

If X is any abelian group we can form the G-module $\operatorname{Hom}(\Lambda, X)$. A G-module of this type is said to be *co-induced.*

By a *cohomological extension* of the functor A^G we mean a sequence of functors $H^q(G, A)$ $(q = 0, 1, \ldots)$, with $H^0(G, A) = A^G$, together with *connecting* (or *boundary*) homomorphisms

$$\delta : H^q(G, C) \to H^{q+1}(G, A)$$

defined functorially for exact sequences (1.2), such that

(i) the sequence

$$\ldots \to H^q(G, A) \to H^q(G, B) \to H^q(G, C) \xrightarrow{\delta} H^{q+1}(G, A) \to \ldots \tag{1.3}$$

is exact; and

(ii) $H^q(G, A) = 0$ for all $q \geq 1$ if A is co-induced.

THEOREM 1. *There exists one and, up to canonical equivalence, only one cohomological extension of the functor A^G.*

The groups $H^q(G, A)$, uniquely determined by Theorem 1, are called the *cohomology groups* of the G-module A.

The existence part of Theorem 1 is established by the following construction. Choose a resolution P of the G-module $\mathbf{Z}$ (G acting trivially on $\mathbf{Z}$) by free G-modules:

$$\ldots \to P_1 \to P_0 \to \mathbf{Z} \to 0$$

and form the complex $K = \operatorname{Hom}_G(P, A)$, i.e.

$$0 \to \operatorname{Hom}_G(P_0, A) \to \operatorname{Hom}_G(P_1, A) \to \ldots.$$

Let $H^q(K)$ $(q \geqslant 0)$ denote the qth cohomology group of this complex. Then $H^q(G, A) = H^q(K)$ satisfies the conditions for a cohomological extension of the functor A^G. For by a basic theorem of homological algebra, the $H^q(G, A)$ so defined satisfy the exactness property (1.3); also $H^0(G, A) = H^0(K) = \operatorname{Hom}_G(\mathbf{Z}, A) = A^G$; finally, if A is co-induced, say $A = \operatorname{Hom}(\Lambda, X)$ where X is an abelian group, then for any G-module B we have

$$\operatorname{Hom}_G(B, A) \cong \operatorname{Hom}(B, X)$$

(the isomorphism being as follows: if $\varphi : B \to A$ is a G-homomorphism, then φ corresponds to the map $B \to X$ defined by $b \mapsto \varphi(b)(1)$, where 1 is the identity element of G). Hence the complex K is now

$$0 \to \operatorname{Hom}(P_0, X) \to \operatorname{Hom}(P_1, X) \to \ldots$$

which is exact at every place after the first, because the P_i are free as abelian groups; and therefore $H^q(G, A) = 0$ for all $q \geqslant 1$.

To prove the uniqueness of the cohomology groups we consider, for each G-module A, the G-module $A^* = \operatorname{Hom}(\Lambda, A)$. There is a natural injection

$A \to A^*$ which maps $a \in A$ to φ_a, where φ_a is defined by $\varphi_a(g) = ga$. Hence we have an exact sequence of G-modules

$$0 \to A \to A^* \to A' \to 0 \tag{1.4}$$

where $A' = A^*/A$; since A^* is co-induced, it follows from (1.3) that

$$\delta : H^q(G, A') \to H^{q+1}(G, A) \tag{1.5}$$

is an isomorphism for all $q \geqslant 1$, and that

$$H^1(G, A) \cong \mathrm{Coker}\,(H^0(G, A^*) \to H^0(G, A')). \tag{1.6}$$

The $H^q(G, A)$ can therefore be constructed inductively from H^0, and so are unique up to canonical equivalence. This procedure could also be used as an inductive definition of the H^q.

Remark. It follows from the uniqueness that the $H^q(G, A)$ are independent of the resolution P of $\mathbf{Z}$ used to construct them. So we may take any convenient choice of P.

2. The Standard Complex

As a particular choice for the resolution P we can take $P_i = \mathbf{Z}[G^{i+1}]$, i.e. P_i is the free $\mathbf{Z}$-module with basis $G \times \ldots \times G$ ($(i+1)$ factors), G acting on each basis element as follows:

$$s(g_0, g_1, \ldots, g_i) = (sg_0, sg_1, \ldots, sg_i).$$

The homomorphism $d: P_i \to P_{i-1}$ is given by the well-known formula

$$d(g_0, \ldots, g_i) = \sum_{j=0}^{i} (-1)^j (g_0, \ldots, g_{j-1}, g_{j+1}, \ldots, g_i), \tag{2.1}$$

and the mapping $\varepsilon: P_0 \to \mathbf{Z}$ is that which sends each generator (g_0) to $1 \in \mathbf{Z}$. (To show that the resulting sequence

$$\ldots \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\varepsilon} \mathbf{Z} \to 0 \tag{2.2}$$

is exact, choose an element $s \in G$ and define $h: P_i \to P_{i+1}$ by the formula

$$h(g_0, \ldots, g_i) = (s, g_0, g_1, \ldots, g_i).$$

It is immediately checked that $dh + hd = 1$ and that $dd = 0$, from which exactness follows.)

An element of $K^i = \mathrm{Hom}_G(P_i, A)$ is then a function $f: G^{i+1} \to A$ such that

$$f(sg_0, sg_1, \ldots, sg_i) = s.f(g_0, g_1, \ldots, g_i).$$

Such a function is determined by its values at elements of G^{i+1} of the form $(1, g_1, g_1g_2, \ldots, g_1g_2 \ldots . g_i)$: if we put

$$\varphi(g_1, \ldots, g_i) = f(1, g_1, g_1g_2, \ldots, g_1 \ldots g_i)$$

the boundary is given by the formula

$$(d\varphi)(g_1,\ldots,g_{i+1})$$
$$= g_1 . \varphi(g_2,\ldots,g_{i+1}) + \sum_{j=1}^{i} (-1)^j \varphi(g_1,\ldots,g_j g_{j+1},\ldots,g_{i+1}) +$$
$$+(-1)^{i+1}\varphi(g_1,\ldots,g_i). \quad (2.3)$$

This shows that a 1-cocycle is a *crossed homomorphism*, i.e. a map $G \to A$ satisfying

$$\varphi(gg') = g . \varphi(g') + \varphi(g)$$

and φ is a coboundary if there exists $a \in A$ such that $\varphi(g) = ga - a$. In particular, if G acts trivially on A then

$$H^1(G, A) = \mathrm{Hom}(G, A). \quad (2.4)$$

From (2.3) we see also that a 2-cocycle is a function $\varphi : G \times G \to A$ such that

$$g_1 \varphi(g_2, g_3) - \varphi(g_1 g_2, g_3) + \varphi(g_1, g_2 g_3) - \varphi(g_1, g_2) = 0.$$

Such functions (called *factor systems*) arise in the problem of group extensions, and $H^2(G, A)$ describes the possible extensions E of G by A, i.e. exact sequences $1 \to A \to E \to G \to 1$, where A is an abelian normal subgroup of E, and G operates on A by inner automorphisms. If E is such an extension, choose a section $\sigma : G \to E$ (a system of coset representatives). Then we have

$$\sigma(g_1) . \sigma(g_2) = \varphi(g_1, g_2)\sigma(g_1 g_2)$$

for some $\varphi(g_1, g_2) \in A$. The function φ is a 2-cocycle of G with values in A; if we change the section σ, we alter φ by a coboundary, so that the class of φ in $H^2(G, A)$ depends only on the extension. Conversely, every element of $H^2(G, A)$ arises from an extension of G by A in this way.

For later use, we give an explicit description of the connecting homomorphism $\delta : H^0(G, C) \to H^1(G, A)$ in the exact sequence (1.3). Let $c \in H^0(G, C) = C^G$, and lift c up to $b \in B$. Then db is the function $s \mapsto sb - b$; the image of $sb - b$ in C is zero, hence $sb - b \in A$ and therefore db is a 1-cocycle of G with values in A. If we change b by the addition of an element of A, we change db by a coboundary, hence the class of db in $H^1(G, A)$ depends only on c, and is the image of c under δ.

3. Homology

If A, B are G-modules, $A \otimes B$ denotes their tensor product over $\mathbf{Z}$, and $A \otimes_G B$ their tensor product over Λ. $A \otimes B$ has a natural G-module structure, defined by $g(a \otimes b) = (ga) \otimes (gb)$.

