ABELIAN GALOIS COHOMOLOGY OF REDUCTIVE GROUPS

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Introduction

In this paper we introduce a new functor $H^1_{\mathrm{ab}}(K,G)$ from the category of connected reductive groups over a field K of characteristic 0 to abelian groups. We call $H^1_{\mathrm{ab}}(K,G)$ the first abelian Galois cohomology group of G. The abelian group $H^1_{\mathrm{ab}}(K,G)$ is related to the pointed set $H^1(K,G)$ by the abelianization map $\mathrm{ab}^1\colon H^1(K,G)\to H^1_{\mathrm{ab}}(K,G)$. When K is a local field or a number field, the map ab^1 is surjective. We use the map ab^1 to give a functorial, almost explicit description of the set $H^1(K,G)$ when K is a number field.

We describe the contents in more detail. For a reductive group G over a field K of characteristic 0, let G^{ss} denote the derived group of G (it is semisimple) and let G^{sc} denote the universal covering group of G (it is simply connected). Following Deligne we consider the composition

$$\rho\colon G^{\mathrm{sc}}\to G^{\mathrm{ss}}\hookrightarrow G.$$

Let $\bar K$ be an algebraic closure of K. We write $\bar G$ for $G_{\bar K}$. In Section 1 we define *the algebraic fundamental group* $\pi_1(\bar G)$ as follows. Let $T\subset G$ be a maximal torus defined over K. We write $T^{(\mathrm{sc})}$ for $\rho^{-1}(T)$ and set

$$\pi_1(\bar G)=X_*(\bar T)/\rho_*X_*(\bar T^{(\mathrm{sc})})$$

where X_* denotes the cocharacter group. The group $\pi_1(\bar G)$ is a finitely generated abelian group endowed with a $\mathrm{Gal}(\bar K/K)$-action. If $K=\mathbf{C}$ then $\pi_1(\bar G)$ is just the usual topological fundamental group $\pi_1^{\mathrm{top}}(G(\mathbf{C}))$. For any K our algebraic fundamental group is related to the invariant $Z(\widehat G)$ of Kottwitz [Ko2], where $\widehat G$ is a connected dual Langlands group for G and $Z(\widehat G)$ is its center. Namely, $\pi_1(\bar G)$ is the character group of the $\mathbf{C}$-group $Z(\widehat G)$.

In Section 2 we define *the abelian Galois cohomology groups*

$$H^i_{\mathrm{ab}}(K,G):=\mathbb{H}^i(K,T^{(\mathrm{sc})}\to T)\qquad (i\geq -1).$$

Here $\mathbb{H}^i$ denotes the Galois hypercohomology of the complex

$$0\to \overset{-1}{T}{}^{(\mathrm{sc})}\to \overset{0}{T}\to 0$$

of tori, where -1 and 0 above the letters denote the degrees. We show that the abelian groups $H^i_{\mathrm{ab}}(K,G)$ depend only on $\pi_1(\bar G)$.

In the third section we construct *the abelianization map*

$$\mathrm{ab}^1 = \mathrm{ab}^1_G\colon H^1(K,G) \to H^1_{\mathrm{ab}}(K,G)$$

with kernel $\rho_* H^1(K, G^{\mathrm{sc}})$. Observe that in the case of a semisimple group G we have

$$G = G^{\mathrm{sc}}/\ker\,\rho, \quad H^1_{\mathrm{ab}}(K,G) = H^2(K,\ker\rho)$$

(where $\ker\rho$ is a finite abelian group), and ab^1 in this case is the connecting homomorphism $\delta\colon H^1(K,G) \to H^2(K,\ker\rho)$. We construct also a homomorphism

$$\mathrm{ab}^0\colon G(K) = H^0(K,G) \to H^0_{\mathrm{ab}}(K,G)$$

with kernel $\rho(G^{\mathrm{sc}}(K))$. When K is a local field, the map ab^1 was earlier constructed by Kottwitz [Ko3] and the homomorphism ab^0 was earlier constructed by Langlands [La1] (see also [Bo], 10.2).

Our construction of the abelianization maps ab^1 and ab^0 differs from those of Kottwitz and Langlands. It is based on the non-abelian hypercohomology theory of groups with coefficients in crossed modules. This theory was initiated by Dedecker [Ded] and developed by the author [Brv5] to be used in the present paper and by Breen [Brn] in a very general setting. In the beginning of Section 3 we state the main facts about crossed modules and their hypercohomology.

In Section 4 we compute explicitly the groups $H^1_{\mathrm{ab}}(K,G)$ for a local field K in terms of $\pi_1(\bar G)$. We write Γ for $\mathrm{Gal}(\bar K/K)$ and M for $\pi_1(\bar G)$. Then

$$H^1_{\mathrm{ab}}(K,G) = \begin{cases} H^{-1}(\Gamma, M) & \text{if } K = \mathbf{R} \\ (M_\Gamma)_{\mathrm{tors}} & \text{if } K \text{ is non-archimedean,} \end{cases}$$

where $(M_\Gamma)_{\mathrm{tors}}$ denotes the torsion subgroup of the group of coinvariants M_Γ. For a number field K we compute $H^i_{\mathrm{ab}}(K,G)$ for $i \geq 3$ and compute it in a sense for $i = 2$. For $i = 1$ we compute the group

$$Ш^1_{\mathrm{ab}}(K,G) := \ker[H^1_{\mathrm{ab}}(K,G) \to \prod_v H^1_{\mathrm{ab}}(K_v,G)]$$

in terms of $\pi_1(\bar G)$. All these results are of abelian nature and generalize the Tate-Nakayama duality theory for tori. The results concerning the case $i = 1$ are essentially due to Kottwitz.

In Section 5 we prove that if K is a local or a number field, then the abelianization map ab^1 is surjective. For local fields this is very close to a result of Kottwitz [Ko3]. This surjectivity means, in particular, that for a local non-archimedean field K

$$H^1(K,G) \simeq (M_\Gamma)_{\mathrm{tors}}$$

([Ko2], 6.4.1). In this case the map ab^1 is not only surjective but also injective.

We use the surjectivity of ab^1 over local and number fields to investigate the usual, non-abelian Galois cohomology $H^1(K,G)$ when K is a number field.

Theorem 5.11. *For any finite subset $\Xi \subset H^1(K,G)$ there exists a K-torus $j\colon T \hookrightarrow G$ such that $\Xi \subset j_* H^1(K,T)$.*

In other words, for a number field K all the $H^1(K,G)$ comes from tori.

Further, we compute $H^1(K,G)$ in terms of $H^1_{\mathrm{ab}}(K,G)$ and the real cohomology:

Theorem 5.12. *The commutative diagram with surjective arrows*

$$\begin{array}{ccc} H^1(K,G) & \xrightarrow{\mathrm{ab}^1} & H^1_{\mathrm{ab}}(K,G) \\ {\scriptstyle \mathrm{loc}_\infty}\big\downarrow & & \big\downarrow \\ \prod_\infty H^1(K_v,G) & \longrightarrow & \prod_\infty H^1_{\mathrm{ab}}(K_v,G) \end{array}$$

identifies $H^1(K,G)$ with the fiber product of $H^1_{\mathrm{ab}}(K,G)$ and $\prod_\infty H^1(K_v,G)$ over $\prod_\infty(K_v,G)$.

This result generalizes a theorem of Sansuc [Sa].

From Theorem 5.12 we obtain

Theorem 5.13. *The restriction of* ab^1 *to the Shafarevich-Tate kernel defines a bijection* $Ш^1(K,G) \to Ш^1_{\mathrm{ab}}(K,G)$.

Thus we see again after Voskresenskii [Vo1], Sansuc [Sa] and Kottwitz [Ko2], that $Ш(G)$ has a natural structure of an abelian group. Combining this bijection with the results of Section 4, we can compute $Ш(G)$ in terms of $\pi_1(\bar{G})$. The obtained formula is equivalent to a formula of Kottwitz [Ko2].

Remark 0.1. The results of this paper can be easily adapted to the case of any, not necessarily reductive, connected K-group. Let G^{u} denote the unipotent radical of G. We set $G^{\mathrm{red}} = G/G^{\mathrm{u}}$; this is a reductive group. We set

$$\pi_1(\bar{G}) = \pi_1(\bar{G}^{\mathrm{red}}), \quad H^1_{\mathrm{ab}}(K,G) = H^1_{\mathrm{ab}}(K,G^{\mathrm{red}})$$

and so on. With this notation almost all the results of the paper remain true for all connected K-groups.

Remark 0.2. In the case of a semisimple group G all the results of this paper were already known (cf. [Sa]). On the other hand, for local fields our results are just a more functorial reformulation of results of Kottwitz [Ko2], [Ko3]. The contribution of the present paper is that we construct the abelian Galois cohomology and the abelianization map for *any* reductive group over an *arbitrary* field of characteristic 0. This enables us to obtain new results concerning usual, non-abelian Galois cohomology of reductive groups over number fields.

Remark 0.3. Our computations in Section 5 of the Galois cohomology of reductive groups over number fields are based on the fundamental results on the Galois cohomology of semisimple groups due to Kneser [Kn1], [Kn2] and Harder [Ha1], [Ha2].

Remark 0.4. The main results of this paper were exposed by J. S. Milne in Appendix B to [Mi3].

Acknowledgements

It is clear from the introduction that the present paper is inspired by the papers [Ko2] and [Ko3] of Kottwitz. I must add that in the summer of 1989 Robert Kottwitz explained to me that my abelian Galois cohomology group $H^1_{\mathrm{ab}}(K,G)$ (which had been previously defined in a rather awkward and non-functorial way) is in fact the Galois hypercohomology group of a complex of tori. This remark greatly influenced the exposition in Sections 2–4. For this I am very grateful to R. Kottwitz.

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Notation

K is a field of characteristic 0, $\bar{K}$ is an algebraic closure of K. We write Γ for $\mathrm{Gal}(\bar{K}/K)$. For an aglebraic variety Xover K we write $\bar{X}$ for $X_{\bar{K}}$.

When K is a number field, let $\mathcal{V}=\mathcal{V}(K)$, $\mathcal{V}_\infty$ and $\mathcal{V}_f$ denote the set of all places, the set of infinite (archimedian) places and the set of finite (non-archimedian) places of K, respectively. We often write just ∞ for $\mathcal{V}_\infty$. If $v\in\mathcal{V}$, we let K_v denote the completion of K at v.

We denote by μ_n the group or roots of unity of order dividing n, and set $\widehat{\mathbf{Z}}(1)=\varprojlim\mu_n$.

G is a reductive K-group. By a reductive K-group we always mean a *connected* reductive K-group. Let G^{ss} denote the derived group of G. We set $G^{\mathrm{tor}}=G/G^{\mathrm{ss}}$. We denote by Z the center of G and set $G^{\mathrm{ad}}=G/Z$. Let G^{sc} denote the universal covering of the semisimple group G^{ss}. We have the canonical homomorphism

$$\rho\colon G^{\mathrm{sc}}\to G^{\mathrm{ss}}\to G.$$

Let $T\subset G$ be a maximal torus (defined over K). We write $T^{(\mathrm{sc})}$ for the maximal torus $\rho^{-1}(T)\subset G^{\mathrm{sc}}$.

We let $Z^{(\mathrm{sc})}$ denote the center of G^{sc}. Then $Z^{(\mathrm{sc})}=\rho^{-1}(Z)$.

Let S be a K-group of multiplicative type, e.g. a torus. We let $X^*(S)$ denote the character group $\mathrm{Hom}(S,\mathbb{G}_m)$ and let $X_*(S)$ denote the cocharacter group $\mathrm{Hom}(\mathbb{G}_m,S)$, where $\mathbb{G}_m$ is the multiplicative group. We usually consider $X^*(\bar{S})$ and $X_*(\bar{S})$.

For a reductive K-group G and a split maximal K-torus T we let $R(G,T)$ denote the root system of G with respect to T. We denote by $R^\vee(G,T)$ the system of coroots. By definition $R(G,T)\subset X^*(T)$ and $R^\vee(G,T)\subset X_*(T)$.

Let L be a torsion free abelian group. We write $L^\vee$ for $\mathrm{Hom}(L,\mathbf{Z})$.

Let M be an abelian group. We let M_{tors} denote the torsion subgroup of M. We set $M_{\mathrm{tf}} = M/M_{\mathrm{tors}}$; this is the maximal torsion free quotient of M.

Let Δ be a group and M a Δ-module. We say that M is a finitely generated (resp. torsion free) Δ-module if M is finitely generated (resp. torsion free) as an abelian group.

Let M be a finitely generated Δ-module. By a short torsion free resolution of M we mean an exact sequence

$$0 \to L^{-1} \to L^0 \to M \to 0$$

of finitely generated Δ-modules such that L^{-1} and L^0 are torsion free. We write $L^{\cdot}$ for the complex $0 \to L^{-1} \to L^0 \to 0$.

Let M be a Δ-module. We write M^Δ and M_Δ to denote the subgroup of invariants and the group of coinvariants of M, respectively. We often consider the functors $(M_\Delta)_{\mathrm{tors}}$ and $(M_\Delta)_{\mathrm{tf}}$.

We often regard a homomorphism $\alpha\colon F \to G$ of groups as a short complex

$$1 \to \overset{-1}{F} \to \overset{0}{G} \to 1$$

where -1 and 0 over the letters denote the degrees: F is in degree -1 and G is in degree 0.

A *crossed module* is a complex of groups $F \xrightarrow{\alpha} G$ endowed with an action of G on F satisfying certain conditions. We often write just $(F \to G)$ for a crossed module.

Let $(F \to G)$ be a crossed module of Δ-groups. We write $\mathbb{H}^i(\Delta, F \to G)$ or just $\mathbb{H}^i(F \to G)$ $(i = 1, 0, 1)$ for the hypercohomology of Δ with coefficients in $F \to G$ (see Section 3 for definitions).

Let G be an algebraic group. As usual, we write $H^i(K, G)$ to denote the Galois cohomology $H^i(\Gamma, G(\bar{K}))$ where $\Gamma = \mathrm{Gal}(\bar{K}/K)$. We denote by $Z^i(K, G)$ the set of i-cocycles and by $B^i(K, G)$ the set of i-cobords.

If $(F \to G)$ is a crossed module of algebraic K-groups, we write $\mathbb{H}^i(K, F \to G)$ or just $\mathbb{H}^i(F \to G)$ for its Galois hypercohomology $\mathbb{H}^i(\Gamma, F(\bar{K}) \to G(\bar{K}))$.

For any Γ-module M (where $\Gamma = \mathrm{Gal}(\bar{K}/K)$) we write $H^i(K, M)$ for $H^i(\Gamma, M)$. Similarly, if F/K is a Galois extension with the Galois group Δ and if M is a Δ-module, we write $H^i(F/K, M)$ for $H^i(\Delta, M)$ and $\widehat{H}^i(F/K, M)$ for $\widehat{H}^i(\Delta, M)$, where $\widehat{H}^i$ are the Tate cohomology groups.

If K is a number field, we use the notation *loc* to denote the localization maps

$$\mathrm{loc}_v\colon H^1(K, G) \to H^1(K_v, G)$$
$$\mathrm{loc}_\infty\colon H^1(K, G) \to \prod_{v \in \mathcal{V}_\infty} H^1(K_v, G)$$

and so on.

1. The algebraic fundamental group of a reductive group

In this section we define the algebraic fundamental group $\pi_1(G_{\bar{K}})$ of a reductive group G defined over a field K of characteristic 0.

1.1 Let G be a (connected) reductive K-group. First suppose that G is split. Choose a maximal split torus $T \subset G$. Consider the canonical morphism $\rho\colon G^{\mathrm{sc}} \to G$. We write $T^{(\mathrm{sc})}$ for $\rho^{-1}(T) \subset G^{\mathrm{sc}}$. Set

$$\pi_1(G,T) = X_*(T)/\rho_* X_*(T^{(\mathrm{sc})}).$$

It is a finitely generated abelian group.

Lemma 1.2. *For two split maximal tori, $T, T' \subset G$, the groups $\pi_1(G,T)$ and $\pi_1(G,T')$ are canonically isomorphic.*

Proof: Choose an element $g \in G(K)$ such that $T' = gTg^{-1}$. The isomorphism $\mathrm{int}(g)\colon T \to T'$ induces an isomorphism $g_*\colon \pi_1(G,T) \to \pi_1(G,T')$. We will show that g_* does not depend on the choice of g.

Let N denote the normalizer of T in G. It suffices to show that if $g \in N(K)$ then the automorphism g_* of $\pi_1(G,T)$ is trivial. The group $N(K)$ acts on T and on $\pi_1(G,T)$ through its quotient group $W := N(K)/T(K)$. One knows that the Weyl group W is generated by the reflections r_α corresponding to the roots $\alpha \in R(G,T)$. It remains to show that for $\alpha \in R(G,T)$ the reflection r_α acts on $\pi_1(G,T)$ trivially.

We have

$$r_\alpha(X) = X - \langle \alpha, X \rangle \alpha^\vee$$

for $X \in X_*(T)$, where $a^\vee$ is the corresponding coroot. Since all the coroots come from $X_*(T^{(\mathrm{sc})})$, we see that

$$r_\alpha(X) \equiv X \mod \rho_* X_*(T^{(\mathrm{sc})}),$$

thus r_α acts on $X_*(T)/\rho_* X_*(T^{(\mathrm{sc})})$ trivially. The lemma is proved.

Definition 1.3. Let G be a split reductive K-group. Let $T \subset G$ be a split maximal K-torus. We set $\pi_1(G) = \pi_1(G,T)$ and call this abelian group the algebraic fundamental group of G.

By Lemma 1.2 this definition is correct.

1.4. Now let G be any (not necessarily split) reductive K-group. By the algebraic fundamental group of G we mean $\pi_1(\bar{G})$ (recall that $\bar{G} = G_{\bar{K}}$).

