Extended Picard complexes for algebraic groups and homogeneous spaces

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Abstract

For a smooth geometrically integral algebraic variety $X$ over a field $k$ of characteristic 0, we define the extended Picard complex $\text{UPic}(X)$. It is a complex of length 2 which combines the Picard group $\text{Pic}(X)$ and the group $U(X) := \bar{k}[X]^\times / \bar{k}^\times$, where $\bar{k}$ is a fixed algebraic closure of $k$ and $X = X \times_k \bar{k}$. For a connected linear $k$-group $G$ we compute the complex $\text{UPic}(G)$ (up to a quasi-isomorphism) in terms of the algebraic fundamental group $\pi_1(G)$. We obtain similar results for a homogeneous space $X$ of a connected $k$-group $G$. To cite this article: M. Borovoi, J. van Hamel, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

Résumé

Complexes de Picard étendus pour des groupes algébriques et des espaces homogènes. Soient $k$ un corps de caractéristique zéro et $X$ une $k$-variété algébrique lisse et géométriquement intègre. Nous définissons le complexe de Picard étendu $\text{UPic}(X)$. C’est un complexe de longueur 2 qui combine le groupe de Picard $\text{Pic}(X)$ et le groupe $U(X) := \bar{k}[X]^\times / \bar{k}^\times$, où $\bar{k}$ est une clôture algébrique fixée de $k$ et $X = X \times_k \bar{k}$. Pour un $k$-groupe linéaire connexe $G$, nous calculons le complexe $\text{UPic}(G)$ (à quasi-isomorphisme près) en termes du groupe fondamental algébrique $\pi_1(G)$. Nous obtenons des résultats similaires pour un espace homogène $X$ d’un $k$-groupe connexe $G$. Pour citer cet article : M. Borovoi, J. van Hamel, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

Throughout the note, $k$ denotes a field of characteristic 0 and $\bar{k}$ is a fixed algebraic closure of $k$. By a $k$-group we mean a linear algebraic group defined over $k$.

Let $G$ be a connected reductive $k$-group. Let $\rho: G^{\text{sc}} \to G^{\text{ss}} \hookrightarrow G$

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be Deligne’s homomorphism, where $G_{ss}$ is the derived subgroup of $G$ (it is semisimple) and $G_{sc}$ is the universal covering of $G_{ss}$ (it is simply connected). Let $T \subset G$ be a maximal torus (defined over $k$) and let $T_{sc} := \rho^{-1}(T)$ be the corresponding maximal torus of $G_{sc}$. The 2-term complex of tori

$$T_{sc} \xrightarrow{\rho} T$$

(with $T_{sc}$ in degree $-1$) plays an important role in the study of the arithmetic of reductive groups. For example, the Galois hypercohomology $H^i(k, T_{sc} \to T)$ of this complex is the abelian Galois cohomology of $G$ (cf. [1]). The corresponding Galois module

$$X_*(\overline{T})/\rho_* X_*(\overline{T}_{sc})$$

(where $X_*$ denotes the cocharacter group of a torus) is called the algebraic fundamental group $\pi_1(G)$ (loc. cit.). The related complex group with holomorphic $\text{Gal}(\overline{k}/k)$-action

$$\text{Hom}(\pi_1(\overline{G}), \mathbb{C}^\times) = \ker(X^*(T) \otimes \mathbb{C}^\times \to X^*(T_{sc}) \otimes \mathbb{C}^\times)$$

(where $X^*$ denotes the character group of an algebraic group) is canonically isomorphic to the center of the connected Langlands dual group $\hat{G}$ considered by Kottwitz [7].

Clearly, the above constructions rely on the linear algebraic group structure of $\overline{G}$. However we show in this note that they are related to a very natural geometric/cohomological construction that works for an arbitrary smooth $k$-variety $X$. The proofs will be published elsewhere.

1. The extended Picard complex

By a $k$-variety we mean a smooth geometrically integral $k$-variety. If $X$ is a $k$-variety, we write $\overline{X}$ for $X \times_k \overline{k}$. We write $k[\overline{X}]$ (resp. $\overline{k}(\overline{X})$) for the ring of regular functions (resp. the field of rational functions) on $\overline{X}$.

For a $k$-variety $X$, consider the cone $\text{UPic}(\overline{X})$ of the morphism

$$G_m(\overline{k}) \to \tau_{\leq 1} R^1 \Gamma(\overline{X}, G_m)$$

in the derived category of discrete Galois modules. More explicitly, this cone is represented by the 2-term complex

$$\overline{k}(X)^\times/\overline{k}^\times \to \text{Div}(\overline{X})$$

(with $\overline{k}(X)^\times/\overline{k}^\times$ in degree 0), where $\text{Div}$ denotes the divisor group. It follows from the definitions that the cohomology groups $\mathcal{H}^i$ of the complex $\text{UPic}(\overline{X})$ vanish for $i \neq 0, 1$, and

$$\mathcal{H}^0(\text{UPic}(\overline{X})) = U(\overline{X}) := k[\overline{X}]^\times/\overline{k}^\times, \quad \mathcal{H}^1(\text{UPic}(\overline{X}) = \text{Pic}(\overline{X})�.