Let I_G be the kernel of the homomorphism $\Lambda \to \mathbf{Z}$ which maps each $s \in G$ to $1 \in \mathbf{Z}$. I_G is an ideal of Λ, generated by all $s - 1$ ($s \in G$). From the exact sequence

$$0 \to I_G \to \Lambda \to \mathbf{Z} \to 0 \quad (3.1)$$

and the right-exactness of $\otimes$ it follows that, for any G-module A,

$$\mathbf{Z} \otimes_G A \cong A/I_G A.$$

The G-module $A/I_G A$ is denoted by A_G. It is the largest quotient module of A on which G acts trivially. Clearly A_G is a right-exact functor of A. For any two G-modules A, B we have

$$A \otimes_G B \cong (A \otimes B)_G. \tag{3.2}$$

A G-module of the form $\Lambda \otimes X$, where X is any abelian group, is said to be *induced.* By interchanging right and left, induced and co-induced, we define a *homological extension* of the functor A_G.

THEOREM 2. *There exists a unique homological extension of the functor* A_G

The homology group $H_q(G, A)$ given by Theorem 2 may be constructed from the standard complex P of § 2 by taking

$$H_q(G, A) = H_q(P \otimes_G A).$$

Uniqueness follows by using the exact sequence

$$0 \to A' \to A_* \to A \to 0 \tag{3.3}$$

where $A_* = \Lambda \otimes A$. The details are exactly similar to those of the proof of Theorem 1.

The connecting homomorphism $\delta : H_1(G, C) \to H_0(G, A)$ may be described explicitly as follows. A 1-cycle of G with values in C is a function $f: G \to C$ such that $f(s) = 0$ for almost all $s \in G$ and such that $df = \sum_{s \in G} (s^{-1} - 1) f(s) = 0$. For each $s \in G$ lift $f(s)$ to $\bar{f}(s) \in B$ (if $f(s) = 0$, choose $\bar{f}(s) = 0$). Then $d\bar{f}$ has zero image in C, hence is an element of A. The class of $d\bar{f}$ in $H_0(G, A)$ is then the image under δ of the class of f.

PROPOSITION 1. $H_1(G, \mathbf{Z}) \cong G/G'$, *where G' is the commutator subgroup of G.*

Proof. From the exact sequence (3.1) and the fact that Λ is an induced G-module, the connecting homomorphism

$$\delta : H_1(G, \mathbf{Z}) \to H_0(G, I_G) = I_G/I_G^2$$

is an isomorphism. On the other hand, the map $s \mapsto s - 1$ induces an isomorphism of G/G' onto I_G/I_G^2.

4. Change of Groups

Let G' be a subgroup of G. If A' is a G'-module, we can form the G-module $A = \mathrm{Hom}_{G'}(\Lambda, A')$: A is really a right G-module, but we turn it into a left G-module as described in § 1 (if $\varphi \in A$, then $g . \varphi$ is the homomorphism $g' \mapsto \varphi(g'g^{-1})$). Then we have

$g' \mapsto \varphi(g'g)$

PROPOSITION 2 (Shapiro's Lemma).

$$H^q(G, A) = H^q(G', A') \quad \text{for all } q \geqslant 0.$$

Proof. If P is a free Λ-resolution of $\mathbf{Z}$ it is also a free Λ'-resolution, and $\mathrm{Hom}_G(P, A) \cong \mathrm{Hom}_{G'}(P, A')$.

The analogous result holds for homology, with Hom replaced by $\otimes$. Note that Prop. 2 may be regarded as a generalization of property (ii) of the cohomology groups (§ 1): if $G' = (1)$, then $\Lambda' = \mathbf{Z}$ and A is a co-induced module, and the $H^q(G', A')$ are zero for $q \geqslant 1$.

If $f: G' \to G$ is a homomorphism of groups, it induces a homomorphism $P' \to P$ of the standard complexes, hence a homomorphism

$$f^* : H^q(G, A) \to H^q(G', A)$$

for any G-module A. (We regard A as a G'-module via f.) In particular, taking $G' = H$ to be a subgroup of G, and f to be the embedding $H \to G$, we have *restriction* homomorphisms

$$\mathrm{Res} : H^q(G, A) \to H^q(H, A).$$

If H is a normal subgroup of G we consider $f: G \to G/H$. For any G-module A we have the G/H-module A^H and hence a homomorphism $H^q(G/H, A^H) \to H^q(G, A^H)$. Composing this with the homomorphism induced by $A^H \to A$ we obtain the *inflation* homomorphisms

$$\mathrm{Inf} : H^q(G/H, A^H) \to H^q(G, A).$$

Similarly, for homology, a homomorphism $f: G' \to G$ gives rise to a homomorphism

$$f_* : H_q(G', A) \to H_q(G, A);$$

in particular, taking $G' = H$ to be a subgroup of G, and $f: H \to G$ the embedding, we have the *corestriction* homomorphisms

$$\mathrm{Cor} : H_q(H, A) \to H_q(G, A).$$

Consider the inner automorphism $s \mapsto tst^{-1}$ of G. This turns A into a new G-module, denoted by A^t, and gives a homomorphism

$$H^q(G, A) \to H^q(G, A^t). \tag{4.1}$$

Now $a \mapsto t^{-1}a$ defines an isomorphism $A^t \to A$ and hence induces

$$H^q(G, A^t) \to H^q(G, A). \tag{4.2}$$

PROPOSITION 3. *The composition of* (4.1) *and* (4.2) *is the identity map of* $H^q(G, A)$.

The proof employs a standard technique, that of dimension-shifting: we verify the result for $q = 0$ and then proceed by induction on q, using (1.5) to shift the dimension downwards.

For $q = 0$, we have $H^0(G, A^t) = (A^t)^G = t.A^G$, and (4.1) is just multiplication by t. Since (4.2) is multiplication by t^{-1}, the composition is the identity.

Now assume that $q > 0$ and that the result is true for $q-1$. Corresponding to the exact sequence (1.4) we have an exact sequence

$$0 \to A^t \to (A^*)^t \to (A')^t \to 0.$$

Since $(A^*)^t$ is G-isomorphic to A^*, it is a co-induced module, hence we have functorial isomorphisms

$$H^q(G, A^t) \cong H^{q-1}(G, (A')^t) \quad (q \geqslant 2)$$

and

$$H^1(G, A^t) \cong \operatorname{Coker}(H^0(G, (A^*)^t) \to H^0(G, (A')^t)).$$

Now apply the inductive hypothesis.

5. The Restriction–Inflation Sequence

PROPOSITION 4. *Let H be a normal subgroup of G, and let A be a G-module. Then the sequence*

$$0 \to H^1(G/H, A^H) \xrightarrow{\text{Inf}} H^1(G, A) \xrightarrow{\text{Res}} H^1(H, A)$$

is exact.

The proof is by direct verification on cocycles.

(1) *Exactness at $H^1(G/H, A^H)$.* Let $f\colon G/H \to A^H$ be a 1-cocycle, then f induces $\bar{f}\colon G \to G/H \to A^H \to A$, which is a 1-cocycle, and the class of $\bar{f}$ is the inflation of the class of f. Hence if $\bar{f}$ is a coboundary, there exists $a \in A$ such that $\bar{f}(s) = sa - a$ $(s \in G)$. But $\bar{f}$ is constant on the cosets of H in G, hence $sa - a = sta - a$ for all $t \in H$, i.e. $ta = a$ for all $t \in H$. Hence $a \in A^H$ and therefore f is a coboundary.

(2) $\text{Res} \circ \text{Inf} = 0$. If $\varphi\colon G \to A$ is a 1-cocycle, then the class of $\varphi|H\colon H \to A$ is the restriction of the class of φ. But if $\varphi = \bar{f}$, it is clear that $\bar{f}|H$ is constant and equal to $f(1) = 0$.

(3) *Exactness at $H^1(G, A)$.* Let $\varphi\colon G \to A$ be a 1-cocycle whose restriction to H is a coboundary; then there exists $a \in A$ such that $\varphi(t) = ta - a$ for all $t \in H$. Subtracting from φ the coboundary $s \mapsto sa - a$, we are reduced to the case where $\varphi|H = 0$. The formula

$$\varphi(st) = \varphi(s) + s.\varphi(t)$$

then shows (taking $t \in H$) that φ is constant on the cosets of H in G, and then (taking $s \in H$, $t \in G$) that the image of φ is contained in A^H. Hence φ is the inflation of a 1-cocycle $G/H \to A^H$, and the proof is complete.

PROPOSITION 5. *Let $q \geqslant 1$, and suppose that $H^i(H, A) = 0$ for $1 \leqslant i \leqslant q-1$. Then the sequence*

$$0 \to H^q(G/H, A^H) \xrightarrow{\text{Inf}} H^q(G, A) \xrightarrow{\text{Res}} H^q(H, A)$$

is exact.