The Galois group $\Gamma = \mathrm{Gal}(\bar{K}/K)$ acts on G and thus on $\pi_1(\bar{G})$. This action can be described as follows.

Choose a maximal torus $T' \subset \bar{G}$. For $\sigma \in \Gamma$ choose an element $g_\sigma \in G(\bar{K})$ such that $g_\sigma \cdot {}^\sigma T' \cdot g_\sigma^{-1} = T'$. Then σ acts on $\pi_1(\bar{G}, T')$ as the composition

$$\pi_1(\bar{G}, T') \xrightarrow{\sigma_*} \pi_1(\bar{G}, {}^\sigma T') \xrightarrow{(g_\sigma)_*} \pi_1(\bar{G}, T')$$

In particular, if $T \subset G$ is a maximal torus defined over K, then the action of Γ on $\pi_1(\bar{G})$ is the action on $X_*(\bar{T})/\rho_* X_*(\bar{T}^{(\mathrm{sc})})$ induced by the action on $X_*(\bar{T})$.

Our algebraic fundamental group is a functor from the category of reductive K-groups and K-homomorphisms to the category of finitely generated Γ-modules. The following lemma shows that this functor is in a sense exact.

Lemma 1.5. *Let $1 \to G_1 \to G_2 \to G_3 \to 1$ be an exact sequence of connected reductive K-groups. Then the sequence*

$$0 \to \pi_1(\bar{G}_1) \to \pi_1(\bar{G}_2) \to \pi_1(\bar{G}_3) \to 0$$

is exact.

Proof: Left to the reader as an easy exercise.

1.6. Examples.

(1) For a K-torus T we have $\pi_1(\bar{T}) = X_*(\bar{T})$.

(2) Suppose G^{ss} to be simply connected. Then the canonical homomorphism $\pi_1(\bar{G}) \to \pi_1(\bar{G}^{\mathrm{tor}})$ is an isomoprhism, thus $\pi_1(\bar{G}) = X_*(\bar{G}^{\mathrm{tor}})$.

(3) Let G be a semisimple group. Then $G = G^{\mathrm{sc}}/\ker\rho$, where $\ker\rho$ is a finite abelian K-group. Let $T \subset G$ be a maximal torus defined over K. Then $T = T^{(\mathrm{sc})}/\ker\rho$. One can easily show that $\pi_1(\bar{G}) = (\ker\rho)(-1) := \mathrm{Hom}(\widehat{\mathbf{Z}}(1), \ker\rho)$. Note that $\pi_1(\bar{G})$ and $\ker\rho$ are isomorphic as abelian groups, but are in general non-isomorphic as Γ-modules. E.g. if $G = \mathrm{PGL}_n$, then $\ker\rho = \mu_n$, but $\pi_1(\bar{G}) = \mathbf{Z}/n\mathbf{Z}$.

Corollary 1.7. *For any reductive K-group G we have an exact sequence*

$$0 \to (\ker\rho)(-1) \to \pi_1(\bar{G}) \to X_*(G^{\mathrm{tor}}_{\bar{K}}) \to 0.$$

Proof: We consider the canonical exact sequence $1 \to G^{\mathrm{ss}} \to G \to G^{\mathrm{tor}} \to 1$ and apply Lemma 1.5 and the statements 1.6(1,3).

Now let $\psi \in Z^1(K, G^{\mathrm{ad}})$ be a cocycle. Consider the twisted form ${}_\psi G$ of G. By definition $({}_\psi G)_{\bar{K}} = G_{\bar{K}}$, but $\sigma \in \mathrm{Gal}(\bar{K}/K)$ acts on $({}_\psi G)_{\bar{K}}$ by $g \mapsto \psi(\sigma)\cdot{}^\sigma g\cdot\psi(\sigma)^{-1}$ where $g \mapsto {}^\sigma g$ is the action of σ on $G_{\bar{K}}$.

Lemma 1.8. *Let $\psi \in Z^1(K, G^{\mathrm{ad}})$ be a cocycle. Then the map $\pi_1(G_{\bar{K}}) \to \pi_1(({}_\psi G)_{\bar{K}})$, induced by the canonical isomorphism $G_{\bar{K}} \to ({}_\psi G)_{\bar{K}}$, is an isomorphism of Galois modules.*

Proof: The assertion follows from the description 1.4 of the Galois action on $\pi_1(\bar{G})$.

In the remaining part of this section we prove some comparison results, which will not be used later.

1.9 Consider the functor $Z(\widehat{G})$ of Kottwitz. Here $\widehat{G}$ is a connected Langlands dual group for G, and $Z(\widehat{G})$ is the center of $\widehat{G}$ (cf. [Ko2]). By definition $\widehat{G}$ is a connected reductive $\mathbf{C}$-group endowed with an algebraic action of $\Gamma = \mathrm{Gal}(\bar{K}/K)$. The group $Z(\widehat{G})$ is an algebraic $\mathbf{C}$-group of multiplicative type; Γ acts on $Z(\widehat{G})$ algebraically. The character group $X^*(Z(G))$ is a finitely generated Γ-module.

Proposition 1.10. *The Γ-modules $\pi_1(\bar{G})$ and $X^*(Z(\widehat{G}))$ are canonically isomorphic.*

Proof: By definition (cf. [Ko2]) there is a maximal torus $\widehat{T} \subset \widehat{G}$ such that $X^*(\widehat{T}) = X_*(T_{\bar{K}})$, where T is a maximal torus of G defined over K. Moreover, $R(\widehat{G}, \widehat{T}) =$

$R^\vee(G_{\bar K},T_{\bar K})$, where R and $R^\vee$ denote the system of roots and the system of coroots, respectively. We have $Z(\widehat G) = \cap \ker[\alpha^\vee\colon \widehat T \to \mathbb{G}_{m\mathbf{C}}]$ where $\alpha^\vee$ runs through $R(\widehat G, \widehat T) = R^\vee(G_{\bar K}, T_{\bar K})$. Hence

$$X^*(Z(\widehat G)) = X^*(\widehat T)/\langle R(\widehat G, \widehat T)\rangle = X_*(T_{\bar K})/\langle R^\vee\rangle$$

where we write $R^\vee$ for $R^\vee(\bar G, \bar T)$ and we use $\langle\ \rangle$ to denote the subgroup of $X_*(T_{\bar K})$ generated by the set in brackets.

All the coroots $\alpha^\vee \in R^\vee \subset X_*(\bar T)$ come from $X_*(\bar T^{(\mathrm{sc})})$; moreover the set $R^\vee \subset \rho_* X_*(\bar T^{(\mathrm{sc})})$ generates $\rho_* X_*(\bar T^{(\mathrm{sc})})$ (cf. [St2], Lemma 25). Thus $X^*(Z(\widehat G)) = X_*(\bar T)/$
$\rho_* X_*(\bar T^{(\mathrm{sc})}) = \pi_1(\bar G)$, which was to be proved.

Remark 1.10.1: Let $\varphi\colon G_1 \to G_2$ be a homomorphism of reductive K-groups. First suppose that φ is normal, i.e. $\varphi(G_1)$ is normal in G_2. Then one can define a homomoprhism $\varphi^*\colon \widehat G_2 \to \widehat G_1$ (cf. [Bo], [Ko2]). But if φ is not normal, then we cannot define φ^*. In other words, $\widehat G$ is functorial with respect to normal homomoprhisms only. Proposition 1.9 shows, however, that the center $Z(\widehat G)$ of $\widehat G$ is functorial with respect to all homomophisms.

Proposition 1.11. *Let $\bar K$ be $\mathbf{C}$ and let K be either $\mathbf{R}$ or $\mathbf{C}$. For a connected reductive K-group G there is a canonical isomoprhism*

$$\pi_1(G) \xrightarrow{\sim} \mathrm{Hom}(\pi_1^{\mathrm{top}}(\mathbb{G}_m(\mathbf{C})), \pi_1^{\mathrm{top}}(G(\mathbf{C})))$$

where π_1^{top} is the usual topological fundamental group.

For brevity we write $\pi_1(G(\mathbf{C}))$ for $\pi_1^{\mathrm{top}}(G(\mathbf{C}))$ and $\pi_1(G(\mathbf{C}))(-1)$ for $\mathrm{Hom}(\pi_1^{\mathrm{top}}(\mathbb{G}_m(\mathbf{C})),$ $\pi_1^{\mathrm{top}}(G(\mathbf{C})))$.

We recall that in the case $K = \mathbf{R}$ the Galois group $\Gamma = \mathrm{Gal}(\mathbf{C}/\mathbf{R})$ acts on $\pi_1(G(\mathbf{C}))$ and (non-trivially) on $\pi_1(\mathbb{G}_m(\mathbf{C}))$. Since $\pi_1(\mathbb{G}_m(\mathbf{C}))$ is isomoprhic to $\mathbf{Z}$ as a group, but not as a Γ-module, we see that $\pi_1(G(\mathbf{C}))$ and $\pi_1(G(\mathbf{C}))(-1)$ are isomorphic as groups, but in general not as Γ-modules.

In the case $K = \mathbf{C}$ we have $\Gamma = 1$, and $\pi_1(G(\mathbf{C}))(-1)$ is isomorphic to $\pi_1(G(\mathbf{C}))$. To fix this isomoprhism it suffices to fix an isomorphism $\pi_1(\mathbf{C}^\times) \xrightarrow{\sim} \mathbf{Z}$ (or a square root of -1 in $\mathbf{C}$).

Proposition 1.11 justifies the term "algebraic fundamental group". The proposition means that $\pi_1(\bar G)$ is "the topological fundamental group, defined algebraically".

Proof: First we consider the case of a torus. Let T, T' be two K-tori. There is a canonical map

$$\mathrm{Hom}(T'_{\mathbf{C}}, T_{\mathbf{C}}) \to \mathrm{Hom}(\pi_1(T'(\mathbf{C})), \pi_1(T(\mathbf{C})))$$

This map is Γ-equivariant, and one can easily see that it is an isomoprhism of groups. Taking $\mathbb{G}_m$ for T' we obtain the required isomoprhism

$$\pi_1(T) = X_*(T_{\mathbf{C}}) \to \pi_1(G(\mathbf{C}))(-1).$$

In the general case we define the map $\pi_1(G) \to \pi_1(G(\mathbf{C}))(-1)$ as follows. Choose a maximal torus $T \subset G$ defined over K; then $\pi_1(\bar G) = X_*(\bar T)/\rho_* X_*(\bar T^{(\mathrm{sc})})$. We consider the composition

$$\alpha_T\colon X_*(\bar T) \to \pi_1(T(\mathbf{C}))(-1) \to \pi_1(G(\mathbf{C}))(-1).$$

One can easily check that $\alpha_T(\rho_*(X_*(\bar{T}^{(\mathrm{sc})}))) = 0$, hence α_T induces a homomorphism

$$(\alpha_T)_*\colon \pi_1(\bar{G}) \to \pi_1(G(\mathbf{C}))(-1)$$

It is not hard to check that $(\alpha_T)_*$ does not depend on the choice of T.

Now we have the diagram

(1.11.3)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(\bar{G}^{\mathrm{ss}}) & \longrightarrow & \pi_1(\bar{G}) & \longrightarrow & \pi(\bar{G}^{\mathrm{tor}}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi_1(G^{\mathrm{ss}}(\mathbf{C}))(-1) & \longrightarrow & \pi_1(G(\mathbf{C}))(-1) & \longrightarrow & \pi_1(G^{\mathrm{tor}}(\mathbf{C}))(-1) & \longrightarrow & 0 \end{array}$$

The upper row is exact by Proposition 1.5. The lower row comes from the exact sequence of the fiber bundle $G(\mathbf{C})$ over $G^{\mathrm{tor}}(\mathbf{C})$.

We have already shown that the right vertical row in (1.11.3) is an isomorphism. The Proposition 1.11 is well known for semisimple groups (cf. e.g. [O-V]), hence the left vertical arrow is an isomorphism. We conclude that the middle vertical arrow is an isomorphism.

1.12. Our definition of $\pi_1(\bar{G})$ uses explicitly the group structure of G. We are now going to show how to define $\pi_1(\bar{G})$ in a more "algebraic-geometrical" way. We make no further use of this construction here.

Let again K be any field of characteristic 0. Consider the algebraic-geometrical fundamental group $\pi_1^{\mathrm{Gr}}(\bar{G})$ defined by Grothendieck [Gr] (see also [Mi1]) (we take $1 \in \mathbb{G}(\bar{K})$ as the base point). Set $\pi_1^{\mathrm{Gr}}(\bar{G})(-1) = \mathrm{Hom}(\widehat{\mathbf{Z}}(1), \pi_1^{\mathrm{Gr}}(\bar{G}))$. Note that $\widehat{\mathbf{Z}}(1) = \pi_1(\mathbb{G}_{m\bar{K}})$. To any regular map $m\colon \mathbb{G}_{m\bar{K}} \to G_{\bar{K}}$ such that $m(1) = 1$, we associate its class $m_* = \mathrm{Cl}(m) \in \pi_1^{\mathrm{Gr}}(\bar{G})(-1) = \mathrm{Hom}(\pi_1^{\mathrm{Gr}}(\mathbb{G}_{m\bar{K}}), \pi_1^{\mathrm{Gr}}(G_{\bar{K}}))$. Let $\pi_1^{\mathrm{Gr}}(\bar{G})(-1)_{\mathrm{alg}}$ denote the subset of such algebraic classes in $\pi_1^{\mathrm{Gr}}(\bar{G})(-1)$.

Proposition 1.13. (i) *$\pi_1^{\mathrm{Gr}}(\bar{G})(-1)_{\mathrm{alg}}$ is a subgroup of the abelian group $\pi_1^{\mathrm{Gr}}(\bar{G})(-1)$.*

(ii) *The map $m \mapsto \mathrm{Cl}(M)$ induces an isomorphism of Γ-modules $\pi_1(\bar{G}) \xrightarrow{\sim} \pi_1^{\mathrm{Gr}}(\bar{G})(-1)_{\mathrm{alg}}$.*

(iii) *$\pi_1^{\mathrm{Gr}}(\bar{G})(-1)$ is isomorphic (as a Γ-module) to the completion of $\pi_1(\bar{G})$ with respect to the topology defined by the subgroups of finite index.*

We omit the proof.

Remark 1.14. Let H be a connected K-subgroup of G. Consider the homogeneous space $X = H\backslash G$. It has a canonical base point, namely the image of the neutral element of G. In this case one can similarly define the algebraic fundamental group $\pi_1(\bar{X})$ as the set of algebraic classes in

$$\pi_1^{\mathrm{Gr}}(\bar{X})(-1) = \mathrm{Hom}(\pi_1^{\mathrm{Gr}}(\mathbb{G}_{m\bar{K}}), \pi_1^{\mathrm{Gr}}(X_{\bar{K}})).$$

One can show that $\pi_1^{\mathrm{Gr}}(\bar{X})$ is an abelian group and that $\pi_1(\bar{X}) = \pi_1^{\mathrm{Gr}}(\bar{X})(-1)_{\mathrm{alg}}$ is a subgroup. In the case $\bar{K} = \mathbf{C}$ we have $\pi_1(\bar{X}) \simeq \pi_1^{\mathrm{top}}(X(\mathbf{C}))(-1)$.

2. Abelian Galois cohomology

2.1. Let K be a field of characteristic 0. We write Γ for $\mathrm{Gal}(\bar{K}/K)$. Let G be a (connected) reductive K-group. Choose a maximal torus $T \subset G$ (defined over K). We consider the complex of tori

$$T^{\cdot} = (\overset{-1}{T}{}^{(\mathrm{sc})} \xrightarrow{\rho} \overset{0}{T})$$

where $T^{(\mathrm{sc})}$ is in degree -1 and T is in degree 0. We define *the abelian Galois cohomology* of G as follows:

Definition 2.2. $H^i_{\mathrm{ab}}(K, G) = \mathbb{H}^i(K, T^{\cdot})$.

Here $\mathbb{H}^i$ means that Galois hypercohomology of the complex $T^{(\mathrm{sc})}(\bar{K}) \to T(\bar{K})$ of $\mathrm{Gal}(\bar{K}/K)$-modules. We may regard $H^{\cdot}_{\mathrm{ab}}(K, G)$ as the hypercohomology of the double complex

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(\bar{K}) & \longrightarrow & C^1(\Gamma, T(\bar{K})) & \longrightarrow & C^2(\Gamma, T(\bar{K})) & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & T^{(\mathrm{sc})}(\bar{K}) & \longrightarrow & C^1(\Gamma, T^{(\mathrm{sc})}(\bar{K})) & \longrightarrow & C^2(\Gamma, T^{(\mathrm{sc})}(\bar{K})) & \longrightarrow & \cdots \end{array} \tag{2.2.1}$$

where C^i are the usual groups of non-homogeneous continuous cochains. Note that the bidegree of $T^{(\mathrm{sc})}(\bar{K})$ is $(-1, 0)$.

We see that the groups $H^i_{\mathrm{ab}}(K, G)$ do not depend on the choice of the algebraic closure $\bar{K}$ of K. We are going to show in this section that they neither depend on the choice of T. Moreover, they depend only on $\pi_1(\bar{G})$.

2.3. Let Δ be a group, and let $(A^{-1} \xrightarrow{\alpha} A^0)$ be a short complex of Δ-modules. A morphism

$$\varepsilon\colon (A_1^{-1} \xrightarrow{\alpha_1} A_1^0) \longrightarrow (A_2^{-1} \xrightarrow{\alpha_2} A_2^0)$$

of complexes is called a quasi-isomorhism if it induces isomorphisms on cohomology, i.e. if the induced homomorphisms $\ker \alpha_1 \to \ker \alpha_2$ and $\operatorname{coker} \alpha_1 \to \operatorname{coker} \alpha_2$ are isomorphisms. It is well known that a quasi-isomorphism ε of complexes of Δ-modules induces isomorphisms

$$\varepsilon^i_*\colon \mathbb{H}^i(\Delta, A_1^{-1} \to A_1^0) \xrightarrow{\sim} \mathbb{H}^i(\Delta, A_2^{-1} \to A_2^0)$$

on hypercohomology.