Hence $\text{UPic}(\overline{X})$ can be regarded as a 2-extension of $\text{Pic}(\overline{G})$ by $U(\overline{X})$. We shall call this complex the extended Picard complex of $X$.

**Lemma 1.1** Let $X_c$ be a smooth compactification of a $k$-variety $X$. Then there is a distinguished triangle

$$\text{UPic}(\overline{X}) \to \text{Div} \overline{X} \to \text{Pic}(\overline{X}_c) \to \text{UPic}(\overline{X})[1]$$

where $\text{Div} \overline{X} \setminus \overline{X}$ is the permutation module of divisors in the complement of $\overline{X}$ in $\overline{X}_c$.

Now we consider $\text{Pic}(X) = H^1(X, G_m)$ and $\text{Br}(X) = H^2_{\text{et}}(X, G_m)$ (over $k$). Let $\text{Br}_1(X)$ denote the kernel of the map $\text{Br}(X) \to \text{Br} \overline{X}$.

**Lemma 1.2** Let $X$ be a $k$-variety.

(i) There is a natural injection $\text{Pic}(X) \hookrightarrow H^1(k, \text{UPic}(\overline{X})), \quad \text{which is an isomorphism if } X(k) \neq \emptyset.$

(ii) There is a natural injection $\text{Br}_1(X)/\text{Br}(k) \hookrightarrow H^2(k, \text{UPic}(\overline{X})), \quad \text{which is an isomorphism if } X(k) \neq \emptyset \text{ or if } H^3(k, G_m) = 0 \quad (\text{e.g. when } k \text{ is a number field}).$
If \( C \) is a complex of \( \text{Gal}(\bar{k}/k) \)-modules, we write \( \Im_\omega(k, C) = \ker[H^i(k, C) \to \prod_\gamma H^i(\gamma, C)] \) where \( \gamma \) runs over all closed procyclic subgroups of \( \text{Gal}(\bar{k}/k) \).

**Proposition 1.3** Let \( X_c \) be a smooth compactification of a smooth \( k \)-variety \( X \). The triangle of Lemma 1.1 gives rise to an isomorphism

\[
\Im_\omega^1(k, \text{Pic}(X_c)) \cong \Im_\omega^2(k, \text{UPic}(X)).
\]

This is particularly interesting for a homogeneous variety \( X \) of a connected \( k \)-group \( G \) with connected geometric stabilizer, for which we have \( \Im_\omega^1(k, \text{Pic}(X_c)) = H^1(k, \text{Pic}(X_c)) \), see [4].

### 2. Algebraic groups and torsors

Let \( G \) be a connected reductive \( k \)-group. We define the dual complex \( \pi_1(G)^D \) to \( \pi_1(G) \) by

\[
\pi_1(G)^D = (X^*(T) \to X^*(T^\text{sc})) \quad \text{(with \( X^*(T) \) in degree 0)}.
\]

**Theorem 2.1** For a connected reductive \( k \)-group \( G \) there is a canonical, functorial in \( G \) isomorphism

\[
\text{UPic}(G) \cong \pi_1(G)^D.
\]

Let \( G \) be any connected linear \( k \)-group, not necessarily reductive. We write \( G^\text{un} \) for the unipotent radical of \( G \), and set \( G^\text{red} = G/G^\text{un} \) (it is reductive). We define \( \pi_1(G) := \pi_1(G^\text{red}) \).

**Corollary 2.2** For any connected linear \( k \)-group \( G \) we have a canonical isomorphism \( \text{UPic}(G) \cong \pi_1(G)^D \).

Combining Corollary 2.2 with Lemma 1.2, we find a new proof of the following result.

**Corollary 2.3** (Kottwitz [7]) For any connected linear \( k \)-group \( G \) we have canonical isomorphisms \( \text{Pic}(G) \cong H^1(k, \pi_1(G)^D) \) and \( \text{Br}_1(G)/\text{Br}(k) \cong H^2(k, \pi_1(G)^D) \).

Theorem 2.1 gives a description of the complex \( \text{UPic} \) for a \( k \)-torsor as well, thanks to the following result which is a straightforward generalization of [8, Lemme 6.7]).