This is another example of dimension-shifting: we reduce to the case $q = 1$, which is Proposition 4. Suppose then that $q > 1$ and that the result is true for $q-1$. In the exact sequence (1.4), the G-module A^* is co-induced *as an H-module* (since $\Lambda = \mathbf{Z}[G]$ is a free $\mathbf{Z}[H]$-module), hence

$$H^i(H, A') \cong H^{i+1}(H, A) = 0 \quad \text{for } 1 \leqslant i \leqslant q-2.$$

Also, since $H^1(H, A) = 0$, the sequence

$$0 \to A^H \to (A^*)^H \to (A')^H \to 0$$

is exact, and $(A^*)^H$ is co-induced as a G/H-module (because $(A^*)^H \cong \mathrm{Hom}\,(\mathbf{Z}[G/H], A)$). Hence in the diagram

$$\begin{array}{ccccccc}
0 \to & H^{q-1}(G/H, (A')^H) & \to & H^{q-1}(G, A') & \to & H^{q-1}(H, A') \\
 & \downarrow\delta & & \downarrow\delta & & \downarrow\delta \\
0 \to & H^q(G/H, A^H) & \to & H^q(G, A) & \to & H^q(H, A)
\end{array}$$

the three vertical arrows are isomorphisms, the diagram is commutative, and by the inductive assumption applied to A', the top line is exact. Hence so is the bottom line.

COROLLARY. *Under the hypotheses of Prop. 5,*

$$H^i(G/H, A^H) \cong H^i(G, A), \quad 1 \leqslant i \leqslant q-1.$$

6. The Tate Groups

From now on we assume that G is finite, and we denote by N the element $\sum_{s \in G} s$ of Λ. For any G-module A, multiplication by N defines an endomorphism $N: A \to A$, and clearly

$$I_G A \subseteq \mathrm{Ker}\,(N), \qquad \mathrm{Im}\,(N) \subseteq A^G.$$

Hence N induces a homomorphism

$$N^* : H_0(G, A) \to H^0(G, A)$$

and we define

$$\hat{H}_0(G, A) = \mathrm{Ker}\,(N^*), \qquad \hat{H}^0(G, A) = \mathrm{Coker}\,(N^*) = A^G/N(A).$$

Since G is finite, we can define a mapping $\mathrm{Hom}\,(\Lambda, X) \to \Lambda \otimes X$ (where X is any abelian group) by the rule

$$\varphi \mapsto \sum_{s \in G} s \otimes \varphi(s),$$

and it is immediately verified that this is a G-module isomorphism. Hence for a finite group the notions of induced and co-induced modules coincide.

PROPOSITION 6. *If A is an induced G-module, then $\hat{H}_0(G, A) = \hat{H}^0(G, A) = 0$.*

Proof. Let $A = \Lambda \otimes X$, X an abelian group. Since Λ is $\mathbf{Z}$-free, every element of A is uniquely of the form $\sum_{s \in G} s \otimes x_s$. If this element is G-invariant, then $\sum_s gs \otimes x_s = \sum_s s \otimes x_s$ for all $g \in G$, from which it follows that all the x_s are equal. Hence such an element is of the form $N.(1 \otimes x)$ and therefore lies in $N(A)$. Hence $\hat{H}^0(G, A) = 0$.

Similarly, if $N.\sum_s s \otimes x_s = 0$, we find that $\sum x_s = 0$, and therefore $\sum s \otimes x_s = \sum (s-1)(1 \otimes x_s) \in I_G A$. Hence $\hat{H}_0(G, A) = 0$.

Now we define the Tate cohomology groups $\hat{H}^q(G, A)$ for *all* integers q by

$$\begin{aligned} \hat{H}^q(G, A) &= H^q(G, A) \qquad \text{for } q \geqslant 1 \\ \hat{H}^{-1}(G, A) &= \hat{H}_0(G, A) \\ \hat{H}^{-q}(G, A) &= H_{q-1}(G, A) \quad \text{for } q \geqslant 2. \end{aligned}$$

THEOREM 3. *For every exact sequence of G-modules*

$$0 \to A \to B \to C \to 0$$

we have an exact sequence

$$\ldots \to \hat{H}^q(G, A) \to \hat{H}^q(G, B) \to \hat{H}^q(G, C) \xrightarrow{\delta} \hat{H}^{q+1}(G, A) \to \ldots$$

Proof. We have to splice together the homology and cohomology sequences. Consider the diagram

$$\begin{array}{ccccccccc} \ldots \to H_1(G, C) & \xrightarrow{\delta} & H_0(G, A) & \to & H_0(G, B) & \to & H_0(G, C) & \to & 0 \\ \downarrow & & \downarrow{\scriptstyle N_A^*} & & \downarrow{\scriptstyle N_B^*} & & \downarrow{\scriptstyle N_C^*} & & \downarrow \\ 0 & \to & H^0(G, A) & \to & H^0(G, B) & \to & H^0(G, C) & \xrightarrow{\delta} & H^1(G, A) \to \ldots \end{array}$$

where N_A^* is the homomorphism N^* relative to A, and so on. It is clear that the inner two squares are commutative, and for the outer two squares commutativity follows immediately from the explicit descriptions of the connecting homomorphism δ given in §§ 2 and 3.

We define $\hat{\delta} : \hat{H}_0(G, C) \to \hat{H}^0(G, A)$ as follows. If $c \in \hat{H}_0(G, C) = \operatorname{Ker}(N_C^*)$, lift c to $b \in H_0(G, B)$, then $N_B^*(b) \in H^0(G, B)$ has zero image in $H^0(G, C)$ and therefore comes from an element $a \in H^0(G, A)$, whose image in $\hat{H}^0(G, A)$ is independent of the choice of b; this element of $\hat{H}^0(G, A)$ we define to be (c). The definitions of the other maps in the sequence

$$\begin{aligned} H_1(G, C) \to \hat{H}_0(G, A) \to \hat{H}_0(G, B) \to \hat{H}_0(G, C) \\ \xrightarrow{\delta} \hat{H}^0(G, A) \to \hat{H}^0(G, B) \to \hat{H}^0(G, C) \to H^1(G, A) \end{aligned}$$

are the obvious ones, and the verification that the whole sequence is exact is a straightforward piece of diagram-chasing.

The Tate groups can be considered as the cohomology groups of a complex constructed out of a *complete resolution* of G. Let P denote a G-resolution of $\mathbf{Z}$ by finitely-generated free G-modules (for example, the standard resolution of § 2), and let $P^* = \operatorname{Hom}(P, \mathbf{Z})$ be its dual, so that we have exact sequences

$$\ldots \to P_1 \to P_0 \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

$$0 \to \mathbf{Z} \xrightarrow{\varepsilon^*} P_0^* \to P_1^* \to \ldots$$

(the dual sequence is exact because each P_i is $\mathbf{Z}$-free). Putting $P_{-n} = P_{n-1}^*$ and splicing the two sequences together we get a doubly-infinite exact sequence

$$L: \qquad \ldots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \ldots.$$

The Tate groups are then the cohomology groups $H^q(\operatorname{Hom}_G(L, A))$ for any G-module A. This assertion is clear if $q \geqslant 1$. If $q \leqslant -2$ we use the following fact: if C is a finitely generated free G-module, let $C^* = \operatorname{Hom}(C, \mathbf{Z})$ be its dual; then the mapping $\sigma: C \otimes A \to \operatorname{Hom}(C^*, A)$ defined as follows:

$$\sigma(c \otimes a) \text{ maps } f \in C^* \text{ to } f(c).a$$

is a G-module isomorphism. Hence the composition

$$\tau: C \otimes_G A = (C \otimes A)_G \xrightarrow{N^*} (C \otimes A)^G \xrightarrow{\sigma} (\operatorname{Hom}(C^*, A))^G = \operatorname{Hom}_G(C^*, A)$$

is an isomorphism (N^* is an isomorphism because $C \otimes A$ is an induced G-module). From this it follows that $\operatorname{Hom}_G(P_{-n}, A) \cong P_{n-1} \otimes_G A$, and hence that $H^{-q}(\operatorname{Hom}_G(L, A)) = H_{q-1}(G, A)$ for $q \geqslant 2$.