Now let $\varepsilon\colon (S_1^{-1} \to S_1^0) \to (S_2^{-1} \to S_2^0)$ be a quasi-isomorphism of complexes of abelian algebraic K-groups. We see that the induced Galois hypercohomology homomorphisms

$$\varepsilon^i_*\colon \mathbb{H}^i(K, S_1^{-1} \to S_1^0) \to \mathbb{H}^i(K, S_2^{-1} \to S_2^0)$$

are isomorphisms.

2.4. Let G be as above. Following Deligne [Del] we consider the complex $(Z^{(\mathrm{sc})} \to Z)$ of abelian K-groups, where Z is the center of G and $Z^{(\mathrm{sc})}$ is the center of G^{sc}.

Lemma 2.4.1. *Let $F \subset G$ be a maximal torus. Then the embedding $(Z^{(\mathrm{sc})} \to Z) \to (T^{(\mathrm{sc})} \to T)$ of complexes of K-groups is a quasi-isomorphism.*

Proof: $\ker \rho \subset Z^{(\mathrm{sc})}$ and $T^{(\mathrm{sc})} \supset Z^{(\mathrm{sc})}$, hence

$$\ker[Z^{(\mathrm{sc})} \to Z] = \ker[\rho\colon G^{\mathrm{sc}} \to G] = \ker[T^{(\mathrm{sc})} \to T]$$

Set $G^{\mathrm{tor}} = G/\rho(G^{\mathrm{sc}}) = \operatorname{coker}\rho$. Since $G = G^{\mathrm{ss}} \cdot T = G^{\mathrm{ss}} \cdot Z$, the maps $\operatorname{coker}[T^{(\mathrm{sc})} \to T] \to G^{\mathrm{tor}}$ and $\operatorname{coker}[Z^{(\mathrm{sc})} \to Z] \to G^{\mathrm{tor}}$ are surjective. Since $\rho(T^{(\mathrm{sc})}) = T \cap G^{\mathrm{ss}}$ and $\rho(Z^{(\mathrm{sc})}) = Z \cap G^{\mathrm{ss}}$, these maps are injective. Thus

$$\operatorname{coker}[Z^{(\mathrm{sc})} \to Z] = \operatorname{coker}[\rho\colon G^{\mathrm{sc}} \to G] = \operatorname{coker}[T^{(\mathrm{sc})} \to T],$$

and we conclude that $(Z^{(\mathrm{sc})} \to Z) \to (T^{(\mathrm{sc})} \to T)$ is a quasi-isomorhism, which was to be proved.

It follows from Lemma 2.4.1 that we have canonical isomorhisms

$$\mathbb{H}^i(Z^{(\mathrm{sc})} \to Z) \xrightarrow{\sim} \mathbb{H}^i(T^{(\mathrm{sc})} \to T)$$

Thus the groups $H^i_{\mathrm{ab}}(K, G) = \mathbb{H}^i(K, T^{(\mathrm{sc})} \to T)$ are defined correctly, i.e. do not depend on the choice of T.

2.5. For a homomorphism $\beta\colon G_1 \to G_2$ we define $\beta_*\colon H^i_{\mathrm{ab}}(K, G_1) \to H^i_{\mathrm{ab}}(K, G_2)$. Let $T_1 \subset G_1$ be a maximal torus, and let $T_2 \subset G_2$ be a maximal torus containing $\beta(T_1)$. The morphism

$$(T_1^{(\mathrm{sc})} \to T_1) \longrightarrow (T_2^{(\mathrm{sc})} \to T_2)$$

defines homomorphisms

$$\beta^i_*\colon H^i_{\mathrm{ab}}(K, G_1) = \mathbb{H}^i(T_1^{(\mathrm{sc})} \to T_1) \to \mathbb{H}^i(T_2^{(\mathrm{sc})} \to T_2) = H^i_{\mathrm{ab}}(K, G_2).$$

We must show that the maps β^i_* are defined correctly, i.e. do not depend on the choice of T_1 and T_2. For this end we are going to show in this section that the group $H^i_{\mathrm{ab}}(K, G)$ depends only on the fundamental group $\pi_1(\bar{G})$ and that the maps β^i_* can be described in terms of the map $\beta_*\colon \pi_1(\bar{G}_1) \to \pi_1(\bar{G}_2)$.

2.6. The functor $\mathcal{H}^i$.

2.6.1. Let Δ be a finite group and M a finitely generated Δ-module. One can easily see that there exists a short torsion free resolution of M, i.e. a short exact sequence

$$0 \to L^{-1} \to L^0 \to M \to 0$$

of Δ-modules such that L^{-1} and L^0 are finitely generated and torsion free as abelian groups.

Let D be a Δ-module. Consider the complex $L^{\cdot} = (L^{-1} \to L^0)$ and the complex

$$L^{\cdot} \underset{\mathbf{Z}}{\otimes} D = (L^{-1} \otimes D \to L^0 \otimes D).$$

Definition 2.6.2. $\mathcal{H}^i(\Delta, M, D) = \mathbb{H}^i(\Delta, L^{\cdot} \otimes D)$.

We observe that the groups $\mathcal{H}^i(\Delta, M, D)$ are defined correctly. Indeed, the complex $L^{\cdot}$ is quasi-isomorphic to $(1 \to M)$. If we choose two short torsion free resolutions $L_1^{\cdot} \to M$ and $L_2^{\cdot} \to M$, then $L_1^{\cdot}$ and $L_2^{\cdot}$ are canonically isomorphic in the appropriate derived category. Since $L_1^{\cdot}$ and $L_2^{\cdot}$ are torsion free, the complexes $L_1^{\cdot}\otimes D$ and $L_2^{\cdot}\otimes D$ are also canonically isomorphic in the derived category. It follows that the groups $\mathbb{H}^i(\Delta, L_1^{\cdot}\otimes D)$ and $\mathbb{H}^i(\Delta, L_2^{\cdot}\otimes D)$ are canonically isomorphic, which was to be proved. Note that in the language of derived categories we have just

$$\mathcal{H}^i(\Delta, M, D) = \mathbb{H}^i(\Delta, M \overset{L}{\underset{\mathbf{Z}}{\otimes}} D),$$

where $\overset{L}{\underset{\mathbf{Z}}{\otimes}}$ denotes the left derived functor of the tensor product.

Remark 2.6.3. We can also define the "Tate groups"

$$\widehat{\mathcal{H}}^i(\Delta, M, D) := \widehat{\mathbb{H}}^i(\Delta, L^{\cdot} \underset{\mathbf{Z}}{\otimes} D) \quad (i \in \mathbf{Z}),$$

where $L^{\cdot} \to M$ is a short torsion free resolution. Here $\widehat{\mathbb{H}}^{\cdot}$ denotes the hypercohomology of the double complex $\mathrm{Hom}(P^{\cdot}, L^{\cdot})$, where $P^{\cdot}$ is *a complete resolution* for Δ (see e.g. [A-W]).

2.6.4. If Δ is a finite group and U is a normal subgroup of Δ, then we have inflation homomorhisms

$$\mathcal{H}^i(\Delta/U, M^U, D^U) \longrightarrow \mathcal{H}^i(\Delta, M, D)$$

Now let Γ be a pro-finite group and M a finitely generated (over $\mathbf{Z}$) discrete Γ-module. Let D be a discrete Γ-module. We set

$$\mathcal{H}^i(\Gamma, M, D) = \varinjlim_{U} \mathcal{H}^i(\Gamma/U, M^U, D^U),$$

where U runs over the open normal subgroups of Γ.

Let $L^{\cdot} \to M$ be *a short torsion free resolution* of M, i.e. an exact sequence

$$0 \to L^{-1} \to L^0 \to M \to 0$$

of discrete Γ-modules, where L^{-1} and L^0 are finitely generated torsion free abelian groups. Let $\mathbb{H}^i(\Gamma, L^{\cdot} \otimes D)$ denote the hypercohomology of the double complex

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^0(\Gamma, L^0 \otimes D) & \longrightarrow & C^1(\Gamma, L^0 \otimes D) & \longrightarrow & C^2(\Gamma, L^0 \otimes D) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C^0(\Gamma, L^{-1} \otimes D) & \longrightarrow & C^1(\Gamma, L^{-1} \otimes D) & \longrightarrow & C^2(\Gamma, L^{-1} \otimes D) & \longrightarrow & \cdots
\end{array}$$

where $C^i(\Gamma, \cdot)$ denotes the groups of *continuous* non-homogeneous cochains. Since $M^U = M$ for sufficiently small U, we have

$$\mathcal{H}^i(\Gamma, M, D) = \mathbb{H}^i(\Gamma, L^{\cdot} \underset{\mathbf{Z}}{\otimes} D).$$

The functor $\mathcal{H}(\Gamma, M, D)$ is a cohomological functor of M in the following sense.

Proposition 2.6.5. *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be a short exact sequence of finitely generated (over $\mathbf{Z}$*)* Γ*-modules. Then we have an exact sequence*

(2.6.5.1)
$$\cdots \to \mathcal{H}^i(\Gamma, M_1, D) \to \mathcal{H}^i(\Gamma, M_2, D) \to \mathcal{H}^i(\Gamma, M_3, D) \to \\ \to \mathcal{H}^{i+1}(\Gamma, M_1, D) \to \cdots$$

Proof: Easy.

Corollary 2.6.6. *Let*

$$0 \to L^{-1} \to L^0 \to M \to 0$$

be a short torsion free resolution of M*. Then there is an exact sequence*

(2.6.6.1)
$$0 \to \mathcal{H}^{-1}(\Gamma, M, D) \to H^0(\Gamma, L^{-1} \otimes D) \to H^0(\Gamma, L^0 \otimes D) \to \\ \to \mathcal{H}^0(\Gamma, M, D) \to H^1(\Gamma, L^{-1} \otimes D) \to \cdots$$

2.7. Let Γ again denote the Galois group $\mathrm{Gal}(\bar{K}/K)$. Let M be a discrete finitely generated Γ-module. We are interested in the groups $\mathcal{H}^i(\Gamma(\bar{K}/K), M; \bar{K}^\times)$; for brevity we write just $\mathcal{H}^i(K, M, \bar{K}^\times)$.

2.7.1. Let $L^\cdot \to M$ be a short torsion free resolution. Consider the complex $T^{-1} \to T^0$ of K-tori such that $L^\cdot = (L^{-1} \to L^0)$ is the complex $X_*(T^{-1}_{\bar{K}}) \to X_*(T^0_{\bar{K}})$ of cocharacter groups of these tori. By definition

$$\mathcal{H}^i(K, M, \bar{K}^\times) = \mathbb{H}^i(K, L^{-1} \otimes \bar{K}^\times \longrightarrow L^0 \otimes \bar{K}^\times) = \mathbb{H}^i(K, T^{-1} \to T^0)$$

Thus $\mathcal{H}^i(K, M, \bar{K}^\times)$ is the Galois hypercohomology of a complex of tori.

2.7.2. Examples.

(1) If M is torsion free, then we set $L^{-1} = 0$, $L^0 = M$, $X_*(T^0) = M$. Thus $\mathcal{H}^1(K, M, \bar{K}^\times) = H^i(K, T^0)$.

(2) Suppose that M is finite. Choose a resolution $L^\cdot \to M$ and define the complex $T^\cdot = T^{-1} \to T^0$ as above. Then the homomorphism $T^{-1}(\bar{K}) \to T^0(\bar{K})$ is surjective. Set $B = \ker[T^{-1} \to T^0]$; it is a finite abelian K-group. Then the morphism of complexes

$$(B(\bar{K}) \to 0) \to (T^{-1}(\bar{K}) \to T^0(\bar{K}))$$

is a quasi-isomorphism. Hence

$$\mathcal{H}^i(K, M, \bar{K}^\times) := \mathbb{H}^i(K, T^{-1}(\bar{K}) \to T^0(\bar{K})) = \mathbb{H}^i(K, B(\bar{K}) \to 0) = H^{i+1}(K, B).$$

Now let G be a connected reductive K-group.

Proposition 2.8. $H^i_{\rm ab}(K,G)=\mathcal{H}^i(K,\pi_1(\bar G),\bar K^\times)$

Proof: Let $T\subset G$ be a maximal torus (defined over K). Set $L^0=X_*(T_{\bar K})$, $L^{-1}=X_*(T^{(\rm sc)})$. Then by definition of $\pi_1(\bar G)$, $(L^{-1}\to L^0)\to\pi_1(\bar G)$ is a resolution of $\pi_1(\bar G)$. Hence, as it was shown in 2.7.1, $\mathcal{H}^i(K,\pi_1(\bar G),\bar K^\times)=\mathbb{H}^i(K,T^{(\rm sc)}\to T)$. By definition $\mathbb{H}^i(K,T^{(\rm sc)}\to T)=H^i_{\rm ab}(K,G)$. This proves the proposition.

We see from Propositioni 2.8 that the groups $H^i_{\rm ab}(K,G)$ depend only on the Galois module $\pi_1(\bar G)$.

Corollary 2.9. *Let $\psi\in H^1(K,G^{\rm ad})$ be a cocycle. There are canonical isomorhisms $H^1_{\rm ab}(K,G)\to H^i_{\rm ab}(K,{}_\psi G)$.*

Proof: The assertion follows from Lemma 1.8 and Proposition 2.8.

Proposition 2.10. *Let $1\to G_1\to G_2\to G_3\to 1$ be an exact sequence of connected reductive K-groups. Then there is a long abelian cohomology exact sequence*

$$0\to H^{-1}_{\rm ab}(K,G_1)\to H^{-1}_{\rm ab}(K,G_2)\to H^{-1}_{\rm ab}(K,G_3)\to H^0_{\rm ab}(K,G_1)\to\cdots \tag{2.10.1}$$

Proof: The assertion follows from Lemma 1.5 and Proposition 2.6.5.

The exact sequence (2.10.1) can be defined more explicitly as follows. Let $T_2\subset G_2$ be a maximal torus. Let T_3 be the image of T_2 in G_3, and let T_1 be the inverse image of T_2 in G_1. We have the short exact sequence

$$0\to(T_1^{(\rm sc)}\to T_1)\to(T_2^{(\rm sc)}\to T_2)\to(T_3^{(\rm sc)}\to T_3)\to 0$$

of complexes of tori. Then (2.10.1) is the corresponding long hypercohomology exact sequence.

2.11 Examples.

(1) G is a torus. Then $(T^{(\rm sc)}\to T)=(1\to G)$, and $H^i_{\rm ab}(K,G)=H^i(K,G)$.

(2) Suppose that $G^{\rm ss}$ is simply connected. By 1.6(2) the homomorphism $\pi_1(\bar G)\to\pi_1(\bar G^{\rm tor})$ is an isomorphism, hence $H^i_{\rm ab}(K,G)=H^i(K,G^{\rm tor})$.

(3) Let G be a semisimple group, $G=G^{\rm sc}/\ker\rho$. Then $\ker(T^{(\rm sc)}\to T)=\ker\rho$, and by 2.7.2(2) $H^i_{\rm ab}(K,G)=H^{i+1}(K,\ker\rho)$. Recall that $\ker\rho$ is a finite abelian K-group.

(4) For any G we have $H^{-1}_{\rm ab}(K,G)=(\ker\rho)(K)$. This follows from the definition.

Proposition 2.12. *Let G be a connected reductive K-group. Let $T\subset G$ be a maximal K-torus. Then there are exact sequences*

(2.12.1)
$$\cdots\to H^{i+1}(K,\ker\rho)\to H^i_{\rm ab}(K,G)\to H^i(K,G^{\rm tor})\to H^{i+2}(K,\ker\rho)\to\cdots$$
(2.12.2)
$$\cdots\to H^i(K,T^{(\rm sc)})\to H^i(K,T)\to H^i_{\rm ab}(K,G)\to H^{i+1}(K,T^{(\rm sc)})\to\cdots$$

Proof: Consider the short exact sequence

$$1\to G^{\rm ss}\to G\to G^{\rm tor}\to 1$$

Applying Proposition 2.10 and calculations 2.11(1,3), we obtain (2.12.1). We obtain (2.12.2) from Proposition 2.8 and Proposition 2.6.5.

3. The abelianization maps

In this section we construct the abelianization maps

$$\begin{aligned}&\mathrm{ab}^0\colon G(K)=H^0(K,G)\to H^0_{\mathrm{ab}}(K,G)\\&\mathrm{ab}^1\colon H^1(K,G)\to H^1_{\mathrm{ab}}(K,G)\end{aligned}$$

for a connected reductive group G over a field K of characteristic 0. For this end we need the non-abelian hypercohomology theory. We give here a short review; for more detail see [Brv5].

3.1. Hypercohomology of complexes of groups.

3.1.1. Let Δ be a group, and let

$$1\to \overset{-1}{F}\xrightarrow{\alpha}\overset{0}{G}\to 1$$

be a short complex of (in general non-abelian) Δ-groups, where F is in degree -1 and G is in degree 0. For brevity we write $F\to G$ for this complex. We define its -1-*hypercohomology group* by

$$\mathbb{H}^{-1}(\Delta,F\to G)=(\ker\alpha)^{\Delta}$$

where $(\quad)^{\Delta}$ denotes the subgroup of invariants.