**Proposition 2.4** Let \( G \) be a connected linear \( k \)-group and let \( X \) be a \( k \)-torsor under \( G \). There is a canonical isomorphism \( \text{UPic}(X) \cong \text{UPic}(G) \), functorial in \( G \) and \( X \), in the derived category of discrete Galois modules.

Combining the fact that \( \Im_\omega^1(k, \text{Pic}(X_c)) = H^1(k, \text{Pic}(X_c)) \) for any smooth compactification \( X_c \) of a \( k \)-torsor \( X \) under \( G \) (cf. [3]) with Proposition 1.3, Proposition 2.4, and Corollary 2.2, we obtain a new proof of the following result.

**Corollary 2.5** (Borovoi–Kunyavskii [2]) With \( G \) and \( X \) as above, \( H^1(k, \text{Pic}(X_c)) \cong \Im_\omega^2(k, \pi_1(G)^D) \).

### 3. Homogeneous spaces

Let \( G \) be a connected \( k \)-group such that \( \text{Pic}(G) = 0 \) (i.e. \( G^\text{red} \) is simply connected). Let \( X \) be a homogeneous space of \( G \) defined over \( k \). Let \( \tilde{x} \in X(\bar{k}) \), and let \( \tilde{H} \) be the stabilizer of \( \tilde{x} \) in \( G \). Then \( \text{Gal}(k/k) \) acts on \( X^*(\tilde{H}) \). We do not assume that \( X \) has a \( k \)-point or that \( \tilde{H} \) is connected.

**Theorem 3.1** For \( G \) and \( X \) as above, there is an isomorphism

\[
\text{UPic}(X) \cong (X^*(\bar{G}) \to X^*(\bar{H})) \quad \text{(with \( X^*(\bar{G}) \) in degree 0)}
\]

in the derived category of discrete Galois modules. In particular, there is an exact sequence

\[
0 \to U(X) \to X^*(\bar{G}) \to X^*(\bar{H}) \to \text{Pic}(X) \to 0.
\]
The exact sequence of Theorem 3.1 generalizes an exact sequence of Fossum–Iversen [6, Prop. 3.1] and Sansuc [8, Prop. 6.10]. Note that the requirement Pic($\overline{G}$) = 0 is not a serious restriction, since for any connected $k$-group $G$ we can find a surjective homomorphism $G' \to G$ with Pic($\overline{G'}$) = 0.

**Corollary 3.2** For $G$ and $X$ as above there are injections $\text{Pic}(X) \hookrightarrow H^1(k, X^*(\overline{G}) \to X^*(\overline{H}))$ and $\text{Br}(k) / \text{Br}(k) \hookrightarrow H^2(k, X^*(\overline{G}) \to X^*(\overline{H}))$, which are isomorphisms if $X(k) \neq \emptyset$.

The corollary follows from Theorem 3.1 and Lemma 1.2.

4. The elementary obstruction

Let $X$ be a $k$-variety. We have an extension of complexes of Galois modules

$$0 \to \bar{k}^* \to (\bar{k}(X))^* \to \text{Div}(X) \to (\bar{k}(X)^*/\bar{k}^* \to \text{Div}(X)) \to 0.$$  

It defines an element $e(X) \in \text{Ext}^1(\text{UPic}(X), k^*)$. If $X$ has a $k$-point, then this extension splits (in the derived category), hence $e(X) = 0$. By slight abuse of terminology we call this class $e(X)$ the elementary obstruction to the existence of a $k$-point in $X$ (cf. [5, Déf. 2.2.1 and Prop. 2.2.4]).

When $X$ is a $k$-torsor under a $k$-group $G$, Proposition 2.4 and Theorem 2.1 give us that $\text{UPic}(X) = \pi_1(\overline{O})^D$. We obtain

$$\text{Ext}^1(\text{UPic}(X), k^*) = H^1(k, \text{Hom}(\pi_1(\overline{O})^D, k^*)) = H^1(k, X^*(T^\text{sc})) \otimes \bar{k}^* \to X^*(T) \otimes \bar{k}^* = H^1(k, T^\text{sc} \to T)$$

(where $T^\text{sc}$ is in degree $-1$). The abelian group $H^1_{\text{ab}}(k, G) := H^1(k, T^\text{sc} \to T)$ is called the first abelian Galois cohomology group of $G$, and in [1] an abelianization map $\text{ab}^1 : H^1(k, G) \to H^1_{\text{ab}}(k, G)$ was constructed. Here we compute the elementary obstruction $e(X) \in H^1_{\text{ab}}(k, G)$ in terms of the cohomology class $\text{cl}(X) \in H^1(k, G)$.

**Theorem 4.1** Let $X$ be a $k$-torsor under a connected $k$-group $G$. With the above notation we have $e(X) = \text{ab}^1(\text{cl}(X))$ (up to sign).

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