Finally, we have to consider the cases $q = 0, 1$. The mapping

$$\operatorname{Hom}_G(P_{-1}, A) \to \operatorname{Hom}_G(P_0, A) \tag{6.1}$$

is induced by the composition $P_0 \xrightarrow{\varepsilon} \mathbf{Z} \xrightarrow{\varepsilon^*} P_{-1}$. If we identify $\operatorname{Hom}_G(P_{-1}, A)$ with $P_0 \otimes_G A$ by means of the isomorphism τ, the mapping (6.1) becomes a mapping from $P_0 \otimes_G A$ to $\operatorname{Hom}_G(P_0, A)$, and from the definition of τ it is not difficult to see that this mapping factorizes into

$$P_0 \otimes_G A \to A_G \xrightarrow{N^*} A^G \to \operatorname{Hom}_G(P_0, A) \tag{6.2}$$

where the extreme arrows are the mappings induced by ε. From this it follows that $H^q(\operatorname{Hom}_G(L, A)) = \hat{H}^q(G, A)$ for $q = 0, 1$.

Remark. Since any G-module can be expressed either as a sub-module or as a quotient module of an induced module, it follows from Prop. 6 and Theorem 3 that the Tate groups $\hat{H}^q$ can be "shifted" both up and down.

If H is a subgroup of G, the restriction homomorphism

$$\operatorname{Res}: H^q(G, A) \to H^q(H, A)$$

has been defined for all $q \geqslant 0$. It is therefore defined for the Tate groups $\hat{H}^q$, $q \geqslant 1$, and commutes with the connecting homomorphism δ. By dimension-shifting it then gets extended to all $\hat{H}^q$ (use the exact sequence (3.3) and the fact that A_* is induced as an H-module). Similarly, the corestriction, which was defined in the first place for H_q (i.e. $\hat{H}^{-q-1}$, $q \geqslant 1$) gets extended by dimension-shifting to all $\hat{H}^q$ (use (1.4) likewise).

PROPOSITION 7. *Let H be a subgroup of G, and let A be a G-module. Then*
(i) Res : $\hat{H}_0(G, A) \to \hat{H}_0(H, A)$ *is induced by* $N'_{G/H} : A_G \to A_H$, *where*

$$N'_{G/H}(a) = \sum_i s_i^{-1} a$$

and (s_i) is a system of coset representatives of G/H;
(ii) Cor : $\hat{H}^0(H, A) \to \hat{H}^0(G, A)$ *is induced by* $N_{G/H} : A^H \to A^G$, *where*

$$N_{G/H}(a) = \sum_i s_i a.$$

We shall prove (i), and leave (ii) to the reader. First of all, since $\hat{\delta} : \hat{H}^0(G, A) \to H^1(G, A')$ is induced by $\delta : H^0(G, A) \to H^1(G, A)$, and since Res : $H^0(G, A) \to H^0(H, A)$ is the embedding $A^G \to A^H$ and is compatible with δ, it follows that Res : $\hat{H}^0(G, A) \to \hat{H}^0(H, A)$ is induced by $A^G \to A^H$. Now let $\nu : \hat{H}_0(G, A) \to \hat{H}_0(H, A)$ be the map induced by $N'_{G/H}$. We have to check that the diagram

$$\begin{array}{ccc} \hat{H}_0(G, A) & \xrightarrow{\delta} & \hat{H}^0(G, A') \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \mathrm{Res}} \\ \hat{H}_0(H, A) & \xrightarrow{\delta} & \hat{H}^0(H, A') \end{array}$$

is commutative. Let $a \in A$ be a representative of $\bar{a} \in \hat{H}_0(G, A)$, so that $N_G(a) = 0$. Lift a to $b \in A_*$, then $N_G(b)$ has zero image in A and is G-invariant, hence belongs to $(A')^G \subseteq (A')^H$. The class of $N_G(b)$ mod. $N_H(A')$ is Res $\circ\, \hat{\delta}(\bar{a})$. On the other hand, $\nu(\bar{a})$ is the class mod. $I_H A$ of $N'_{G/H}(a)$, which lifts to $N'_{G/H}(b)$, and $\hat{\delta} \circ \nu(\bar{a})$ is represented by $N_H \circ N'_{G/H}(b) = N_G(b)$.

Note: For $q = -2$ and $A = \mathbf{Z}$ we have $\hat{H}^{-2}(G, \mathbf{Z}) = H_1(G, \mathbf{Z}) \cong G/G'$; Res : $G/G' \to H/H'$ is classically called the *transfer* and can be defined as follows. G/G' is dual to Hom $(G, \mathbf{C}^*)$, hence the transfer will be dual to a homomorphism

$$\mathrm{Hom}\,(H, \mathbf{C}^*) \to \mathrm{Hom}\,(G, \mathbf{C}^*).$$

This homomorphism is given by

$$\rho \mapsto \det(i_*\rho)/\det(\iota_* 1),$$

where $i_*\rho$ is the representation of G *induced* by ρ, and det denotes the corresponding one-dimensional representation obtained by taking determinants (Hom $(G, \mathbf{C}^*)$ is here written multiplicatively).

PROPOSITION 8. *If* $(G:H) = n$, *then*

$$\mathrm{Cor} \circ \mathrm{Res} = n.$$

Proof. For $\hat{H}^0$ this follows from Prop. 7(ii): Res is induced by the embedding $A^G \to A^H$, and Cor by $N_{G/H}: A^H \to A^G$; and $N_{G/H}(a) = na$ for all $a \in A^G$. The general case then follows by dimension-shifting.

COROLLARY 1. *If G has order n, all the groups $\hat{H}^q(G, A)$ are annihilated by n.*

Proof. Take $H = (1)$ in Prop. 8, and use the fact that $\hat{H}^q(H, A) = 0$ for all q.

COROLLARY 2. *If A is a finitely-generated G-module, all the groups $\hat{H}^q(G, A)$ are finite.*

Proof. The calculation of the $\hat{H}^q(G, A)$ from the standard complete resolution L shows that they are finitely generated abelian groups; since by Cor. 1 they are killed by $n = \mathrm{Card}\,(G)$, they are therefore finite.

COROLLARY 3. *Let S be a Sylow p-subgroup of G. Then*

$$\mathrm{Res}: \hat{H}^q(G, A) \to \hat{H}^q(S, A)$$

is a monomorphism on the p-primary component of $\hat{H}^q(G, A)$.

Proof. Let $\mathrm{Card}\,(G) = p^a.m$ where m is prime to p. Let x belong to the p-primary component of $\hat{H}^q(G, A)$, and suppose that $\mathrm{Res}\,(x) = 0$. Then

$$mx = \mathrm{Cor} \circ \mathrm{Res}(x) = 0$$

by Prop. 8, since $m = (G:S)$. On the other hand, we have $p^a x = 0$ by Cor. 1; since $(p^a, m) = 1$, it follows that $x = 0$.

COROLLARY 4. *If an element x of $\hat{H}^q(G, A)$ restricts to zero in $\hat{H}^q(S, A)$ for all Sylow subgroups S of G, then $x = 0$.*

7. Cup-products

THEOREM 4. *Let G be a finite group. Then there exists one and only one family of homomorphisms*

$$\hat{H}^p(G, A) \otimes \hat{H}^q(G, B) \to \hat{H}^{p+q}(G, A \otimes B)$$

(denoted by $(a \otimes b) \to a.b$), defined for all integers p, q and all G-modules A, B, such that:

(i) *These homomorphisms are functorial in A and B;*

(ii) *For $p = q = 0$ they are induced by the natural product*

$$A^G \otimes B^G \to (A \otimes B)^G;$$

(iii) *If $0 \to A \to A' \to A'' \to 0$ is an exact sequence of G-modules, and if $0 \to A \otimes B \to A' \otimes B \to A'' \otimes B \to 0$ is exact, then for $a'' \in \hat{H}^p(G, A'')$ and $b \in \hat{H}^q(G, B)$ we have*

$$(\delta a'').b = \delta(a''.b) \quad (\in \hat{H}^{p+q+1}(G, A \otimes B));$$

(iv) *If* $0 \to B \to B' \to B'' \to 0$ *is an exact sequence of G-modules, and if* $0 \to A \otimes B \to A \otimes B' \to A \otimes B'' \to 0$ *is exact, then for* $a \in \hat{H}^p(G, A)$ *and* $b'' \in \hat{H}^q(G, B'')$ *we have*

$$a.(\delta b'') = (-1)^p \delta(a.b'') \quad (\in \hat{H}^{p+q+1}(G, A \otimes B)).$$

Let $(P_n)_{n\in Z}$ be a complete resolution for G, as in § 6. The proof of existence depends on constructing G-module homomorphisms

$$\varphi_{p,q} : P_{p+q} \to P_p \otimes P_q$$

for all pairs of integers p, q, satisfying the following two conditions:

$$\varphi_{p,q} \circ d = (d \otimes 1) \circ \varphi_{p+1,q} + (-1)^p (1 \otimes d) \circ \varphi_{p,q+1}; \tag{7.1}$$

$$(\varepsilon \otimes \varepsilon) \circ \varphi_{0,0} = \varepsilon, \tag{7.2}$$

where $\varepsilon : P_0 \to \mathbf{Z}$ is defined by $\varepsilon(g) = 1$ for all $g \in G$.