We define the 0-*hypercohomology set* $\mathbb{H}^0(\Delta,F\to G)$ in terms of cocycles. We write $\mathrm{Maps}(\Delta,F)$ for the set of the maps $\varphi\colon\Delta\to F$ and set

$$\begin{aligned}&C^0=\mathrm{Maps}(\Delta,F)\times G\quad\text{(we regard } C^0 \text{ as a set)}\\&Z^0=\{\varphi,g\in C^0\ \big|\ \varphi(\sigma\tau)=\varphi(\sigma)\cdot{}^{\sigma}\varphi(\tau),\ {}^{\sigma}g=\alpha(\varphi(\sigma))^{-1}\cdot g\}\end{aligned}$$

The sets C^0 and Z^0 are the sets of cochains and cocycles, respectively. The group F acts on Z^0 on the right by

$$(\varphi,g)*f=(\varphi',g'),\quad \varphi'(\sigma)=f^{-1}\cdot\varphi(\sigma)\cdot{}^{\sigma}f,\ g'=\alpha(f)^{-1}\cdot g,$$

and we set

$$\mathbb{H}^0(\Delta,F\to G)=Z^0/F.$$

The set $\mathbb{H}^0$ has a *neutral element*, namely, the class of the cocycle $(1,1)\in Z^0$.

We write $\mathbb{H}^{-1}(F\to G)$ and $\mathbb{H}^0(F\to G)$ for $\mathbb{H}^{-1}(\Delta,F\to G)$ and $\mathbb{H}^0(\Delta,F\to G)$, respectively.

3.1.2 Examples.

(1) $\mathbb{H}^0(1\to G)=H^0(G)$.

(2) $\mathbb{H}^0(F\to 1)=H^1(F)$. To $\mathrm{Cl}(\varphi,1)\in\mathbb{H}^0(F\to 1)$ we associated $\mathrm{Cl}(\varphi)\in H^1(F)$.

(3) If $\alpha\colon F\to G$ is injective, then $\mathbb{H}^0(F\to G)=H^0(\mathrm{coker}\,\alpha)$.

(4) If α is surjective, then $\mathbb{H}^0(F\to G)=H^1(\ker\alpha)$.

3.1.3. Let $\varepsilon\colon (F_1 \to G_1) \to (F_2 \to G_2)$ be a *morphism* of complexes of Δ-groups, i.e. a commutative diagram

$$\begin{array}{ccc} F_1 & \longrightarrow & F_2 \\ \big\downarrow & & \big\downarrow \\ G_1 & \longrightarrow & G_2 \end{array}$$

of groups. We have induced map:

$$\begin{aligned} \varepsilon_*^{(-1)}\colon \mathbb{H}^{-1}(F_1 \to G_1) &\longrightarrow \mathbb{H}^{-1}(F_2 \to G_2), \\ \varepsilon_*^{0}\colon \mathbb{H}^{0}(F_1 \to G_1) &\longrightarrow \mathbb{H}^{0}(F_2 \to G_2), \end{aligned}$$

where $\varepsilon_*^{(-1)}$ is a group homomoprhism and ε_*^0 is a morphism of pointed sets.

3.2 Crossed modules.

To define 1-hypercohomology we need crossed modules, introduced by J. H. C. Whitehead [Wh] (see [Bro], [Br-H] for a survey).

Definition 3.2.1. A (left) crossed module is a short complex $\alpha\colon F \to G$ endowed with a left action $G \times F \to F$ of G on F, denoted $(g, f) \mapsto {}^g f$, such that

$$ff'f^{-1} = {}^{\alpha(f)}f' \tag{3.2.1.1}$$

$$\alpha({}^g f) = g \cdot \alpha(f) \cdot g^{-1} \tag{3.2.1.2}$$

for any $f, f' \in F$, $g \in G$.

We say that a group Δ acts on a crossed module $(F \to G)$ if Δ acts on F and G so that

$$\alpha({}^\sigma f) = {}^\sigma(\alpha(f)),\ \ {}^\sigma({}^g f) = {}^{{}^\sigma g}({}^\sigma f)$$

for any $f \in F$, $g \in G$, $\sigma \in \Delta$.

3.2.2 Examples.

(1) An *abelian* crossed module: F and G are abelian groups, the action of G on F is trivial.

(2) F is an abelian group (G-module),α is trivial.

(3) F is a normal subgroup of G, $\alpha\colon F \hookrightarrow G$ is the inclusion, ${}^g f = gfg^{-1}$.

(4) $F \to G$ is any surjective homomoprhism with central kernel.

(5) $F \to \operatorname{Aut} F$ for any group F.

Any crossed module is in a sense a combination of examples 3.2.2(3) and 3.2.2(4). We have

Lemma 3.2.3. (cf. [Br-H]). *Let $F \xrightarrow{\alpha} G$ be a crossed module. Then*

(i) $\ker\alpha$ *is central in* F*;*

(ii) im α *is normal in* G *(hence coker* α *is defined);*

(iii) coker α *acts on* $\ker\alpha$.

3.2.4. A *morphism* of crossed modules $\varepsilon\colon (F_1 \to G_1) \to (F_2 \to G_2)$ is a morphism of complexes

$$\begin{array}{ccc} F_1 & \xrightarrow{\varepsilon_F} & F_2 \\ \downarrow & & \downarrow \\ G_1 & \xrightarrow[\varepsilon_G]{} & F_2 \end{array}$$

such that the homomorphism ε_F is ε_G-equivariant.

3.3. 1-hypercohomology with coefficients in a crossed module.

3.3.1. Let $F \to G$ be a crossed module of Δ-groups. We define a group structure on $C^0 = C^0(\Delta, F \to G)$ as follows.

Let $(\varphi_1, g_1), (\varphi_2, g_2) \in C^0$. We set

$$(\varphi_1, g_1) \cdot (\varphi_2, g_2) = (\varphi', g_1 g_2) \quad \text{where} \quad \varphi'(\sigma) = {}^{g_1}\varphi_2(\sigma) \cdot \varphi_1(\sigma)$$

One can check that Z^0 is a subgroup of C^0 with respect to this group structure. In this way we obtain a group structure on $\mathbb{H}^0(F \to G)$.

3.3.2. Let $F \to G$ be a crossed module of Δ-groups. We define *the* 1*-hypercohomology set* $\mathbb{H}^1(F \to G)$ in terms of cocycles. We follow an idea of Dedecker [Ded2].

Let Z^1 be the set of pairs (u, ψ), $u \in \mathrm{Maps}(\Delta \times \Delta, F)$, $\psi \in \mathrm{Maps}(\Delta, G)$, such that

$$\psi(\sigma\tau) = \alpha(u(\sigma,\tau)) \cdot \psi(\sigma) \cdot {}^{\sigma}\psi(\tau) \tag{3.3.2.1}$$

$$u(\sigma, \tau\upsilon) \cdot {}^{\psi(\sigma)\sigma}u(\tau, \upsilon) = u(\sigma\tau, \upsilon) \cdot u(\sigma, \tau) \tag{3.3.2.2}$$

We define a right action $*$ of the group C^0 on the set of 1-cocycles Z^1. For $(a, g) \in C^0$ we set

$$(u, \psi) * (a, g) = (u', \psi')$$

where

$$\psi'(\sigma) = g^{-1} \cdot \alpha(a(\sigma)) \cdot \psi(\sigma) \cdot {}^{\sigma}g$$

$$u'(\sigma,\tau) = {}^{g^{-1}}[a(\sigma\tau) \cdot u(\sigma,\tau) \cdot {}^{\psi(\sigma)\sigma}a(\tau)^{-1} \cdot a(\sigma)^{-1}]$$

We set $\mathbb{H}^1(F \to G) = Z^1/C^0$.

The set $\mathbb{H}^1(F \to G)$ has a *neutral element*, namely, the class of the cocycle $(1, 1) \in Z^1$.

A morphism $\varepsilon\colon (F_1 \to G_1) \to (F_2 \to G_2)$ of crossed complexes of Δ-groups induces a morhism of pointed sets $\varepsilon^1_*\colon \mathbb{H}^1(F_1 \to G_1) \to \mathbb{H}^1(F_2 \to G_2)$.

Remark 3.3.3. Our notation here slightly differs from that of [Brv5]: in [Brv5] we write an element of $Z^1(F \to G)$ in the form (h, ψ) where $h(\sigma, \tau) = u(\sigma, \tau)^{-1}$.

3.3.4 Examples.

(1) $\mathbb{H}^1(1 \to G) = H^1(G)$. To $\mathrm{Cl}(1,\psi) \in \mathbb{H}^1(1 \to G)$ we associate $\mathrm{Cl}(\psi) \in H^1(G)$.

(2) $\mathbb{H}^1(F \to 1) = H^2(F)$. To $\mathrm{Cl}(u,1) \in \mathbb{H}^1(F \to 1)$ we associate $\mathrm{Cl}(u) \in H^2(F)$. Note that in this case F is abelian, and therefore $H^2(F)$ makes sense.

(3) If $F \xrightarrow{\alpha} G$ is a crossed module and α is injective, then $\mathbb{H}^1(F \to G) = \mathbb{H}^1(\operatorname{coker}\alpha)$.

(4) If α is surjective, then the embedding $(\ker\alpha \to 1) \hookrightarrow (F \to G)$ induces the canonical bijection $H^2(\ker\alpha) = H^1(F \to G)$.

3.4. Hypercohomology exact sequences.

3.4.1. *A short exact sequence of complexes of groups*

$$1 \to (F_1 \to G_1) \to (F_2 \to G_2) \to (F_3 \to G_3) \to 1$$

is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & F_3 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & 1 \end{array}$$

with exact rows.

Proposition 3.4.2. ([Brv5], [Brn]). *Let*

$$1 \to (F_1 \to G_1) \xrightarrow{i} (F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3) \to 1$$

be an exact sequence of complexes of Δ-groups. Assume that $(F_1 \to G_1) \to (F_2 \to G_2)$ is a morhism of crossed modules with Δ-action and that the subgroup $i(F_1) \subset F_2$ is G_2-invariant.

(i) *There is a hypercohomology exact sequence*

(3.4.2.1)
$$\begin{aligned} 1 \to \mathbb{H}^{-1}(F_1 \to G_1) &\xrightarrow{i_*} \mathbb{H}^{-1}(F_2 \to G_2) \xrightarrow{j_*} \mathbb{H}^{-1}(F_3 \to G_3) \\ &\xrightarrow{\delta_{-1}} \mathbb{H}^{0}(F_1 \to G_1) \xrightarrow{i_*} \mathbb{H}^{0}(F_2 \to G_2) \xrightarrow{j_*} \mathbb{H}^{0}(F_3 \to G_3) \\ &\xrightarrow{\delta_{0}} \mathbb{H}^{1}(F_1 \to G_1) \xrightarrow{i_*} \mathbb{H}^{1}(F_2 \to G_2) \end{aligned}$$

(ii) *The group $\mathbb{H}^0(F_2 \to G_2)$ acts on the set $\mathbb{H}^0(F_3 \to G_3)$, and δ_0 defines a bijection*

(3.4.2.2) $$\mathbb{H}^0(F_2 \to G_2)\backslash\mathbb{H}^0(F_3 \to G_3) \xrightarrow{\sim} \ker[\mathbb{H}^1(F_1 \to G_1) \to \mathbb{H}^1(F_2 \to G_2)]$$

(iii) *If moreover $(F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3)$ is a morphism of crossed modules, then the exact sequence* (3.4.2.1) *can be prolonged by the term $\xrightarrow{j_*} \mathbb{H}^1(F_3 \to G_3)$, i.e. the sequence*

(3.4.2.3) $$\mathbb{H}^1(F_1 \to G_1) \xrightarrow{i_*} \mathbb{H}^1(F_2 \to G_2) \xrightarrow{j_*} \mathbb{H}^1(F_3 \to G_3)$$

is exact.

The connecting maps δ_1 and δ_0 are defined as follows.

We identify the crossed module $(F_1 \to G_1)$ with its image in $(F_2 \to G_2)$. Let $f_3 \in \mathbb{H}^{-1}(F_3 \to G_3) = (\ker \alpha_3)^\Delta$. We lift f_3 to some element $f \in F_2$ and set

$$\varphi_1(\sigma) = f \cdot {}^\sigma f^{-1}, \ g_1 = \alpha_2(f)$$

Then $(\varphi_1, g_1) \in Z^0(F_1 \to G_1)$. We set

$$\delta_{-1}(f_3) = \mathrm{Cl}(\varphi_1, g_1) \in \mathbb{H}^0(F_1 \to G_1).$$

Let $\xi_3 \in \mathbb{H}^0(F_3 \to G_3)$, $\xi_3 = \mathrm{Cl}(\varphi_3, g_3)$. We lift φ_3 to some map $\varphi\colon \Delta \to F_2$ and lift g_3 to some element $g \in G_2$. We set

$$\begin{aligned} \psi_1(\sigma) &= g^{-1} \cdot \alpha_2(\varphi(\sigma)) \cdot {}^\sigma g \\ u_1(\sigma, \tau) &= {}^{g^{-1}}[\varphi(\sigma\tau) \cdot {}^\sigma\varphi(\tau)^{-1} \cdot \varphi(\sigma)^{-1}] \end{aligned}$$

Then $(u_1, \psi_1) \in Z^1(F_1 \to G_1)$. We set

$$\delta_0(\xi_3) = \mathrm{Cl}(u_1, \psi_1) \in \mathbb{H}^1(F_1 \to G_1)$$

One can check that the connecting maps δ_{-1} and δ_0 are defined correctly.

The group $\mathbb{H}^0(F_2 \to G_2)$ acts on the left on $\mathbb{H}^0(F_3 \to G_3)$ by

$$\mathrm{Cl}(\varphi_2, g_2) \cdot \mathrm{Cl}(\varphi_3, g_3) = \mathrm{Cl}({}^{j(g_2)}\varphi_3 \cdot \varphi_2, j(g_2)g_3)$$

where $(\varphi_2, g_2) \in Z^0(F_2 \to G_2)$, $(\varphi_3, g_3) \in Z^0(F_3 \to G_3)$.

Corollary 3.4.3. *Let $(F \to G)$ be a crossed module of Δ-groups.*

(i) *There is an exact sequence*

(3.4.3.1)
$$\begin{aligned} 1 \to \mathbb{H}^{-1}(F \to G) \to H^0(F) \to H^0(G) \to \mathbb{H}^0(F \to G) \\ \to H^1(F) \to H^1(G) \to \mathbb{H}^1(F \to G) \end{aligned}$$

(ii) *The group $\mathbb{H}^0(F \to G)$ acts on $H^1(F)$, and there is a canonical bijection*

(3.4.3.2)
$$\mathbb{H}^0(F \to G)\backslash H^1(F) \xrightarrow{\sim} \ker[H^1(G) \to \mathbb{H}^1(F \to G)].$$

3.5 Quasi-isomorphisms of crossed modules.

3.5.1. A morphism $(F_1 \xrightarrow{\alpha_1} G_1) \to (F_2 \xrightarrow{\alpha_2} G_2)$ of crossed modules is called a *quasi-isomorphism* if the induced homomorphisms $\ker \alpha_1 \to \ker \alpha_2$ and $\operatorname{coker} \alpha_1 \to \operatorname{coker} \alpha_2$ are isomprhisms.

3.5.2 Examples.

(1) If $F \subset G$ is a normal subgroup then $(F \to G) \to (1 \to G/F)$ is a quasi-isomorphism.

(2) If $F \xrightarrow{\alpha} G$ is a crossed module and α is surjective, then $(\ker \alpha \to 1) \to (F \to G)$ is a quasi-isomorphism.

Theorem 3.5.3. ([Brv5]) *Let $\varepsilon\colon (F_1 \to G_1) \to (F_2 \to G_2)$ be a quasi-isomorphism of crossed modules of Δ-groups. Then the induced maps*

$$\varepsilon^0_*\colon \mathbb{H}^0(F_1 \to G_1) \to \mathbb{H}^0(F_2 \to G_2)$$
$$\varepsilon^1_*\colon \mathbb{H}^1(F_1 \to G_1) \to \mathbb{H}^1(F_2 \to G_2)$$

are bijective.

3.6. Let now $F \xrightarrow{\alpha} G$ be *a crossed module of algebraic groups* over a field K of characteristic 0. This means that α is a homomorhism of K-groups, and a left action $G \times F \to F$, $(g, f) \mapsto {}^g f$ is given such that the axioms (3.2.1.1) and (3.2.1.2) are satisfied.

For a finite Galois extension K'/K the Galois group $\mathrm{Gal}(K'/K)$ acts on the crossed module $F(K') \to G(K')$, and we set

$$\mathbb{H}^i(K'/K, F \to G) = \mathbb{H}^i(\mathrm{Gal}(K'/K), F(K') \to G(K'))\ (i = -1, 0, 1).$$

Set

$$\mathbb{H}^i(K, F \to G) = \varinjlim_{K'} \mathbb{H}^i(K'/K, F \to G)$$

where K' runs over the Galois extensions of K contained in a fixed algebraic closure $\bar{K}$ of K, and the inductive limit is taken with respect to the obvious inflation maps $\mathbb{H}^i(K'/K, F \to G) \to \mathbb{H}^i(K''/K, F \to G)$ for $K'' \supset K'$. We may also write

$$\mathbb{H}^i(K, F \to G) = \mathbb{H}^i_{\mathrm{cont}}(\Gamma, F(\bar{K}) \to G(\bar{K}))$$

where $\Gamma = \mathrm{Gal}(\bar{K}/K)$, and the subscript $_{\mathrm{cont}}$ means that we define hypercohomology in terms of continuous cocycles.

We now return to the Galois cohomology of reductive groups.

3.7. Let G be a reductive K-group. Consider the complex of K-groups

$$G^{\mathrm{sc}} \xrightarrow{\rho} G.$$

The canonical homomorphism

$$G \to G^{\mathrm{ad}} = (G^{\mathrm{sc}})^{\mathrm{ad}} \hookrightarrow \mathrm{Aut}(G^{\mathrm{sc}})$$

defines an action of G on G^{sc}. We denote this action by $(g, s) \mapsto {}^g s$ where $g \in G$,$s \in G^{\mathrm{sc}}$.

Lemma 3.7.1. *$G^{\mathrm{sc}} \xrightarrow{\rho} G$ is a crossed module.*

Indeed, we have obvious equalities

$$\rho({}^g s) = g \cdot \rho(s) \cdot g^{-1}$$
$$ss's^{-1} = {}^{\rho(s)}s'\ (s, s' \in G^{\mathrm{sc}}, g \in G),$$

which was to be proved.

We define *the relative Galois cohomology* of G with respect to G^{sc} by

$$H^i_{\mathrm{rel}}(K,G) = \mathbb{H}^i(K, G^{\mathrm{sc}} \xrightarrow{\rho} G) \ (i = -1,0,1)$$

To any homomorphism $\beta\colon G_1 \to G_2$ of connected reductive K-groups we associate the morphism $(G_1^{\mathrm{sc}} \to G_1) \to (G_2^{\mathrm{sc}} \to G_2)$ of crossed modules. We see that $G \mapsto (G^{\mathrm{sc}} \to G)$ is a functor with values in the category of crossed modules of K-groups. Hence $G \mapsto H^i_{\mathrm{rel}}(K,G)$ is a functor with values in the category of abelian groups for $i=-1$, of groups for $i=0$, and of pointed sets for $i=1$.

We want to relate $H^i_{\mathrm{rel}}(K,G)$ to the abelian Galois cohomology$H^i_{\mathrm{ab}}(K,G)$ defined in Section 2.

3.8. Let $T \subset G$ be a maximal torus defined over K. Let Z be the center of G.

Lemma 3.8.1. *All the arrows in the commutative diagram of crossed modules*

$$\begin{array}{ccc} (Z^{(\mathrm{sc})} \to Z) & \hookrightarrow & (T^{(\mathrm{sc})} \to GT \\ & & \downarrow \\ & & (G^{\mathrm{sc}} \to G) \end{array} \tag{3.8.1.1}$$

are quasi-isomorphisms.

Proof: We have already proved the assertion when proving Lemma 2.4.1.

Now it follows from Theorem 3.4 and Lemma 3.8.1 that all the maps in the commutative diagram

$$\begin{array}{ccc} \mathbb{H}^i(K, Z^{(\mathrm{sc})} \to Z) & \xrightarrow{\sim} & \mathbb{H}^i(K, T^{(\mathrm{sc})} \to T) \\ & \sim & \big\downarrow{\scriptstyle\sim} \\ H^i_{\mathrm{rel}}(K,G) & = & \mathbb{H}^i(K, G^{\mathrm{sc}} \to G) \end{array} \tag{3.8.2}$$

are bijections $(i=-1,0,1)$.

The sets $\mathbb{H}^i(K, Z^{(\mathrm{sc})} \to Z)$ and $\mathbb{H}^i(K, T^{(\mathrm{sc})} \to T)$ are canonically abelian groups. In the case $i=0$ we conclude that the group $H^0_{\mathrm{rel}}(K,G)$ is abelian. In the case $i=1$ we obtain a structure of abelian groups on the pointed set $H^1_{\mathrm{rel}}(K,G)$. This abelian group structure does not depend on the choice of T because it comes from $\mathbb{H}^1(K, Z^{(\mathrm{sc})} \to Z)$. With this group structure on $H^1_{\mathrm{rel}}(K,G)$ all the arrows in the commutative diagram (3.8.2) become isomorphisms of abelian groups.

We show that this abelian group structure on $H^1_{\mathrm{rel}}(K,G)$ is functorial. Let $\beta\colon G_1 \to G_2$ be a homomorphism. Let $T_1 \subset G_1$ be a maximal torus. Let $T_2 \subset G_2$ be a maximal torus such that $\beta(T_1) \subset T_2$. From the commutative diagram of crossed modules

$$\begin{array}{ccc} (T_1^{(\mathrm{sc})} \to T_1) & \hookrightarrow & (G_1^{\mathrm{sc}} \to G_1) \\ \downarrow & & \downarrow \\ (T_2^{(\mathrm{sc})} \to T_2) & \hookrightarrow & (G_2^{\mathrm{sc}} \to G_2) \end{array} \tag{3.8.3}$$

We get a commutative diagram

$$
\begin{array}{ccc}
H^1(K, T_1^{(\mathrm{sc})} \to T_1) & \xrightarrow{\sim} & H^1_{\mathrm{rel}}(K, G_1) \\
\downarrow & & \downarrow{\scriptstyle \beta_*} \\
H^1(K, T_2^{(\mathrm{sc})} \to T_2) & \xrightarrow{\sim} & H^1_{\mathrm{rel}}(K, G_2)
\end{array}
\tag{3.8.4}
$$

The horizontal arrows in the diagram (3.8.4) are isomorphisms of abelian groups. The left vertical arrow is a homomorphism. Hence the right vertical arrow is a homomorphism, which was to be proved.

3.9. In Section 2 we defined the abelian Galois cohomology groups $H^i_{\mathrm{ab}}(K, G)$ for $i \geq -1$. By definition $H^i_{\mathrm{ab}}(K, G) = \mathbb{H}^i(K, T^{(\mathrm{sc})} \to T)$ where $T \subset G$ is a maximal torus. The vertical arrow in the diagram (3.8.2) defines the canonical isomorphisms of abelian groups $H^i_{\mathrm{ab}}(K, G) \xrightarrow{\sim} H^i_{\mathrm{rel}}(K, G)$ for $i = -1, 0, 1$. The commutative diagtrams (3.8.3) and (3.8.4) show that these isomorhisms define isomorphisms of functors. We will henceforth identify $H^i_{\mathrm{ab}}(K, G)$ and $H^i_{\mathrm{rel}}(K, G)$ for $i = -1, 0, 1$.

3.10. The morphism of crossed modules $(1 \to G) \to (G^{\mathrm{sc}} \to G)$ induces *the abelianization maps*

$$\mathrm{ab}^i \colon H^i(K, G) \to \mathbb{H}^i(K, G^{\mathrm{sc}} \to G) = H^i_{\mathrm{ab}}(K, G) \quad (i = 0, 1).$$

The map

$$\mathrm{ab}^0 \colon G(K) = H^0(K, G) \to H^0_{\mathrm{ab}}(K, G)$$

is a homomorphism. The map

$$\mathrm{ab}^1 \colon H^1(K, G) \to H^1_{\mathrm{ab}}(K, G)$$

is a morphism of pointed sets. By Corollary 3.4.3 we have an exact sequence

$$
\begin{aligned}
1 \to (\ker \rho)(K) \to G^{\mathrm{sc}}(K) \xrightarrow{\rho} G(K) \xrightarrow{\mathrm{ab}^0} H^0_{\mathrm{ab}}(K, G) \\
\to H^1(K, G^{\mathrm{sc}}) \xrightarrow{\rho_*} H^1(K, G) \xrightarrow{\mathrm{ab}^1} H^1_{\mathrm{ab}}(K, G).
\end{aligned}
\tag{3.10.1}
$$

In the rest of this section we investigate the maps ab^0 and ab^1.

Proposition 3.11. *$ab^i \colon H^i(K, G) \to H^i_{\mathrm{ab}}(K, G)$ are morphisms of functors* ($i = 0, 1$).

Note that $H^0(K, G) = G(K)$ is a functor from (connected reductive) K-groups to groups; $H^1(K, G)$ is a functor to pointed sets; $H^i_{\mathrm{ab}}(K, G)$ are functors to abelian groups.

Proof: Let $\beta \colon G_1 \to G_2$ be a homomorphism. From the commutative diagram of crossed modules

$$
\begin{array}{ccc}
(1 \to G_1) & \longrightarrow & (G_1^{\mathrm{sc}} \to G_1) \\
{\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \beta_*} \\
(1 \to G_2) & \longrightarrow & (G_2^{\mathrm{sc}} \to G_2)
\end{array}
$$

we get a commutative diagram

$$\begin{array}{ccc} H^i(K,G_1) & \xrightarrow{\mathrm{ab}^i} & H^i_{\mathrm{ab}}(K,G_1) \\ {\scriptstyle\beta_*}\downarrow & & \downarrow{\scriptstyle\beta_*} \\ H^i(K,G_2) & \xrightarrow{\mathrm{ab}^i} & H^i_{\mathrm{ab}}(K,G_2). \end{array}$$

The proposition is proved.

We show that abelianization maps ab^i take cohomology exact sequences to abelian cohomology exact sequences.

Proposition 3.12. *Let*

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

be a short exact sequence of connected reductive K-groups. Then the diagram

(3.12.1)

$$\begin{array}{ccccccccccccc} 0 \longrightarrow & G^1(K) & \longrightarrow & G_2(K) & \longrightarrow & G_3(K) & \xrightarrow{\delta_0} & H^1(G_1) & \longrightarrow & H^1(G_2) & \longrightarrow & H^1(G_3) \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & H^0_{\mathrm{ab}}(G_1) & \longrightarrow & H^0_{\mathrm{ab}}(G_2) & \longrightarrow & H^0_{\mathrm{ab}}(G_3) & \xrightarrow{\delta_0} & H^1_{\mathrm{ab}}(G_1) & \longrightarrow & H^1_{\mathrm{ab}}(G_2) & \longrightarrow & H^1_{\mathrm{ab}}(G_3) \end{array}$$

with exact rows is commutative.

(ii) *Let*

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

be a short exact sequence of reductive K-groups where G_2 and G_3 are connected and G_1 is a group of multiplicative type (i.e. abelian). Then the diagram

$$\text{(3.12.2)}\quad \begin{array}{ccccccccccccccc} 0 \longrightarrow & G_1(K) & \longrightarrow & G_2(K) & \longrightarrow & G_3(K) & \xrightarrow{\delta_0} & H^1(G_1) & \longrightarrow & H^1(G_2) & \longrightarrow & H^1(G_3) & \longrightarrow & H^2(G_1) \\ & \downarrow\wr & & \downarrow & & \downarrow & & \downarrow\wr & & \downarrow & & \downarrow & & \downarrow\wr \\ & G_1(K) & \longrightarrow & H^0_{\mathrm{ab}}(G_2) & \longrightarrow & H^0_{\mathrm{ab}}(G_3) & \xrightarrow{\delta_0} & H^1(G_1) & \longrightarrow & H^1_{\mathrm{ab}}(G_2) & \longrightarrow & H^1_{\mathrm{ab}}(G_3) & \xrightarrow{\delta_1} & H_2(G_1) \end{array}$$

with exact rows is commutative.

Proof: (i) Consider the commutative diagram

(3.12.3)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (1 \to G_1) & \longrightarrow & (1 \to G_2) & \longrightarrow & (1 \to G_3) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & (G_1^{\mathrm{sc}} \to G_1) & \longrightarrow & (G_2^{\mathrm{sc}} \to G_2) & \longrightarrow & (G_3^{\mathrm{sc}} \to G_3) & \longrightarrow & 1 \end{array}$$

The morhism (3.12.3) of short exact sequences of crossed complexes yields the morphism (3.12.1) of hypercohomology exact sequences.

(ii) The morphism

$$\begin{array}{ccccccccc}
1 & \longrightarrow & (1\to G_1) & \longrightarrow & (1\to G_2) & \longrightarrow & (1\to G_3) & \longrightarrow & 1\\
 & & \big\downarrow\wr & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & (1\to G_1) & \longrightarrow & (G_2^{\mathrm{sc}}\to G_2) & \longrightarrow & (G_3^{\mathrm{sc}}\to G_3) & \longrightarrow & 1
\end{array}$$

of short exact sequences of crossed modules yields (3.12.2).

3.13. We describe the identifications $\lambda^i\colon \mathbb{H}^i(K, G^{\mathrm{sc}}\to G)\simeq \mathbb{H}^i(K, Z^{(\mathrm{sc})}\to Z)$ and the abelianization maps ab^i $(i=0,1)$ in terms of cocycles.

3.13.0. $i=0$. We describe λ^0. Let $\xi\in\mathbb{H}^0(G^{\mathrm{sc}}\to G)$, $\xi=\mathrm{Cl}(\varphi,g)$, where $\varphi\colon\Gamma\to G^{\mathrm{sc}}(\bar K)$ is a continuous map and $g\in G(\bar K)$. Since $G=\rho(G^{\mathrm{sc}})\cdot Z$, we may write $g=\rho(f)\cdot g'$, where $f\in G^{\mathrm{sc}}(\bar K)$,$g'\in Z(K')$. Acting on the cocycle (φ,g) by $f\in G^{\mathrm{sc}}(\bar K)$, we get

$$(\varphi,g)*f=(\varphi',g')$$

where

$$\varphi'(\sigma)=f^{-1}\cdot\varphi(g)^{\sigma}f \tag{3.13.0.1}$$

(cf. 3.1.1). Since $(\varphi',g')\in Z^0(G^{\mathrm{sc}}\to G)$, we have ${}^{\sigma}g'=\rho(\varphi'(\sigma))^{-1}\cdot g'$, hence $\rho(\varphi'(\sigma))\in Z(\bar K)$ and $\varphi'(\sigma)\in Z^{(\mathrm{sc})}(\bar K)$. Thus $(\varphi',g')\in Z^0(Z^{(\mathrm{sc})}\to Z)$ and $\lambda^0(\xi)=\mathrm{Cl}(\varphi',g')$.

We describe ab^0. Let $g\in G(K)$. There exist $g'\in Z(\bar K)$ and $f\in G^{\mathrm{sc}}(\bar K)$ such that $g=\rho(f)\cdot g'$. Then $\mathrm{ab}^0(g)=\mathrm{Cl}(\varphi',g')\in\mathbb{H}^0(Z^{(\mathrm{sc})}\to Z)$ where

$$\varphi'(\sigma)=f^{-1}\cdot{}^{\sigma}f \tag{3.13.0.2}$$

3.13.1. $i=1$. We describe λ^1. Let $\eta\in\mathbb{H}^1(G^{\mathrm{sc}}\to G)$, $\eta=\mathrm{Cl}(u,\psi)$, where $(u,\psi)\in Z^1(G^{\mathrm{sc}}\to G)$, the maps $u\colon\Gamma\times\Gamma\to G^{\mathrm{sc}}(\bar K)$ and $\psi\colon\Gamma\to G(\bar K)$ are continuous. Since $G=\rho(G^{\mathrm{sc}})\cdot Z$, there exist continuous maps $s\colon\Gamma\to G^{\mathrm{sc}}(\bar K)$ and $\psi'\colon\Gamma\to Z(\bar K)$ such that $\psi(\sigma)=\rho(s(\sigma))\cdot\psi'(\sigma)$ for $\sigma\in\Gamma$. Acting on the cocycle (u,ψ) by $(s^{-1},1)\in C^0(G^{\mathrm{sc}}\to G)$, we get

$$(u,\psi)*(s^{-1},1)=(u',\psi')$$

where

$$u'(\ \sigma,\tau)=s(\sigma\tau)^{-1}\cdot u(\sigma,\tau)\cdot{}^{\psi(\sigma)\sigma}s(\tau)\cdot s(\sigma)$$

(cf. 3.2.2). Since $(u',\psi)\in Z^1(G^{\mathrm{sc}}\to G)$,

$$\psi'(\sigma,\tau)=\rho(u'(\sigma,\tau))\cdot\psi'(\sigma)\cdot{}^{\sigma}\psi'(\tau),$$

hence $\rho(u'(\sigma,\tau))\in Z(\bar K)$ and $u'(\sigma,\tau)\in Z^{(\mathrm{sc})}(\bar K)$. Therefore

(3.13.1.1)

$$\begin{aligned}u'(\sigma,\tau)&=s(\sigma\tau)\cdot u'(\sigma,\tau)\cdot s(\sigma\tau)^{-1}=u(\sigma,\tau)\cdot{}^{\rho(s(\sigma))\psi'(\sigma)\sigma}s(\tau)\cdot s(\sigma)\cdot s(\sigma\tau)^{-1}.\\&=u(\sigma,\tau)\cdot s(\sigma)\cdot{}^{\sigma}s(\tau)\cdot s(\sigma\tau)^{-1}\end{aligned}$$

We see that $(u', \psi') \in Z^1(Z^{(\mathrm{sc})} \to Z)$, hence $\lambda^1(\eta) = \mathrm{Cl}(u', \psi') \in \mathbb{H}^1(Z^{(\mathrm{sc})} \to Z)$, where u' is defined by (3.13.1.1).

We describe ab^1. Let $\eta \in H^1(K, G)$, $\eta = \mathrm{Cl}(\psi)$, $\psi \in Z^1(K, G)$. We write $\psi(\sigma) = \rho(s(\sigma)) \cdot \psi'(\sigma)$ where $s\colon \Gamma \to G^{\mathrm{sc}}(\bar{K})$ and $\psi'\colon \Gamma \to Z(\bar{K})$ are continuous maps. Then $\mathrm{ab}^1(\eta) = \mathrm{Cl}(u', \psi') \in \mathbb{H}^1(Z^{(\mathrm{sc})} \to Z)$ where

$$u'(\sigma, \tau) = s(\sigma) \cdot {}^{\sigma}s(\tau) \cdot s(\sigma\tau)^{-1}. \tag{3.13.1.2}$$

3.14. Examples (In these examples $i = 0, 1$).

(1) G is a torus. Then $G^{\mathrm{sc}} = 1$, $H^i_{\mathrm{ab}}(K, G) = H^i(K, G)$, and ab^i is the identity map.

(2) Suppose that G^{ss} is simply connected, hence ρ is injective. Then $H^i_{\mathrm{ab}}(K, G) = H^i(K, G^{\mathrm{tor}})$. The abelianization map ab^i is the map $t_*\colon H^i(K, G) \to H^i(K, G^{\mathrm{tor}})$ induced by the canonical homomorphism $t\colon G \to G/G^{\mathrm{ss}} = G^{\mathrm{tor}}$.