Once the $\varphi_{p,q}$ have been defined, we proceed as follows. Let $f \in \operatorname{Hom}_G(P_p, A)$, $g \in \operatorname{Hom}_G(P_q, B)$ be cochains, and define the product cochain $f \circ g \in \operatorname{Hom}_G(P_{p+q}, A \otimes B)$ by

$$f \cdot g = (f \otimes g) \circ \varphi_{p,q}.$$

Then it follows immediately from (7.1) that

$$d(f \cdot g) = (df) \cdot g + (-1)^p f \cdot (dg). \tag{7.3}$$

Hence if f, g are cocycles, so is $f \cdot g$, and the cohomology class of $f \cdot g$ depends only on the classes of f and g: in other words, we have a homomorphism

$$\hat{H}^p(G, A) \otimes \hat{H}^q(G, B) \to \hat{H}^{p+q}(G, A \otimes B).$$

Clearly condition (i) is satisfied, and (ii) is a consequence of (7.2). Consider (iii). We have an exact sequence

$$0 \to \operatorname{Hom}_G(P_p, A) \to \operatorname{Hom}_G(P_p, A') \to \operatorname{Hom}_G(P_p, A'') \to 0.$$

Let $\alpha'' \in \operatorname{Hom}_G(P_p, A'')$ be a representative cocycle of the class a'', and lift α'' back to $\alpha' \in \operatorname{Hom}_G(P_p, A')$; $d\alpha'$ has zero image in $\operatorname{Hom}_G(P_{p+1}, A'')$ and therefore lies in $\operatorname{Hom}_G(P_{p+1}, A)$. The class of $d\alpha'$ in $\hat{H}^{p+1}(G, A)$ is $\delta(\alpha'')$. Hence if $\beta \in \operatorname{Hom}_G(P_q, B)$ is a cocycle in the class b, then $\alpha''.\beta$ represents the class $a''.b$; $d(\alpha'.\beta)$ represents $\delta(a''.b)$; and $(d\alpha').\beta$ represents $(\delta a'').b$. But (since $d\beta = 0$) we have $d(\alpha'.\beta) = (d\alpha').\beta$ from (7.3); hence $\delta(a''.b) = (\delta a'').b$. The proof of (iv) is similar.

Thus it remains to define the $\varphi_{p,q}$, which we shall do for the standard complete resolution ($P_q = \mathbf{Z}[G^{q+1}]$ if $q \geqslant 0$; $P_{-q} =$ dual of P_{q-1} if $q \geqslant 1$). If $q \geqslant 1$, $P_{-q} = P^*_{q-1}$ has a basis (as $\mathbf{Z}$-module) consisting of all $(g^*_1, \ldots, g^*_q)$, where $(g^*_1, \ldots, g^*_q)$ maps $(g_1, \ldots, g_q) \in P_{q-1}$ to $1 \in \mathbf{Z}$, and every other basis element of P_{q-1} to 0. In terms of this basis of P_{-q}, $d : P_{-q} \to P_{-q-1}$ is given by

$$d(g^*_1, \ldots, g^*_q) = \sum_{s \in G} \sum_{i=0}^{q} (-1)^i (g^*_1, \ldots, g^*_i, s^*, g^*_{i+1}, \ldots, g^*_q)$$

and $d: P_0 \to P_{-1}$ by $d(g_0) = \sum_{s \in G} (s^*)$.

We define $\varphi_{p,q}: P_{p+q} \to P_p \otimes P_q$ as follows:

(1) if $p \geqslant 0$ and $q \geqslant 0$,

$$\varphi_{p,q}(g_0, \ldots, g_{p+q}) = (g_0, \ldots, g_p) \otimes (g_p, \ldots, g_{p+q});$$

(2) if $p \geqslant 1$ and $q \geqslant 1$,

$$\varphi_{-p,-q}(g_1^*, \ldots, g_{p+q}^*) = (g_1^*, \ldots, g_p^*) \otimes (g_{p+1}^*, \ldots, g_{p+q}^*);$$

(3) if $p \geqslant 0$ and $q \geqslant 1$,

$$\varphi_{p,-p-q}(g_1^*, \ldots, g_q^*) = \sum (g_1, s_1, \ldots, s_p) \otimes (s_p^*, \ldots, s_1^*, g_1^*, \ldots, g_q^*);$$
$$\varphi_{-p-q,p}(g_1^*, \ldots, g_q^*) = \sum (g_1^*, \ldots, g_q^*, s_1^*, \ldots, s_p^*) \otimes (s_p, \ldots, s_1, g_q);$$
$$\varphi_{p+q,-q}(g_0, \ldots, g_p) = \sum (g_0, \ldots, g_p, s_1, \ldots, s_q) \otimes (s_q^*, \ldots, s_1^*);$$
$$\varphi_{-q,p+q}(g_0, \ldots, g_p) = \sum (s_1^*, \ldots, s_q^*) \otimes (s_q, \ldots, s_1, g_0, \ldots, g_p).$$

(In the sums on the right-hand side, the s_i run independently through G.) The verification that the $\varphi_{p,q}$ satisfy (7.1) is tedious, but entirely straightforward.

This completes the existence part of the proof of Theorem 4. The uniqueness is proved by starting with (ii) and shifting dimensions by (iii) and (iv): the point is that the exact sequence (3.3), namely

$$0 \to A' \to A_* \to A \to 0,$$

splits over $\mathbf{Z}$, as the $\mathbf{Z}$-homomorphism $A \to A_* = \Lambda \otimes A$ defined by $a \mapsto 1 \otimes a$ shows; hence the result of tensoring it with any G-module B is still exact, and $A_* \otimes B = \Lambda \otimes A \otimes B = (A \otimes B)_*$. Similarly for the exact sequence (1.4).

Note the following properties of the cup-product, which are easily proved by dimension-shifting:

PROPOSITION 9.

(i) $(a.b).c = a.(b.c)$ *(identifying* $(A \otimes B) \otimes C$ *with* $A \otimes (B \otimes C)$*).*

(ii) $a.b = (-1)^{\dim a.\dim b} b.a$ *(identifying* $A \otimes B$ *with* $B \otimes A$*).*

(iii) $\operatorname{Res}(a.b) = \operatorname{Res}(a).\operatorname{Res}(b)$.

(iv) $\operatorname{Cor}(a.\operatorname{Res}(b)) = \operatorname{Cor}(a).b$.

As an example, let us prove (iv). Here H is a subgroup of G, $a \in \hat{H}^p(H, A)$, $b \in \hat{H}^q(G, B)$, so that both sides of (iv) are elements of $\hat{H}^{p+q}(G, A \otimes B)$. If $p = q = 0$, a is represented by say $\alpha \in A^H$, and by Prop. 7(ii) $\operatorname{Cor}(a)$ is represented by $N_{G/H}(\alpha) = \sum_i s_i \alpha \in A^G$; b is represented by $\beta \in B^G$, hence $\operatorname{Cor}(a).b$ is represented by

$$N_{G/H}(\alpha) \otimes \beta = (\sum s_i \alpha) \otimes \beta = \sum s_i(\alpha \otimes \beta) = N_{G/H}(\alpha \otimes \beta).$$

On the other hand, $a.\operatorname{Res}(b)$ is represented by $\alpha \otimes \beta \in (A \otimes B)^H$, hence

Cor $(a.\mathrm{Res}(b))$ by $N_{G/H}(\alpha \otimes \beta)$. This establishes (iv) for $p = q = 0$. Now use dimension-shifting as in the proof of the uniqueness of the cup-products and the fact that both Cor and Res commute with the connecting homomorphisms relative to exact sequences of the types (3.3) and (1.4).

We shall later have to consider cup-products of a slightly more general type. Let A, B, C be G-modules, $\varphi : A \otimes B \to C$ a G-homomorphism. If we compose the cup-product with the cohomology homomorphism φ^* induced by φ, we have mappings

$$\hat{H}^p(G, A) \otimes \hat{H}^q(G, B) \to \hat{H}^{p+q}(G, C);$$

explicitly, $a \otimes b \mapsto \varphi^*(a.b)$. $\varphi^*(a.b)$ is the *cup-product of a, b relative to φ.*

8. Cyclic Groups; Herbrand Quotient

If G is a cyclic group of order n, and s is a generator of G, we can define a particularly simple complete resolution K for G. Each K_i is isomorphic to Λ, and $d: K_{i+1} \to K_i$ is multiplication by $T = s-1$ if i is even (resp. by N if i is odd). The kernel of T is $\Lambda^G = N.\Lambda =$ image of N, and the image of T is $I_G =$ kernel of N. Hence for any G-module A the complex $\mathrm{Hom}_G(K, A)$ is

$$\ldots \leftarrow A \xleftarrow{N} A \xleftarrow{T} A \xleftarrow{N} A \xleftarrow{T} \ldots$$

and therefore

$$\hat{H}^{2q}(G, A) = \hat{H}^0(G, A) = A^G/NA,$$
$$\hat{H}^{2q+1}(G, A) = \hat{H}_0(G, A) = {}_NA/I_GA,$$

where ${}_NA$ is the kernel of $N: A \to A$.

In particular, $H^2(G, \mathbf{Z}) = \mathbf{Z}^G/N\mathbf{Z} = \mathbf{Z}/n\mathbf{Z}$ is cyclic of order n.

THEOREM 5. *Cup-product by a generator of $H^2(G, \mathbf{Z})$ induces an isomorphism*

$$\hat{H}^q(G, A) \to \hat{H}^{q+2}(G, A)$$

for all integers q and all G-modules A.

Proof. The exact sequences

$$0 \to I_G \to \Lambda \to \mathbf{Z} \to 0, \tag{8.1}$$
$$0 \to \mathbf{Z} \xrightarrow{N} \Lambda \xrightarrow{T} I_G \to 0, \tag{8.2}$$

give rise to isomorphisms

$$\hat{H}^0(G, \mathbf{Z}) \xrightarrow{\delta} H^1(G, I_G) \xrightarrow{\delta} H^2(G, \mathbf{Z}).$$

Since both (8.1), (8.2) split over $\mathbf{Z}$, they remain exact when tensored with A, and we are therefore reduced to showing that cup-product by a generator of $\hat{H}^0(G, \mathbf{Z})$ induces an automorphism of $\hat{H}^q(G, A)$. By dimension-shifting again, we reduce to the case $q = 0$. Since $\hat{H}^0(G, \mathbf{Z}) = \mathbf{Z}/n\mathbf{Z}$, a generator b of $\hat{H}^0(G, \mathbf{Z})$ is represented by an integer β prime to n, and cup-product with b is multiplication by β. Now β is prime to n, hence there is an integer γ such that $\beta\gamma \equiv 1 \pmod{n}$; $\hat{H}^0(G, A)$ is killed by n, hence multiplication by β is an automorphism of $\hat{H}^0(G, A)$.

Let $h_q(A)$ denote the order of $\hat{H}^q(G, A)$ $(q = 0, 1)$ whenever this is finite. If both are finite we define the *Herbrand quotient*

$$h(A) = h_0(A)/h_1(A).$$

PROPOSITION 10. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of G-modules (G a cyclic group). Then if two of the three Herbrand quotients $h(A)$, $h(B)$, $h(C)$ are defined, so is the third and we have*

$$h(B) = h(A) . h(C).$$

Proof. In view of the periodicity of the $\hat{H}^q$, the cohomology exact sequence is an exact hexagon:

$$\begin{array}{ccccc} & H^0(A) & \to & H^0(B) & \\ \nearrow & & & & \searrow \\ H^1(C) & & & & H^0(C) \\ \nwarrow & & & & \swarrow \\ & H^1(B) & \leftarrow & H^1(A) & \end{array}$$

where $H^0(A)$ means $\hat{H}^0(G, A)$, and so on. Suppose for example that $H^0(A)$, $H^1(A)$, $H^0(B)$, $H^1(B)$ are finite. Let M_1 be the image of $H^0(A)$ in $H^0(B)$, and so on in clockwise order round the hexagon. Then the sequence $0 \to M_2 \to H^0(C) \to M_3 \to 0$ is exact, and M_2, M_3 are finite groups (M_2 because it is a homomorphic image of $H^0(B)$, M_3 because it is a subgroup of $H^1(A)$). Hence $H^0(C)$ is finite, and similarly $H^1(C)$ is finite. The orders of the groups $H^0(A), \ldots, H^1(C)$ are respectively $m_6 m_1, m_1 m_2, \ldots, m_5 m_6$ (m_i = order of M_i), hence $h(B) = h(A) . h(C)$.

PROPOSITION 11. *If A is a finite G-module, then $h(A) = 1$.*

Proof. Consider the exact sequences

$$0 \to A^G \to A \xrightarrow{T} A \to A_G \to 0,$$

$$0 \to H^1(A) \to A_G \xrightarrow{N^*} A^G \to H^0(A) \to 0.$$

The first one shows that A^G and A_G have the same order, and then the second one shows that $H^0(A)$ and $H^1(A)$ have the same order.

COROLLARY. *Let A, B be G-modules, $f: A \to B$ a G-homomorphism with finite kernel and cokernel. Then if either of $h(A)$, $h(B)$ is defined, so is the other, and they are equal.*

Proof. Suppose for example that $h(A)$ is defined. From the exact sequences

$$0 \to \operatorname{Ker}(f) \to A \to f(A) \to 0$$
$$0 \to f(A) \to B \to \operatorname{Coker}(f) \to 0$$

it follows from Prop. 10 and 11 that $h(f(A))$ is defined and equal to $h(A)$, then that $h(B)$ is defined and equal to $h(f(A))$.

PROPOSITION 12. *Let E be a finite-dimensional real representation space of G, and let L, L' be two lattices of E which span E and are invariant under C Then if either of $h(L)$, $h(L')$ is defined, so is the other, and they are equal.*

For the proof of Prop. 12 we need the following lemma:

LEMMA. *Let G be a finite group and let M, M' be two finite-dimensional $\mathbf{Q}[G]$-modules such that $M_{\mathbf{R}} = M \otimes_{\mathbf{Q}} \mathbf{R}$ and $M'_{\mathbf{R}} = M' \otimes_{\mathbf{Q}} \mathbf{R}$ are isomorphic as $\mathbf{R}[G]$-modules. Then M, M' are isomorphic as $\mathbf{Q}[G]$-modules.*

Proof. Let K be any field, L any extension field of K, A a K-algebra. If V is any K-vector space let V_L denote the L-vector space $V \otimes_K L$. Let M, M' be A-modules which are finite-dimensional as K-vector spaces. An A-homomorphism $\varphi: M \to M'$ induces an A_L-homomorphism $\varphi \otimes 1 : M_L \to M'_L$, and $\varphi \mapsto \varphi \otimes 1$ gives rise to an isomorphism (of vector spaces over L)

$$(\operatorname{Hom}_A(M, M'))_L \cong \operatorname{Hom}_{A_L}(M_L, M'_L). \tag{8.3}$$

In the case in point, take $K = \mathbf{Q}$, $L = \mathbf{R}$, $A = \mathbf{Q}[G]$, so that $A_L = \mathbf{R}[G]$. The hypotheses of the lemma imply that M and M' have the same dimension over $\mathbf{Q}$, hence by choosing bases of M and M' we can speak of the *determinant* of an element of $\operatorname{Hom}_{\mathbf{Q}[G]}(M, M')$, or of $\operatorname{Hom}_{\mathbf{R}[G]}(M_{\mathbf{R}}, M'_{\mathbf{R}})$. (It will of course depend on the bases chosen.)

From (8.3) it follows that if ξ_i are a $\mathbf{Q}$-basis of $\operatorname{Hom}_{\mathbf{Q}[G]}(M, M')$, they are also an $\mathbf{R}$-basis of $\operatorname{Hom}_{\mathbf{R}[G]}(M_{\mathbf{R}}, M'_{\mathbf{R}})$. Since $M_{\mathbf{R}}$, $M'_{\mathbf{R}}$ are $\mathbf{R}[G]$-isomorphic, there exist $a_i \in \mathbf{R}$ such that $\det(\sum a_i \xi_i) \neq 0$. Hence the polynomial

$$F(t) = \det(\textstyle\sum t_i \xi_i) \in \mathbf{Q}[t_1, \ldots, t_m],$$

where t_i are independent indeterminates over $\mathbf{Q}$, is not identically zero, since $F(a) \neq 0$. Since $\mathbf{Q}$ is infinite, there exist $b_i \in \mathbf{Q}$ such that $F(b) \neq 0$, and then $\sum b_i \xi_i$ is a $\mathbf{Q}[G]$-isomorphism of M onto M'.