(3) Suppose that G is semisimple, hence ρ is surjective and $H^i_{\mathrm{ab}}(K, G) = H^{i+1}(K, \ker \rho)$. Then the map ab^i is the connecting map $\delta_i\colon H^i(K, G) \to H^{i+1}(K, \ker \rho)$ associated with the short exact sequence

$$1 \to \ker \rho \to G^{\mathrm{sc}} \to G \to 1$$

This assertion follows immediately from the explicit formulae (3.3.0.2) and (3.3.1.2) (see [Se], Ch. I, §5 for the definitions of δ_0 and δ_1).

(4) In the general case from the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G^{\mathrm{ss}} & \longrightarrow & G & \longrightarrow & G^{\mathrm{tor}} & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & (G^{\mathrm{sc}} \to G^{\mathrm{ss}}) & \longrightarrow & (G^{\mathrm{sc}} \to G) & \longrightarrow & (1 \to G^{\mathrm{tor}}) & \longrightarrow & 1
\end{array}$$

we get a commutative diagram

$$\begin{array}{ccccccccccc}
G^{\mathrm{ss}}(K) & \longrightarrow & G(K) & \longrightarrow & G^{\mathrm{tor}}(K) & \longrightarrow & H^1(G^{\mathrm{ss}}) & \longrightarrow & H^1(G) & \longrightarrow & H^1(G^{\mathrm{tor}}) \\
\big\downarrow{\scriptstyle \delta_0} & & \big\downarrow{\scriptstyle \mathrm{ab}^0} & & \Big\| & & \big\downarrow{\scriptstyle \delta_1} & & \big\downarrow{\scriptstyle \mathrm{ab}^1} & & \Big\| \\
H^1(\ker \rho) & \longrightarrow & H^0_{\mathrm{ab}}(G) & \longrightarrow & G^{\mathrm{tor}}(K) & \longrightarrow & H^2(\ker \rho) & \longrightarrow & H^1_{\mathrm{ab}}(G) & \longrightarrow & H^1(G^{\mathrm{tor}})
\end{array}$$

3.15. We consider twisting. For $\psi \in Z^1(K, G)$ consider the twisted K-group ${}_{\psi}G$ and the groups

$$H^1_{\mathrm{ab}}(K, {}_{\psi}G) = \mathbb{H}^i(K, {}_{\psi}(G^{\mathrm{sc}} \to G)) = \mathbb{H}^i(K, {}_{\psi}(Z^{(\mathrm{sc})} \to Z)$$

Since ${}_{\psi}(Z^{(\mathrm{sc})} \to Z) = (Z^{(\mathrm{sc})} \to Z)$, we see that $H^i_{\mathrm{ab}}(K, {}_{\psi}G) = H^i_{\mathrm{ab}}(K, G)$. This identification coincides with that of Corollary 2.9.

There is a canonical map

$$t_{\psi}\colon H^1(K, {}_{\psi}G) \to H^1(K, G)$$

defined by

$$t_{\psi}(\mathrm{Cl}(\psi')) = \mathrm{Cl}(\psi' \cdot \psi) \;\; \psi' \in Z^1(K, {}_{\psi}G)$$

(cf. [Se], Ch. I, 5.3, Prop. 3.5 bis). There is also a canonical map

$$t_{\psi}\colon \mathbb{H}^1(K, {}_{\psi}(G^{\mathrm{sc}} \to G)) \to \mathbb{H}^1(K, G^{\mathrm{sc}} \to G)$$

defined by

$$t_{\psi}(\mathrm{Cl}(u', \psi')) = \mathrm{Cl}(u', \psi' \cdot \psi),$$

where $(u', \psi') \in Z^1(K, {}_{\psi}(G^{\mathrm{sc}} \to G))$ (cf. [Brv5], 2.14).

Lemma 3.15.1. *Let $\psi \in Z^1(K,G)$. The diagram*

$$\begin{array}{ccc} \mathbb{H}^1(K, Z^{(\mathrm{sc})} \to Z) & \xrightarrow{x\mapsto x+a(\psi)} & \mathbb{H}^1(K, Z^{(\mathrm{sc})} \to Z) \\ \sim\downarrow & & \downarrow\sim \\ \mathbb{H}^1(K, {}_\psi(G^{\mathrm{sc}} \to G)) & \xrightarrow{t_\psi} & \mathbb{H}^1(K, G^{\mathrm{sc}} \to G) \end{array}$$

commutes, where $a(\psi) = \mathrm{ab}^1(\mathrm{Cl}(\psi)) \in \mathbb{H}^1(K, Z^{(\mathrm{sc})} \to Z)$.

Proof: Let $x = \mathrm{Cl}(u_0, \psi_0)$, where $(u_0, \psi_0) \in Z^1(Z^{(\mathrm{sc})} \to Z)$. The image (say y) of x in $\mathbb{H}^1(K, {}_\psi(G^{\mathrm{sc}} \to G))$ is again $\mathrm{Cl}(u_0, \psi_0)$. Then $t_\psi(y) = \mathrm{Cl}(u_0, \psi_0\psi) = \mathrm{Cl}(u_0, \psi\psi_0)$ (because $\psi_0(\sigma) \in Z(\bar{K})$).

We write $\psi = \rho(s) \cdot \psi_*$ as in 3.13.1. Then

$$(u_0, \psi\psi_0) * (s^{-1}, 1) = (u_1, \psi_*\psi_0)$$

where

$$u_1(\sigma,\tau) = s(\sigma) \cdot {}^\sigma s(\tau) \cdot s(\sigma\tau)^{-1} \cdot u_0(\sigma,\tau).$$

Set $u_*(\sigma,\tau) = s(\sigma) \cdot {}^\sigma s(\tau) \cdot s(\sigma\tau)^{-1} \in Z^{\mathrm{sc}}(\bar{K})$; then $u_1 = u_* u_0$. We have

$$t_\psi(y) = \mathrm{Cl}(u_1, \psi_*\psi_0) = \mathrm{Cl}(u_* u_0, \psi_*\psi_0) = \mathrm{Cl}(u_*, \psi_*) + x,$$

where $\mathrm{Cl}(u_*, \psi_*) \in \mathbb{H}^1(K, Z^{(\mathrm{sc})} \to Z)$. But by 3.13.1 $\mathrm{Cl}(u_*, \psi_*) = \mathrm{ab}^1(\mathrm{Cl}(\psi)) = a(\psi)$. This proves the lemma.

Now consider the maps

$$\mathrm{ab}^i \colon H^i(K,G) \to H^i_{\mathrm{ab}}(K,G) \quad (i = 0, 1).$$

By definition the map ab^0 is a homomorphism. We show that the map ab^1 has the following multiplicativity property:

Proposition 3.16. *Let $\psi \in Z^1(K,G)$. Then the diagram*

$$\begin{array}{ccc} H^1(K, {}_\psi G) & \xrightarrow{t_\psi} & H^1(K,G) \\ \mathrm{ab}^1\downarrow & & \downarrow\mathrm{ab}^1 \\ H^1_{\mathrm{ab}}(K, {}_\psi G) = H^1_{\mathrm{ab}}(K,G) & \xrightarrow{z\mapsto x+a(\psi)} & H^1_{\mathrm{ab}}(K,G) \end{array}$$

commutes, where $a(\psi) = \mathrm{ab}^1_G(\mathrm{Cl}(\psi))$.

Proof: For any morphism $\beta \colon (F \to G) \to (F' \to G')$ of crossed modules of K-groups and a cocycle $\psi \in Z^1(K,G)$ we have a commutative diagram

$$\begin{array}{ccc} \mathbb{H}^1({}_\psi(F \to G)) & \xrightarrow{t_\psi} & \mathbb{H}^1(F \to G) \\ {}_\psi\beta\downarrow & & \downarrow\beta_* \\ \mathbb{H}^1({}_{\psi'}(F' \to G')) & \xrightarrow{t_{\psi'}} & \mathbb{H}^1(F' \to G') \end{array}$$

where $\psi' = \beta(\psi) \in Z^1(K, G)')$. In particular, for the morphism $(1 \to G) \to (G^{\mathrm{sc}} \to G)$ we obtain the commutative diagram

$$\begin{array}{ccc}
H^1(K, {}_\psi G) & \xrightarrow{t_\psi} & H^1(K, G) \\
\downarrow & & \downarrow \\
H^1_{\mathrm{ab}}(K, {}_\psi G) & \xrightarrow{t_\psi} & H^1_{\mathrm{ab}}(K, G)
\end{array}$$

By Lemma 3.15.1 the lower horizontal arrow in this diagram is $x \mapsto x + a(\psi)$. The proposition is proved.

We can now compute the fibers of the map ab^1.

Corollary 3.17. *Let $\xi \in H^1(K, G)$, $\xi = \mathrm{Cl}(\psi)$, $\psi \in Z^1(K, G)$. Let ${}_\psi\rho\colon {}_\psi G^{\mathrm{sc}} \to {}_\psi G$ denote the twist of ρ. Then*

(i) $\ker \mathrm{ab}^1 = \rho_* H^1(G^{\mathrm{sc}}) \simeq H^0_{\mathrm{ab}}(K, G)\backslash H^1(K, G^{\mathrm{sc}})$
(ii) $(\mathrm{ab}^1)^{-1}(\mathrm{ab}^1(\xi)) = t_\psi(({}_\psi\rho_*)H^1(K, {}_\psi G^{\mathrm{sc}})) \simeq H^0_{\mathrm{ab}}(K, G)\backslash H^1(K, {}_\psi G^{\mathrm{sc}})$

Proof: (i) follows from (3.10.1) and (3.4.3.2); (ii) follows from (i) and Proposition 2.16; we take in account that $H^0_{\mathrm{ab}}(K, {}_\psi G) = H^0_{\mathrm{ab}}(K, G)$.

4. Computation of abelian Galois cohomology

In Section 3 we have defined the abelianization map $\mathrm{ab}^1\colon H^1(K,G)\to H^1_{\mathrm{ab}}(K,G)$. By Proposition 2.8 $H^1_{\mathrm{ab}}(K,G)=\mathcal{H}^1(K,M,\bar{K}^\times)$. In this section we try to calculate $\mathcal{H}^1(K,M,\bar{K}^\times)$ for $i\geq 1$. We compute $\mathcal{H}^1(K,M,\bar{K}^\times)$ for local fields. For a number field K we compute $\mathcal{H}^1(K,M,\bar{K}^\times)$ for $i\geq 2$. For $i=1$ we compute the kernel and the cokernel of the localization map $\mathcal{H}^1(K,M,\bar{K}^\times)\to\oplus\mathcal{H}^1(K_v,M,\bar{K}^\times{}_v)$.

All this stuff is a kind of Tate-Nakayama theory. The results in the case $i=1$ are essentially due to Kottwitz.

In this section K is a local or global field of charactristic 0, $\Gamma=\mathrm{Gal}(\bar{K}/K)$, M is a finitely generated Γ-module.

Proposition 4.1. *Let K be a non-archimedian local field. There are canonical isomorhisms:*

(i) $\lambda^1_K\colon\mathcal{H}^1(K,M,\bar{K}^\times)\xrightarrow{\sim}(M_\Gamma)_{\mathrm{tors}}$

(ii) $\lambda^2_K\colon\mathcal{H}^2(K,M,\bar{K}^\times)\xrightarrow{\sim}(M_\Gamma)_{\mathrm{tf}}\underset{\mathbf{Z}}{\otimes}\mathbf{Q}/\mathbf{Z}$

(iii) $\mathcal{H}^i(K,M,\bar{K}^\times)=0$ *for* $i\geq 3$.

Recall that $(M_\Gamma)_{\mathrm{tf}}=M_\Gamma/(M_\Gamma)_{\mathrm{tors}}$.

Proof:

4.1.1. We prove (iii). Let $L^\cdot\to M$ be a short torsion free resolution, where $L^\cdot=(L^{-1}\to L^0)$. In the exact sequence (2.6.5.1)

$$\cdots\to H^i(K,L^0\otimes\bar{K}^\times)\to\mathcal{H}^i(K,M,\bar{K}^\times)\to H^{i+1}(K,L^{-1}\otimes\bar{K}^\times)\to\cdots$$

we have $H^i(K,L^0\otimes\bar{K}^\times)=0$, $H^{i+1}(K,L^{-1}\otimes\bar{K}^\times)=0$ for $i\geq 3$ (cf. [Mi], Ch. 1, 1.11). Hence $\mathcal{H}^i(K,M,\bar{K}^\times)=0$, which proves (iii).

4.1.2. We begin proving (i) and (ii). Let $L^\cdot\to M$ be a short torsion free resolution. We consider the dual complex

$$L^{\cdot\vee}=\mathrm{Hom}(L^\cdot,\mathbf{Z})=(L^{0\vee}\to L^{-1\vee})$$

(recall that $^\vee$ denotes $\mathrm{Hom}(\,\cdot,\mathbf{Z})$). Here $L^{0\vee}$ is in degree 0 and $L^{-1}\vee$ is in degree +1.

We have by definition

$$\mathcal{H}^i(K,M,\bar{K}^\times)=\mathbb{H}^i(K,L^\cdot\otimes\bar{K}^\times).$$

The cup product pairing

$$\mathbb{H}^i(K,L^\cdot\otimes\bar{K}^\times)\otimes\mathbb{H}^{2-i}(K,L^{\cdot\vee})\to H^2(K,\bar{K}^\times)=\mathrm{Br}(K)$$

defines canonical isomorphisms

$$\mathcal{H}^i(K,M,\bar{K}^\times)=\mathbb{H}^{2-i}(K,L^{\cdot\vee})^B,\tag{4.1.2.1}$$

where ^{B} denotes $\mathrm{Hom}(\,\cdot,\mathrm{Br}(K))$.

Lemma 4.1.3. *Homomorphisms* (4.1.2.1) *are isomorphisms for* $i \geq 1$.

Proof: If M is torsion free then this is the Tate-Nakayama duality theorem. In the general case we can write down the exact sequence (2.6.5.1) and the corresponding commutative diagram. Applying the five-lemma we obtain the desired result.

4.1.4. We compute $\mathbb{H}^0(K, L^{\cdot\vee})^B$. By definition

$$\mathbb{H}^0(K, L^{\cdot\vee})^B = \ker[(L^{0\vee})^\Gamma \to (L^{-1\vee})^\Gamma]^B = \operatorname{coker}[(L^{-1\vee})^{\Gamma B} \to (L^{0\vee})^{\Gamma B}]$$

We have

$$(L^{0\vee})^\Gamma = \operatorname{Hom}_\Gamma(L^0, \mathbf{Z}) = \operatorname{Hom}(L^0_\Gamma, \mathbf{Z}) = \operatorname{Hom}((L^0_\Gamma)_{\mathrm{tf}}, \mathbf{Z}) = (L^0_\Gamma)^\vee_{\mathrm{tf}}$$

Hence $(L^{0\vee})^{\Gamma B} = (L^0_\Gamma)_{\mathrm{tf}} \underset{\mathbf{Z}}{\otimes} \operatorname{Br}(K) = L^0_\Gamma \underset{\mathbf{Z}}{\otimes} \operatorname{Br}(K)$.

Similarly

$$(L^{-1\vee})^{\Gamma B} = L^{-1}_\Gamma \underset{\mathbf{Z}}{\otimes} \operatorname{Br}(K)$$

Further

$$\begin{aligned}\operatorname{coker}[(L^{-1\vee})^{\Gamma B} \to (L^{0\vee})^{\Gamma B}] &= \operatorname{coker}[L^{-1}_\Gamma \underset{\mathbf{Z}}{\otimes} \operatorname{Br}(K) \to L^0_\Gamma \otimes \operatorname{Br}(K)] \\ = \operatorname{coker}[L^{-1}_\Gamma \to L^0_\Gamma] \underset{\mathbf{Z}}{\otimes} \operatorname{Br}(K) &= M_\Gamma \underset{\mathbf{Z}}{\otimes} \operatorname{Br}(K) = (M_\Gamma)_{\mathrm{tf}} \underset{\mathbf{Z}}{\otimes} \operatorname{Br}(K)\end{aligned}$$

There is a canonical isomorphism $\operatorname{Br}(K) \xrightarrow{\sim} \mathbf{Q}/\mathbf{Z}$. Now 4.1 (ii) follows from Lemma 4.1.3.

4.1.5. We compute $\mathbb{H}^1(K, L^{\cdot\vee})^B$. Following an idea of Kottwitz [Ko2], we consider the short exact sequence

$$0 \to L^{\cdot\vee} \to L^{\cdot\vee} \underset{\mathbf{Z}}{\otimes} \mathbf{Q} \to L^{\cdot\vee} \underset{\mathbf{Z}}{\otimes} \mathbf{Q}/\mathbf{Z} \to 0$$

which gives rise to the hypercohomology exact sequence

$$\mathbb{H}^0(K, L^{\cdot\vee} \otimes \mathbf{Q}) \to \mathbb{H}^0(K, L^{\cdot\vee} \otimes \mathbf{Q}/\mathbf{Z}) \to \mathbb{H}^1(K, L^{\cdot\vee}) \to 0$$

(because $L^{\cdot\vee} \otimes \mathbf{Q}$ is a complex of injective Γ-modules).