For the proof of Prop. 12, let $M = L \otimes \mathbf{Q}$, $M' = L' \otimes \mathbf{Q}$. Then $M_{\mathbf{R}}$ and $M'_{\mathbf{R}}$ are both $\mathbf{R}[G]$-isomorphic to E. Hence by the lemma there is a $\mathbf{Q}[G]$-isomorphism $\varphi : L \otimes \mathbf{Q} \to L' \otimes \mathbf{Q}$. L is mapped injectively by φ to a lattice contained in $(1/N)L'$ for some positive integer N. Hence $f = N.\varphi$ maps L injectively into L'; since L, L' are both free abelian groups of the same (finite) rank, $\operatorname{Coker}(f)$ is finite. The result now follows from the Corollary to Prop. 11.

9. Cohomological Triviality

A G-module A is *cohomologically trivial* if, for every subgroup H of G, $\hat{H}^q(H, A) = 0$ for all integers q. For example, an induced module is cohomologically trivial.

LEMMA 1. *Let p be a prime number, G a p-group and A a G-module such that $pA = 0$. Then the following three conditions are equivalent:*

(i) $A = 0$;
(ii) $H^0(G, A) = 0$;
(iii) $H_0(G, A) = 0$.

Proof. Clearly (i) implies (ii) and (iii).

(ii) $\Rightarrow$ (i): Suppose $A \neq 0$, let x be a non-zero element of A. Then the submodule B generated by x is finite, of order a power of p. Consider the G-orbits of the elements of B; they are all of p-power order (since the order of G is a power of p), and there is at least one fixed point, namely 0. Hence there are at least p fixed points, so that $H^0(G, A) = A^G \neq 0$.

(iii) $\Rightarrow$ (i). Let $A' = \mathrm{Hom}(A, \mathbf{F}_p)$ be the dual of A, considered as a vector-space over the field $\mathbf{F}_p$ of p elements. Then

$$H^0(G, A') = (A')^G = \mathrm{Hom}_G(A, \mathbf{F}_p)$$

is the dual of $H_0(G, A)$. Hence $H^0(G, A') = 0$, so that $A' = 0$ and therefore $A = 0$.

LEMMA 2. *With the same hypotheses as in Lemma 1, suppose that $H_1(G, A) = 0$. Then A is a free module over $\mathbf{F}_p[G] = \Lambda/p\Lambda$.*

Proof. Since $pA = 0$, we have $p.H_0(G, A) = 0$ and therefore $H_0(G, A)$ is a vector space over $\mathbf{F}_p$. Take a basis e_λ of this space and lift each e_λ to $a_\lambda \in A$. Let A' be the submodule of A generated by the a_λ, and let $A'' = A/A'$. Then we have an exact sequence

$$H_0(G, A') \xrightarrow{\alpha} H_0(G, A) \to H_0(G, A'') \to 0$$

in which by construction α is an isomorphism. Hence $H_0(G, A'') = 0$ and therefore $A'' = 0$ by Lemma 1, so that the a_λ generate A as a G-module. Hence they define a G-epimorphism $\varphi : L \to A$, where L is a free $\mathbf{F}_p[G]$-module. By construction, φ induces an isomorphism

$$\beta : H_0(G, L) \to H_0(G, A).$$

Let $R = \mathrm{Ker}(\varphi)$. Then since $H_1(G, A) = 0$, the sequence

$$0 \to H_0(G, R) \to H_0(G, L) \xrightarrow{\beta} H_0(G, A) \to 0$$

is exact; since β is an isomorphism, $H_0(G, R) = 0$ and therefore $R = 0$ by Lemma 1. Hence φ is an isomorphism.

THEOREM 6. *Let G be a p-group and let A be a G-module such that $pA = 0$. Then the following conditions are equivalent:*

(i) *A is a free $\mathbf{F}_p[G]$-module;*
(ii) *A is an induced module;*
(iii) *A is cohomologically trivial;*
(iv) *$\hat{H}^q(G, A) = 0$ for some integer q.*

Proof. Clearly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (i). By dimension-shifting we construct a module B such that $pB = 0$ and $\hat{H}^{q+r}(G, A) = \hat{H}^{r-2}(G, B)$ for all r. Hence $H_1(G, B) = 0$ and therefore (Lemma 2) B is free over $\mathbf{F}_p[G]$; hence

$$\hat{H}^{-2}(G, A) = \hat{H}^{-q-4}(G, B) = 0$$

and therefore (Lemma 2 again) A is free over $\mathbf{F}_p[G]$.

THEOREM 7. *Let G be a p-group and A a G-module without p-torsion. Then the following conditions are equivalent:*

(i) *A is cohomologically trivial;*
(ii) *$\hat{H}^q(G, A) = \hat{H}^{q+1}(G, A) = 0$ for some integer q;*
(iii) *A/pA is a free $\mathbf{F}_p[G]$-module.*

Proof. (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii): From the exact sequence

$$0 \to A \xrightarrow{p} A \to A/pA \to 0$$

we have an exact sequence $\hat{H}^q(G, A) \to \hat{H}^q(G, A/pA) \to \hat{H}^{q+1}(G, A)$, hence $\hat{H}^q(G, A/pA) = 0$. Hence, by Theorem 6, A/pA is free over $\mathbf{F}_p[G]$.

(iii) $\Rightarrow$ (i): From the same exact sequence it follows that

$$\hat{H}^q(H, A) \xrightarrow{p} \hat{H}^q(H, A)$$

is an isomorphism for all integers q and all subgroups H of G. But $\hat{H}^q(H, A)$ is a p-group (Prop. 8, Cor. 1), hence $\hat{H}^q(H, A) = 0$.

COROLLARY. *Let A be a G-module which is $\mathbf{Z}$-free and satisfies the equivalent conditions of Theorem 7. Then, for any torsion-free G-module B, the G-module $N = \operatorname{Hom}(A, B)$ is cohomologically trivial.*

Proof. Since A is $\mathbf{Z}$-free, the exact sequence

$$0 \to B \xrightarrow{p} B \to B/pB \to 0$$

gives an exact sequence

$$0 \to N \xrightarrow{p} N \to \operatorname{Hom}(A, B/pB) \to 0,$$

so that N has no p-torsion and $N/pN \cong \operatorname{Hom}(A/pA, B/pB)$. Since A/pA is a free $\mathbf{F}_p[G]$-module, it is induced, hence is the direct sum of the

$s.A'$ ($s \in G$), where A' is a subgroup of A/pA. Hence N/pN is the direct sum of the subgroup $s.\mathrm{Hom}(A', B/pB)$ and is therefore induced. Therefore N is cohomologically trivial by Theorems 6 and 7.

A G-module A is *projective* if $\mathrm{Hom}_G(A,\ \ )$ is an exact functor, or equivalently if A is a direct summand of a free G-module. A projective G-module is cohomologically trivial.

THEOREM 8. *Let G be a finite group, A a G-module which is $\mathbf{Z}$-free, G_p a Sylow p-subgroup of G. Then the following are equivalent:*

(i) *For each prime p, the G_p-module A satisfies the equivalent conditions of Theorem 7;*
(ii) *A is a projective G-module.*

Proof. (ii) $\Rightarrow$ (i) is clear.

(i) $\Rightarrow$ (ii): Choose an exact sequence $0 \to Q \to F \to A \to 0$, where F is a free G-module. Since A is $\mathbf{Z}$-free, this gives an exact sequence

$$0 \to \mathrm{Hom}(A, Q) \to \mathrm{Hom}(A, F) \to \mathrm{Hom}(A, A) \to 0$$

By the Corollary to Theorem 7, $\mathrm{Hom}(A, Q)$ is cohomologically trivial as a G_p-module for each p, hence $H^1(G, \mathrm{Hom}(A, Q)) = 0$ by Prop. 8, Cor. 4. Bearing in mind that $H^0(G, \mathrm{Hom}(A, Q)) = (\mathrm{Hom}(A, Q))^G = \mathrm{Hom}_G(A, Q)$, and so on, it follows that $\mathrm{Hom}_G(A, F) \to \mathrm{Hom}_G(A, A)$ is surjective, hence the identity map of A extends to a G-homomorphism $A \to F$. Consequently A is a direct summand of F and is therefore projective.