We observe that

$$L^{\cdot\vee} \otimes \mathbf{Q} = \operatorname{Hom}(L^\cdot, \mathbf{Q}),\ L^{\cdot\vee} \otimes (\mathbf{Q}/\mathbf{Z}) = \operatorname{Hom}(L^\cdot, \mathbf{Q}/\mathbf{Z}).$$

Since $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$ are $\mathbf{Z}$-injective, the sequences

$$0 \to \operatorname{Hom}(M, \mathbf{Q}) \to \operatorname{Hom}(L^0, \mathbf{Q}) \to \operatorname{Hom}(L^{-1}, \mathbf{Q}) \to 0$$
$$0 \to \operatorname{Hom}(M, \mathbf{Q}/\mathbf{Z}) \to \operatorname{Hom}(L^0, \mathbf{Q}/\mathbf{Z}) \to \operatorname{Hom}(L^{-1}, \mathbf{Q}/\mathbf{Z}) \to 0$$

are exact. Thus

$$\begin{aligned}\mathbb{H}^0(K, L^{\cdot\vee} \otimes \mathbf{Q}) = \mathbb{H}^0(K, \operatorname{Hom}(L^\cdot, \mathbf{Q})) &= H^0(K, \operatorname{Hom}(M, \mathbf{Q})) = \operatorname{Hom}_\Gamma(M, \mathbf{Q}) \\ &= \operatorname{Hom}(M_\Gamma, \mathbf{Q}) = \operatorname{Hom}((M_\Gamma)_{\mathrm{tf}}, \mathbf{Q})\end{aligned}$$

and similarly

$$\mathbb{H}^0(K, L^{\cdot\vee} \underset{\mathbf{Z}}{\otimes} \mathbf{Q}/\mathbf{Z}) = \mathrm{Hom}_\Gamma(M, \mathbf{Q}/\mathbf{Z}) = \mathrm{Hom}(M_\Gamma, \mathbf{Q}/\mathbf{Z})$$

We see that

$$\begin{aligned}\mathbb{H}^1(K, L^{\cdot\vee}) =& \mathrm{coker}\,[\mathrm{Hom}(M_\Gamma)_{\mathrm{tf}}, \mathbf{Q}) \to \mathrm{Hom}(M_\Gamma, \mathbf{Q}/\mathbf{Z})] \\ =& \mathrm{coker}\,[\mathrm{Hom}((M_\Gamma)_{\mathrm{tf}}, \mathbf{Q}/\mathbf{Z}) \to \mathrm{Hom}(M_\Gamma, \mathbf{Q}/\mathbf{Z})] \\ =& \mathrm{Hom}(\ker[M_\Gamma \to (M_\Gamma)_{\mathrm{tf}}], \mathbf{Q}/\mathbf{Z}) = \mathrm{Hom}((M_\Gamma)_{\mathrm{tors}}, \mathbf{Q}/\mathbf{Z})\end{aligned}$$

Using the canonical isomorphism $\mathrm{Br}(K) \xrightarrow{\sim} \mathbf{Q}/\mathbf{Z}$, we conclude that

$$\mathbb{H}^1(K, L^{\cdot\vee})^B = \mathrm{Hom}((\mathrm{Hom}(M_\Gamma)_{\mathrm{tors}}, \mathbf{Q}/\mathbf{Z}), \mathrm{Br}(K)) \simeq (M_\Gamma)_{\mathrm{tors}}.$$

Now 4.1 (i) follows from Lemma 4.1.3.

Proposition 4.1 is proved.

The exposition in the remaining part of this section is somewhat sketchy.

Proposition 4.2. *For $K = \mathbf{R}$ there are canonical isomorphisms*

$$\lambda^i_{\mathbf{R}} \colon \mathcal{H}^i(\mathbf{R}, M, \mathbf{C}^\times) \xrightarrow{\sim} \widehat{H}^{i-2}(\mathbf{R}, M) \text{ for } i \geq 1.$$

In particular

$$\mathcal{H}^i(\mathbf{R}, M, \mathbf{C}^\times) \simeq \begin{cases} H^1(\mathbf{R}, M) & \text{if } i \text{ is odd} \\ \widehat{H}^0(\mathbf{R}, M) & \text{if } i \text{ is even} \quad (i > 0). \end{cases}$$

Proof: Similar to that of Proposition 4.1.

4.3. Now let K be a number field. Set $\bar{\mathbf{A}} = \mathbf{A} \underset{K}{\otimes} \bar{K}$, where $\mathbf{A}$ is the adèle ring of K. We set $\bar{C} = \bar{\mathbf{A}}^\times / \bar{K}^\times$.

Let M be a finitely generated Γ-module. Let $L^\cdot \to M$ be a short torsion free resolution. We consider the short exact sequences

$$1 \to \bar{K}^\times \to \bar{\mathbf{A}}^\times \to \bar{C} \to 1$$
$$0 \to L^\cdot \otimes \bar{K}^\times \to L^\cdot \otimes \bar{\mathbf{A}}^\times \to L^\cdot \otimes \bar{C} \to 0$$

and the corresponding long exact sequence

$$(4.3.1) \qquad \cdots \to \mathcal{H}^i(K, M, \bar{K}^\times) \to \mathcal{H}^i(K, M, \bar{\mathbf{A}}^\times) \to \mathcal{H}^i(K, M, \bar{C}) \to \cdots$$

We would like to compute this exact sequence.

Proposition 4.4. *There are canonical isomorphisms*

(i) $\lambda^1 \colon \mathcal{H}^1(K, M, \bar{C}) \xrightarrow{\sim} (M_\Gamma)_{\mathrm{tors}}$
(ii) $\lambda^2 \colon \mathcal{H}^2(K, M, \bar{C}) \xrightarrow{\sim} (M_\Gamma)_{\mathrm{tf}} \otimes \mathbf{Q}/\mathbf{Z}$
(iii) $\mathcal{H}^i(K, M, \bar{C}) = 0$ *for* $i \geq 3$.

Proof: The same as that of Proposition 4.1.

Lemma 4.5. *There is a canonical isomorphism*

$$\mathrm{loc}\colon \mathcal{H}^i(K, M, \bar{\mathbf{A}}^\times) \simeq \oplus \mathcal{H}^i(K_v, M, \bar{K_v}^\times) \quad \textit{for} \quad i \geq 1.$$

Proof: The embedding $\oplus(K_v \underset{K}{\otimes} \bar{K}) \hookrightarrow \bar{\mathbf{A}}^\times$ induces the homomorphism

$$\oplus \mathcal{H}^i(K_v \otimes \bar{K})^\times) \to \mathcal{H}^i(K, M, \bar{\mathbf{A}}).$$

By Shapiro's lemma

$$\mathcal{H}^i(K, M, (K_v \otimes \bar{K})^\times) = \mathcal{H}^i(K_v, M, \bar{K}_v^\times).$$

Thus we obtain a homomorphism

$$\oplus \mathcal{H}^i(K, M, \bar{K}_v^\times) \to \mathcal{H}^i(K, M, \bar{\mathbf{A}}^\times).$$

We must prove that it is an isomoprhism. Using the exact sequences (2.6.5.1) we reduce the assertion to the well known (cf. [Vo2], 6.25) case of a torsion free module M. The lemma is proved.

Corollary 4.6. *For any $h \in \mathcal{H}^i(K, M, \bar{K}^\times)$ $(i \geq 0)$ there exists a finite set $S \subset \mathcal{V}(K)$ such that $\mathrm{loc}_v(h) \in \mathcal{H}^i(K_v, M, \bar{K}_v^\times)$ is zero for $v \notin S$.*

Proof: It follows from the proof of Lemma 4.5 that for any $\xi \in \mathcal{H}^i(K, M, \bar{\mathbf{A}}^\times)$ there exists a finite set $S \subset \mathcal{V}$ such that ξ comes from $\mathcal{H}^1(K, M, \underset{S}{\oplus}(K_v \underset{K}{\otimes} \bar{K})^\times)$. This implies the corollary.

4.7. We want to describe the map

$$\mathcal{H}^i(K_v, M, \bar{K}_v^\times) = \mathcal{H}^i(K, M, (\bar{K} \otimes K_v)^\times) \to \mathcal{H}^i(K, M, \bar{\mathbf{A}}) \to \mathcal{H}^i(K, M, \bar{C})$$

for $i = 1, 2$.

Set

$$\begin{aligned} & \mathcal{T}^{-1}(M) = (M_\Gamma)_{\mathrm{tors}},\ \mathcal{T}^0(M) = M_\Gamma \otimes \mathbf{Q}/\mathbf{Z}; \\ \text{if } v \in \mathcal{V}_f\colon \quad & \mathcal{T}_v^{-1}(M) = (M_{\Gamma_v})_{\mathrm{tors}},\ \mathcal{T}_v^0(M) = M_{\Gamma_v} \otimes \mathbf{Q}/\mathbf{Z}; \\ \text{if } v \in \mathcal{V}_\infty\colon \quad & \mathcal{T}^{-1}(M) = H^{-1}(\Gamma_v, M),\ \mathcal{T}_v^0(M) = \widehat{H}^0(\Gamma_v, M) \end{aligned}$$

We have canonical corestriction maps $\mathrm{cor}_v^j\colon \mathcal{T}_v^j(M) \to \mathcal{T}^j(M)$, which for $v \in \mathcal{V}_f$ are defined in the obvious way and for $v \in \mathcal{V}_\infty$ are defined as follows:

$$\begin{aligned} \mathrm{cor}_v^{-1}\colon \mathcal{T}_v^{-1}(M) = H^{-1}(\Gamma_v, M) \hookrightarrow (M_{\Gamma_v})_{\mathrm{tors}} \to (M_\Gamma)_{\mathrm{tors}} = \mathcal{T}^{-1}(M) \\ \mathrm{cor}_v^0\colon \mathcal{T}_v^0(M) = \widehat{H}^0(\Gamma_v, M) \to M_{\Gamma_v} \otimes \left(\frac{1}{2}\mathbf{Z}/\mathbf{Z}\right) \to M_\Gamma \otimes \mathbf{Q}/\mathbf{Z} = \mathcal{T}^0(M) \end{aligned}$$

Proposition 4.8. *The diagram*

$$\begin{array}{ccc} \mathcal{H}^i(K_v, M, \bar{K}^\times{}_v) & \longrightarrow & \mathcal{H}^i(K, M, \bar{C}) \\ \lambda_v^i \Big\downarrow \wr & & \wr \Big\downarrow \lambda_i \\ \mathcal{T}_v^{i-2}(M) & \xrightarrow{\mathrm{cor}_v^{i-2}} & \mathcal{T}^{i-2}(M) \end{array} \tag{4.8.1}$$

commutes ($i = 1, 2$), *where* λ^i *and* λ^i_v *are the isomorphisms of Propositions* 4.1, 4.2 *and* 4.4.

Proof: We consider the case $i = 1$; the case $i = 2$ can be treated similarly. Let $v \in \mathcal{V}_f$. Consider the map

$$\operatorname{Br} K_v = H^2(K_v, \bar{K}^\times{}_v) = H^2(K, \bar{K} \otimes K_v) \to H^2(K, (\bar{K} \otimes \mathbf{A})^\times) = H^2(K, \bar{\mathbf{A}}^\times) \to H^2(K, \bar{C})$$

This map is known to be an isomorphism compatible with the isomorphisms

$$\begin{aligned} \operatorname{inv}_v &: \operatorname{Br} K_v \xrightarrow{\sim} \mathbf{Q}/\mathbf{Z} \\ \operatorname{inv} &: H^2(K, \bar{C}) \xrightarrow{\sim} \mathbf{Q}/\mathbf{Z} \end{aligned}$$

We have

$$\mathcal{H}^1(K_v, M, \bar{K}_v{}^\times) = \mathcal{T}_v^{-1}(M) \underset{\mathbf{Q}/\mathbf{Z}}{\otimes} \operatorname{Br} K_v$$

(see the proof of Proposition 4.1), ,and, similarly,

$$\mathcal{H}^1(K, M, \bar{C}) = \mathcal{T}^{-1}(M) \otimes H^2(K, \bar{C}),$$

hence the diagram (4.8.1) commutes for $i = 1$, $v \in \mathcal{V}_f$.

The case $v \in \mathcal{V}_\infty$ can be treated similarly.

Proposition 4.9. *Consider the localization homomorphism*

$$\operatorname{loc}^i_\infty \colon \mathcal{H}^i(K, M, \bar{K}^\times) \to \prod_\infty \mathcal{H}^i(K_v, M, \bar{K}^\times{}_v). \tag{4.9.0}$$

Then

(i) $\operatorname{loc}^0_\infty$ *has dense image;*

(ii) $\operatorname{loc}^i_\infty$ *is an epimorphism for* $i = 1, 2$;

(iii) $\operatorname{loc}^i_\infty$ *is an isomoprhism for* $i \geq 3$.

Proof:

4.9.1. We prove (iii). The assertion follows from the exact sequence (4.3.1) and Propositions 4.1(iii) and 4.4(iii).

4.9.2. We prove (ii). The isomorphisms λ^i_v of Propositions 4.2 and 4.4 define an isomorphism $\mathcal{H}^i(K, M, \bar{\mathbf{A}}^\times) = \bigoplus_{v \in \mathcal{V}} \mathcal{H}_i(K_v, M, \bar{K}^\times) \xrightarrow{\sim} \bigoplus_{v \in \mathcal{V}} \mathcal{T}^i_v(M)$, and we see from the exact sequence (4.3.1) and Proposition 4.8 that the image of $\mathcal{H}^i(K, M, \bar{K}^\times)$ in $\mathcal{H}^i(K, M, \bar{\mathbf{A}}^\times)$ is isomorphic to

$$\ker \sum_{v \in \mathcal{V}} \operatorname{cor}^i_v \colon \oplus \mathcal{T}^i_v(M) \to \mathcal{T}(M).$$

Let Δ be the image of Γ in $\operatorname{Aut} M$. Let K' be the corresponding Galois extension (so that $\operatorname{Gal}(K'/K) = \Delta$), and for $v \in \mathcal{V}$ let Δ_v be a decomposition group of v (defined up to conjugation).

Let $v \in \mathcal{V}$. If $v' \in \mathcal{V}_f$ and $\Delta_v = \Delta_{v'}$ (up to conjugation), then $\operatorname{im} \operatorname{cor}_v^i \subset \operatorname{im} \operatorname{cor}_{v'}^i$. (These images are equal if both v, $v \in \mathcal{V}_f$.) Let $v \in \mathcal{V}_\infty$; then Δ_v is cyclic. By Chebotorev's density theorem there are infinitely many places $v' \in \mathcal{V}_f$ such that $\Delta_{v'}$ is conjugate to Δ_v. It follows that the projection

$$\ker \sum_{v \in \mathcal{V}} \operatorname{cor}_v^i \to \bigoplus_\infty \mathcal{T}_v^i(M)$$

is surjective, which proves (ii).

4.9.3. We prove (i). Choose a set S of generators of the abelian group M, and set

$$L^0 = \mathbf{Z}^S,\ L^{-1} = \ker(L^0 \to M).$$

Let $T_0 = \operatorname{Hom}(L^0, \mathbb{G}_m)$ and $T_{-1} = \operatorname{Hom}(L^{-1}, \mathbb{G}_m)$ be the corresponding tori. Then $H^1(K', T_0) = 0$ for any extension k' of K. We have the commutative diagram

$$\begin{array}{ccccccc}
T_0(K) & \longrightarrow & \mathcal{H}^0(K, M, \bar{K}^\times) & \longrightarrow & H^1(K, T_{-1}) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
\prod_\infty T_0(K_v) & \longrightarrow & \prod_\infty \mathcal{H}^0(K_v, M, \bar{K}_v^\times) & \longrightarrow & \prod_\infty H^1(K_v, T_{-1}) & \longrightarrow & 1
\end{array}$$

The right vertical arrow in this diagram is surjective (cf. [Ha1], II, A.1.2 or [Sa], 1.8), and the left vertical arrow has dense image (by the real approximation theorem, see [Vo2], 6.36 or [Sa], 3.5(iii)). Hence the middle vertical arrow has dense image, which was to to be proved.

Corollary 4.10 (Tate-Poitou). *If $i = 2$ and M is finite then* (4.9.0) *is an isomorphism.*

Proof: This follows from the exact sequence (4.3.1) and Propositions 4.1(ii) and 4.4(ii).

Proposition 4.11. *The canonical homomorphisms*

$$\mathbf{tf}_*\colon \mathcal{H}^2(K, M, \bar{K}^\times) \to H^2(K, M_{\mathrm{tf}} \otimes \bar{K}^\times)$$
$$\mathrm{loc}_\infty\colon \mathcal{H}^2(K, M, \bar{K}^\times) \to \prod_\infty \mathcal{H}^2(K_v, M, \bar{K}^\times{}_v)$$

define an isomorphism of $\mathcal{H}^2(K, M, \bar{K}^\times)$ on the fiber product of $H^2(K, M_{\mathrm{tf}} \otimes \bar{K}^\times)$ over $\prod_\infty \mathcal{H}^2(K_v, M, \bar{K}_v{}^\times)$ and $\prod_\infty H^2(K_v, M_{\mathrm{tf}} \otimes \bar{K}^\times)$.

Let T_M be the K-torus such that $X_*(\bar{T}) = M_{\mathrm{tf}}$. We have computed $\mathcal{H}^2(K, M, K^\times)$ in terms of the Galois cohomology $H^2(K, T_M)$ of this torus and of the real cohomology groups $\mathcal{H}^2(K, M, \bar{K}^\times{}_v) \simeq \widehat{H}^0(K_v, M)$.

Proof: Consider the canonical short exact sequence

$$0 \to M_{\mathrm{tors}} \xrightarrow{i} M \xrightarrow{\mathrm{tf}} M_{\mathrm{tf}} \to 0$$

and the corresponding commutative diagram

$$\begin{array}{ccccccc}
H^1(K,T_M) & \xrightarrow{\delta} & H^3(K,M_{\rm tors}(1)) & \xrightarrow{i_*} & \mathcal{H}^2(K,M,\bar{K}^\times) & \xrightarrow{\rm tf_*} & H^2(K,T_M) \\
\downarrow{\scriptstyle \rm loc_\infty} & & \downarrow{\scriptstyle \rm loc_\infty} & & \downarrow{\scriptstyle \rm loc_\infty} & & \downarrow{\scriptstyle \rm loc_\infty} \\
\prod\limits_\infty H^1(K_v,T_M) & \xrightarrow{\delta} & \prod\limits_\infty H^3(K,M_{\rm tors}(1)) & \xrightarrow{i_*} & \prod\limits_\infty \mathcal{H}^2(K,M,\bar{K}^\times) & \xrightarrow{\rm tf_*} & \prod\limits_\infty H^2(K,T_M)
\end{array}$$

with exact rows. It is clear that

$$\mathrm{tf}_* \times \mathrm{loc}_\infty\colon \mathcal{H}^2(K,M,\bar{K}^\times) \to H^2(K,T_M) \times \prod_\infty \mathcal{H}^2(K_v,M,\bar{K}^\times{}_v)$$

define a homomorphism j from $\mathcal{H}^2(K,M,\bar{K}^\times)$ to the fiber product over $\prod\limits_\infty H^2(K_v,T_M)$.