THEOREM 9. *Let A be any G-module. Then the following are equivalent:*

(i) *For each prime p, $\hat{H}^q(G_p, A) = 0$ for two consecutive values of q (which may depend on p);*
(ii) *A is cohomologically trivial;*
(iii) *There is an exact sequence $0 \to B_1 \to B_0 \to A \to 0$ in which B_0 and B_1 are projective G-modules.*

Proof. (ii) $\Rightarrow$ (i) is clear; so is (iii) $\Rightarrow$ (ii), since a projective G-module is cohomologically trivial.

(i) $\Rightarrow$ (iii): Choose an exact sequence of G-modules

$$0 \to B_1 \to B_0 \to A \to 0,$$

with B_0 a free G-module. Then $\hat{H}^q(G_p, B_1) \cong \hat{H}^{q-1}(G_p, A)$ for all q and all p, hence $\hat{H}^q(G_p, B_1) = 0$ for two consecutive values of q. Also B_1 is $\mathbf{Z}$-free (because B_0 is); hence, by Theorem 8, B_1 is projective.

10. Tate's Theorem

THEOREM 10. *Let G be a finite group, B and C two G-modules and $f: B \to C$ a G-homomorphism. For each prime p, let G_p be a Sylow p-subgroup of G,* *and suppose that there exists an integer n_p such that*

$$f_q^*: \hat{H}^q(G_p, B) \to \hat{H}^q(G_p, C)$$

is surjective for $q = n_p$, bijective for $q = n_p+1$ and injective for $q = n_p+2$. Then for any subgroup H of G and any integer q,

$$f_q^*: \hat{H}^q(H, B) \to \hat{H}^q(H, C)$$

is an isomorphism.

Proof. Let $B^* = \operatorname{Hom}(\Lambda, B)$ and let $i: B \to B^*$ be the injection (defined by $i(b)(g) = g.b$). Then $(f, i): B \to C \oplus B^*$ is injective, so that we have an exact sequence

$$0 \to B \to C \oplus B^* \to D \to 0.$$

Since B^* is cohomologically trivial, the cohomology of $C \oplus B^*$ is the same as that of C. Hence the cohomology exact sequence and the hypotheses of the theorem imply that $\hat{H}^q(G_p, D) = 0$ for $q = n_p$ and $q = n_p+1$. It follows from Theorem 9 that D is cohomologically trivial, whence the result.

THEOREM 11. *Let A, B, C be three G-modules and $\varphi: A \otimes B \to C$ a G-homomorphism. Let q be a fixed integer and a a given element of $\hat{H}^q(G, A)$. Assume that for each prime p there exists an integer n_p such that the map $\hat{H}^n(G_p, B) \to \hat{H}^{n+q}(G_p, C)$ induced by cup-product with $\operatorname{Res}_{G/Gp}(a)$ (relative to φ) is surjective for $n = n_p$, bijective for $n = n_p+1$ and injective for $n = n_p+2$. Then, for all subgroups H of G and all integers n, the cup-product with $\operatorname{Res}_{G/H}(a)$ induces an isomorphism*

$$\hat{H}^n(H, B) \to \hat{H}^{n+q}(H, C).$$

(Explicitly, this mapping is $b \mapsto \varphi_{n+q}^*(\operatorname{Res}_{G/H}(a).b)$.)

Proof. The case $q = 0$ is essentially Theorem 10. We have $a \in \hat{H}^0(G, A)$: choose $\alpha \in A^G$ representing a (then α also represents $\operatorname{Res}_{G/H}(a)$ for every subgroup H of G). Define $f: B \to C$ by $f(\beta) = \varphi(\alpha \otimes \beta)$; f is a G-homomorphism, since α is G-invariant. We claim that, for every $b \in \hat{H}^n(H, B)$,

$$\varphi^*(\operatorname{Res}_{G/H}(a).b) = f^*(b). \tag{10.1}$$

Indeed, this is clear for $n = 0$ (from the definition of f), and the general case then follows by dimension-shifting. To shift downwards, for example, assume (10.1) true for $n+1$, and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & B' & \to & B_* & \to & B & \to & 0 \\ & & {\scriptstyle f'}\downarrow & & {\scriptstyle 1\otimes f}\downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \to & C' & \to & C_* & \to & C & \to & 0 \end{array} \tag{10.2}$$

where $B_* = \Lambda \otimes B$, $C_* = \Lambda \otimes C$, and the rows are exact. B_*, C_* are induced modules and therefore cohomologically trivial, hence the connecting homomorphisms $\hat{\delta}$ are isomorphisms, and the diagram

$$\begin{array}{ccc} \hat{H}^n(H,B) & \xrightarrow{\delta} & \hat{H}^{n+1}(H,B') \\ \downarrow f^* & & \downarrow f'^* \\ \hat{H}^n(H,C) & \xrightarrow{\delta} & \hat{H}^{n+1}(H,C') \end{array}$$

is commutative. Moreover, the rows of (10.2) split over $\mathbf{Z}$, hence (10.2) remains exact (and commutative) when tensored with A (over $\mathbf{Z}$). Let $\varphi'' : A \otimes B' \to C'$ be the homomorphism induced by $\varphi : A \otimes B \to C$. Then using the inductive hypothesis and the compatibility of cup-products with connecting homomorphisms, we have

$$\begin{aligned}\hat{\delta} \circ f^*(b) = f'^* \circ \hat{\delta}(b) &= \varphi''^*(\mathrm{Res}_{G/H}(a) . \hat{\delta}(b)) \\ &= \varphi''^* \circ \hat{\delta}(\mathrm{Res}_{G/H}(a) . b) = \hat{\delta} \circ \varphi^*(\mathrm{Res}_{G/H}(a) . b).\end{aligned}$$

Since $\hat{\delta}$ is an isomorphism, (10.1) is proved.

Now f satisfies the hypotheses of Theorem 10, hence f_n^* is an isomorphism. This establishes Theorem 11 for the case $q = 0$.

The general case now follows by another piece of dimension-shifting. To shift downwards from $q+1$ to q, for example, consider the exact sequence

$$0 \to A' \to A_* \to A \to 0$$

where $A_* = \Lambda \otimes A$; this gives rise to isomorphisms $\hat{\delta} : \hat{H}^q(H, A) \to \hat{H}^{q+1}(H, A')$. Let $u = \mathrm{Res}_{G/H}(a) \in \hat{H}^q(H, A)$; then $u' = \hat{\delta}(u) = \mathrm{Res}_{G/H}(\hat{\delta}(a))$. Also $\varphi : A \otimes B \to C$ induces $\varphi' : A' \otimes B \to C'$. Consider the diagram

$$\begin{array}{ccccc} \hat{H}^n(H,B) & \xrightarrow{u.} & \hat{H}^{n+q}(H, A \otimes B) & \xrightarrow{\varphi^*} & \hat{H}^{n+q}(H,C) \\ \| & & & & \downarrow \delta \\ \hat{H}^n(H,B) & \xrightarrow{u'.} & \hat{H}^{n+q+1}(H, A' \otimes B) & \xrightarrow{\varphi'^*} & \hat{H}^{n+q+1}(H,C'); \end{array}$$

it is commutative, because

$$\hat{\delta} \circ \varphi^*(u . b) = \varphi'^* \circ \hat{\delta}(u . b) = \varphi'^*(\hat{\delta}(u) . b) = \varphi'^*(u' . b);$$

by the inductive hypothesis, the bottom line is an isomorphism, and $\hat{\delta}$ is an isomorphism; hence the top line is an isomorphism.

THEOREM 12 (Tate). *Let A be a G-module, $a \in H^2(G, A)$. For each prime p let G_p be a Sylow p-subgroup of G, and assume that*

(i) $H^1(G_p, A) = 0$;

(ii) $H^2(G_p, A)$ *is generated by* $\mathrm{Res}_{G/G_p}(a)$ *and has order equal to that of* G_p.

Then for all subgroups H of G and all integers n, cup-product with $\mathrm{Res}_{G/H}(a)$ *induces an isomorphism*

$$\hat{H}^n(H, \mathbf{Z}) \to \hat{H}^{n+2}(H, A).$$

Proof. Take $B = \mathbf{Z}$, $C = A$, $q = 2$, $n_p = -1$ in Theorem 11. For $n = -1$ the surjectivity follows from (i). For $n = 0$, $\hat{H}^0(G_p, \mathbf{Z})$ is cyclic of order equal to the order of G_p, so the bijectivity follows from (iii). For $n = 1$, the injectivity follows from the fact that $H^1(G_p \mathbf{Z}) = \mathrm{Hom}\,(G_p, \mathbf{Z}) = 0$. Thus all the hypotheses of Theorem 11 are satisfied.