We prove that j is injective. Suppose $\xi \in \ker j$. Then $\xi \in \ker \mathrm{tf}_*$, hence $\xi = i_*(\eta)$ for some $\eta \in H^3(K,M,(1))$. Now, since $\xi \in \ker \mathrm{loc}_\infty$, $i_*(\mathrm{loc}_\infty(\eta)) = 0$, hence $\mathrm{loc}_\infty(\eta) = \delta(\zeta_\infty)$ for some $\zeta_\infty \in \prod\limits_\infty H^1(K_v,T_M)$. Since the map

$$\mathrm{loc}^1_\infty\colon H^1(K,T_M) \to \prod_\infty H^1(K_v,T_M)$$

is surjective ([Ha], II, A.1.2, see also [Sa], 1.8), there exists $\zeta \in H^1(K,T_M)$ such that $\zeta_\infty = \mathrm{loc}_\infty(\zeta)$. We see that $\mathrm{loc}_\infty(\delta(\zeta)) = \mathrm{loc}_\infty(\eta)$. By Corollary 4.7 the map $\mathrm{loc}^3_\infty\colon H^3(K,M_{\rm tors}(1))$
$\to \prod\limits_\infty H^3(K_v,M_{\rm tors}(1))$ is bijective, hence $\delta(\zeta) = \eta$. By construction $\xi = i_*(\eta)$. We conclude that $\xi = 0$. This proves the injectivity of j.

The proof of the surjectivity of j is left to the reader.

4.12 Let F/K be a finite Galois extension such that $\mathrm{Gal}(\bar{K},F)$ acts on M trivially. We set $\Delta = \mathrm{Gal}(F/K)$. Then M is a Δ-module. Consider the cokernel

$$c_1(F/K,M) = \mathrm{coker}\left[\bigoplus_v H_1(\ \Delta_v,M) \xrightarrow{\Sigma\,\mathrm{cor}_v} H_1(\Delta,M)\right]$$

where cor_v is the corestriction map, and Δ_v is a decomposition group of v in F. One can show that $c_1(F/K,M)$ does not depend on the choice of F. We write $c_1(K,M)$ for $c_1(F/K,M)$. We set

$$Ш^1_{\mathcal{H}}(K,M) = \ker[\mathrm{loc}\colon \mathcal{H}^1(K,M,\bar{K}^\times) \to \bigoplus_v \mathcal{H}^1(K_v,M,\bar{K}^\times{}_v)].$$

Proposition 4.13. *There is a canonical isomorphism*

$$c_1(K,M) \xrightarrow{\sim} Ш^1_{\mathcal{H}}(K,M)$$

Idea of proof: One can show that $Ш^1_{\mathcal{H}}(K,M)$ is canonically isomorphic to

$$Ш^1_{\mathcal{H}}(F/K,M) := \ker[\mathcal{H}^1(F/K,M,F^\times) \to \mathcal{H}^1(F/K,M,(\mathbf{A} \underset{K}{\otimes} F)^\times)],$$

where F/K is as in 4.12. We write Δ for $\mathrm{Gal}(F/K)$. This kernel is the cokernel of

$$\widehat{\mathcal{H}}^0(\Delta,M,(\mathbf{A} \otimes F)^\times) \to \widehat{\mathcal{H}}^0(\Delta,M,(\mathbf{A} \otimes F)^\times/F^\times)$$

(see Remark 2.5.3 for the definitions of the groups $\widehat{\mathcal{H}}^i$). Then we compute the groups and the homomorphism by the method used in the proof of Propositions 4.1, 4.4 and Lemma 4.5. We show that this homomorphism is

$$\oplus H_1(\Delta_v,M) \xrightarrow{\Sigma\,\mathrm{cor}_v} H_1(\Delta,M).$$

This proves the assertion.

5. Galois cohomology over local and number fields

In this section we apply the results of Sections 3 and 4 to the study of the usual (non-abelian) Galois cohomology of connected reductive groups over local and (especially) number fields.

5.0. We will need the following fundamental results on Galois cohomology over local and global fields.

Theorem 5.0.1 ([Kn1], [Kn3])**.** *Let G be a simply connected group over a non-archimedian local field K. Then $H^1(K,G)=1$.*

Another proof of this result appeared in [Br-T].

5.0.2. Let K be a number field. A K-group G is said to satisfy the Hasse principle if

$$Ш(G):=\ker[H^1(K,G)\to \prod_{v\in\mathcal{V}} H^1(K_v,G)]=0.$$

Theorem 5.0.3 (Kneser-Harder-Chernousov)**.** *For any semisimple simply connected*

group G over a number field K, the map

$$H^1(K,G)\to \prod_{\infty} H^1(K_v,G)$$

is bijective.

In particular, the Hasse principle is valid for such a group.

The classical groups were treated by Kneser (cf. [Kn2], [Kn3]), and the exceptional ones, excepting E_8, by Harder [Ha1]. The proof in the most difficult case, E_8, initiated by Harder [Ha1], has recently been completed by Chernousov [Ch].

We begin with proving that the maps ab^0 and ab^1 are in some cases surjective.

Proposition 5.1. *Let K be a non-archimedian local field. Then for any connected reductive group G the homomorphism $\mathrm{ab}^0\colon G(K)\to H^0_{\mathrm{ab}}(K,G)$ is surjective.*

Proof: We have an exact sequence

$$G(K)\xrightarrow{\mathrm{ab}^0} H^0_{\mathrm{ab}}(K)\to H^1(K,G^{\mathrm{sc}})$$

(cf. 3.10), and by Theorem 5.0.1 $H^1(K,G^{\mathrm{sc}})=0$. This proves the proposition.

Remark 5.1.1. For $K=\mathbf{R}$ the homomorphism ab^0 is in general non-surjective. For example let $\mathfrak{A}$ denote the algebra of the Hamiltonian quaternions over $\mathbf{R}$. Set $G=\mathfrak{A}^\times$; then G^{ss} is simply connected and $G^{\mathrm{tor}}=\mathbb{G}_m$. Hence

$$\mathrm{ab}^0\colon G(\mathbf{R})\to H^0_{\mathrm{ab}}(\mathbf{R},G)=\mathbb{G}_m(\mathbf{R})=\mathbf{R}^\times$$

is the reduced norm

$$\mathrm{Nm}_{\mathfrak{A}/\mathbf{R}}\colon \mathfrak{A}^\times\to\mathbf{R}^\times$$

We see that

$$\mathrm{im}\ \mathrm{ab}^0_G=\mathbf{R}^\times_+\neq\mathbf{R}^\times=H^0_{\mathrm{ab}}(\mathbf{R},G).$$

Corollary 5.2. *If K is a non-archimedian local field, then $H^0_{\mathrm{ab}}(K,G)=G(K)/\rho(G^{\mathrm{sc}}(K))$.*

To prove the surjectivity of ab^1 for local and global fields we need the notion of a fundamental torus.

5.3. Fundamental tori (a survey).

Let K be a local field and let G be a connected reductive K-group.

Definition 5.3.1 [Ko3]**.** A fundamental torus $T \subset G$ is a maximal torus of minimal K-rank.

There is a one-to-one correspondence between the maximal K-tori of G and maximal K-tori of G^{sc}:

$$T \subset G \longmapsto T^{(\mathrm{sc})} \subset G^{\mathrm{sc}}$$
$$T' \subset G^{\mathrm{sc}} \longmapsto \rho(T') \cdot Z(G)^0$$

where $Z(G)^0$ is the connected component of the center of G. We see that a maximal torus $T \subset G$ is fundamental in G if and only if $T^{(\mathrm{sc})}$ is fundamental in G^{sc}.

Proposition 5.3.2 ([Kn1], II, p. 271)**.** *If $T \subset G$ is a fundamental torus of a semisimple group over a non-archimedian field, then T is anisotropic.*

In other words, in this case G contains anisotropic maximal tori.

Lemma 5.3.3 [Ko3]**.** *Let T be a fundamental torus of a simply connected semisimple group G over a local field K. Then $H^2(K,T) = 0$.*

Proof: If K is non-archimedian, then T is anisotropic, and by Tate-Nakayama duality $H^2(K,T) = 0$. Now suppose $K = \mathbf{R}$. Then T is isomorphic to a product of a compact torus and a torus fo the form $(R_{\mathbf{C}/\mathbf{R}}\mathbb{G}_m)^n$ (cf. e.g. [Ko3], Lemma 10.4), hence $H^2(\mathbf{R},T) = 0$.

Lemma 5.3.4 ([Ko3], 10.1, see also [Brv1])**.** *Let $T \subset G$ be a fundamental torus of a reductive $\mathbf{R}$-group. Then the map $H^1(\mathbf{R},T) \to H^1(\mathbf{R},G)$ is surjective.*

Theorem 5.4. *If K is a local field, then the map $\mathrm{ab}^1_G \colon H^1(K,G) \to H^1_{\mathrm{ab}}(K,G)$ is surjective.*

This result is essentially due to Kottwitz [Ko3].

Proof: It suffices to find a maximal torus $T \subset G$ such that the map

$$H^1(K,T) \to H^1_{\mathrm{ab}}(K,G) = \mathbb{H}^1(K, T^{(\mathrm{sc})} \to T)$$

is surjective. Let T be a fundamental torus of G; then $T^{(\mathrm{sc})}$ is a fundamental torus of G^{sc}. From the exact sequence (2.12.2)

$$H^1(K,T) \to H^1_{\mathrm{ab}}(K,G) \to H^2(K,T^{(\mathrm{sc})}),$$

where $H^2(K,T^{(\mathrm{sc})}) = 0$ by Lemma 5.3.3, we see tht for such T the map $H^1(K,T) \to H^1_{\mathrm{ab}}(K,G)$ is surjective. The theorem is proved.

Corollary 5.4.1. *If K is a non-archimedian local field, then the map ab^1_G of Theorem 5.4 is bijective.*

Proof: By Corollary 3.17 any fiber of ab^1_G comes from $H^1(K, {}_\psi G^{\mathrm{sc}})$ for some cocycle $\psi \in Z^1(K,G)$. Since ${}_\psi G^{\mathrm{sc}}$ is simply connected, by Theorem 5.0.1 $H^1(K, {}_\psi G^{\mathrm{sc}}) = 1$. Hence the map ab^1_G is injective. By Theorem 5.4 ab^1_G is surjective. Thus ab^1_G is bijective, which was to be proved.

Corollary 5.5 [Ko3]**.** *Let G be a connected reductive group over a local field K. Set $M = \pi_1(\bar{G})$.*

(i) *If K is non-archimedean, then there is a canonical, functorial in G bijection $H^1(K, G)$
$\to (M_\Gamma)_{\mathrm{tors}}$, where $\Gamma = \mathrm{Gal}(\bar{K}/K)$.*

(ii) *If $K = \mathbf{R}$, then there is a canonical, functorial in G surjective map*

$$H^1(\mathbf{R}, G) \to \widehat{H}^{-1}(\mathbf{R}, M) = H^1(\mathbf{R}, M)$$

Proof: (i) By Corollary 5.4.1 the map ab^1_G is bijective. By Proposition 4.1 (i) $H^1_{\mathrm{ab}}(K, G) = (M_\Gamma)_{\mathrm{tors}}$. The assertion (i) is proved.

(ii) By Theorem 5.4 ab^1_G is surjective, and by Proposition 4.2 $H^1_{\mathrm{ab}}(\mathbf{R}, G) = \widehat{H}^{-1}(\mathbf{R}, M) = H^1(\mathbf{R}, M)$, which proves the assertion (ii).

5.6 To investigate Galois cohomology over number fields we need some lemmas. Throughout this subsection K is a number field.

Lemma 5.6.1 (Kneser-Harder)**.** *Let G be a connected K-group. Then the map*

$$\mathrm{loc}_\infty \colon H^1(K, G) \to \prod_\infty H^1(K_v, G)$$

is surjective.

Proof: See [Ha1], II, 5.5.1. See also [Kn3].

Lemma 5.6.2 (Kneser-Harder)**.** *Let T be a K-torus. Suppose that there is a place v_0 of K such that T is anisotropic over K_{v_0}. Then*

$$\text{Ш}^2(K, T) := \ker[H^2(K, T) \to \prod_{v \in \mathcal{V}} H^2(K_v, T)] = 0.$$

Proof: See[Ha1], II, p. 408, or [Kn3], 3.2, Thm. 7, p. 58, or [Sa], 1.9.3.

Lemma 5.6.3 (Harder)**.** *Let G be a K-group. Let $\Sigma \subset \mathcal{V}$ be a finite set of places of K. For any $v \in \Sigma$ let $T_v \subset G_{K_v}$ be a maximal torus. Then there exists a maximal torus $T \subset G$ such that T_{K_v} is conjugate to T_v under $G(K_v)$ for any $v \in \Sigma$.*

Proof: See [Ha], II, Lemma 5.5.3.

Lemma 5.6.4. *Let G be a semisimple simply connected K-group. Let $j \colon T \hookrightarrow G$ be a maximal torus of G such that for every $v \in \mathcal{V}_\infty$ the torus T_{K_v} is fundamental in G_{K_v}. Then the map*

$$j_* \colon H^1(K, T) \to H^1(K, G)$$

is surjective.

Proof: Let $\xi \in H^1(K, G)$. By Lemma 5.3.4 the map $j_* \colon H^1(K_v, T) \to H^1(K_v, G)$ is surjective for $v \in \mathcal{V}_\infty$. Hence for any $v \in \mathcal{V}_\infty$ there exists an element $\eta_v \in H^1(K_v, T)$ such that $j_*(\eta_v) = \mathrm{loc}_v(\xi)$. By Lemma 5.6.1 the homomorphism $\mathrm{loc}_\infty \colon H^1(K, T) \to \prod_\infty H^1(K_v, T)$ is surjective. Hence there is an element $\eta \in H^1(K, T)$ such that $\eta_v = \mathrm{loc}_v(\eta)$ for all $v \in \mathcal{V}_\infty$. We see that $\mathrm{loc}_\infty(j_*(\eta)) = \mathrm{loc}_\infty(\xi)$. By Theorem 5.0.3 it follows that $\xi = j_*(\eta)$. The lemma is proved.

Lemma 5.6.5. *Let G be a semisimple simply connected K-group and let $\Sigma \subset \mathcal{V}(K)$ be a finite set of places of K. Then there exists a maximal K-torus $j\colon T \hookrightarrow G$ with the following properties:*

(i) $H^2(K_v, T) = 0$ *for* $v \in \Sigma$*;*

(ii) $Ш^2(K, T) = 0$*;*

(iii) *the map* $j_*\colon H^1(K, T) \to H^1(K, G)$ *is surjective.*

Proof: We may and will assume that $\Sigma \supset \mathcal{V}_\infty$ and that Σ contains at least one non-archimedian place v_0 of K. For every place $v \in \Sigma$ choose a fundamental torus $T_v \subset G_{K_v}$. By Lemma 5.6.3 there exists a K-torus $T \subset G$ such that T_{K_v} is conjugate to T_v for all $v \in \Sigma$. We see that T_{K_v} is fundamental for any $v \in \Sigma$. Hence by Lemma 5.3.3 $H^2(K_v, T) = 0$, which proves (i). The torus T is fundamental over K_{v_0}, where $v_0 \in \mathcal{V}_f(K)$, hence by Lemma 5.3.2 T is K_{v_0}-anisotropic. By Lemma 5.6.2 $Ш^2(K, T) = 0$, which proves (ii). Since $\Sigma \supset \mathcal{V}_\infty$, the assertion (iii) follows from Lemma 5.6.4. The lemma is proved.

Now we can prove an analogue of Theorem 5.4 for number fields.

Theorem 5.7. *Let G be a connected reductive group over a number field K. Then the map* $\mathrm{ab}^1\colon H^1(K, G) \to H^1_{\mathrm{ab}}(K, G)$ *is surjective.*

Proof: Let $h \in H^1_{\mathrm{ab}}(K, G)$. It suffices to construct a torus $T \subset G$ such that the image of $H^1(K, T)$ in $\mathbb{H}^1(K, T^{(\mathrm{sc})} \to T) = H^1_{\mathrm{ab}}(K, G)$ contains h.

By Corollary 4.6 there exists a finite set S of places of K such that $\mathrm{loc}_v(h) = 0$ for $v \notin S$. Let $T' \subset G^{\mathrm{sc}}$ be a maximal torus such as in Lemma 5.6.5. We set $T = \rho(T^{(\mathrm{sc})}) \cdot Z(G)^0$; then $T^{(\mathrm{sc})} = T'$. Consider the exact sequence (2.12.2)

$$\cdots \to H^1(K, T) \to H^1_{\mathrm{ab}}(K, G) \xrightarrow{\delta} H^2(K, T^{(\mathrm{sc})}) \to \cdots$$

Set $\eta = \delta(h)$; then $\mathrm{loc}_v(\eta) = 0$ for $v \notin S$. Since $H^2(K_v, T^{(\mathrm{sc})}) = 0$ for $v \in S$ by 5.6.5 (i), we see that $\mathrm{loc}_v(\eta) = 0$ for $v \in S$ as well. Thus $\eta \in Ш^2(K, T^{(\mathrm{sc})})$. By 5.6.5 (ii) $Ш^2(K, T^{(\mathrm{sc})}) = 0$. We conclude that $\eta = 0$. Hence h comes from $H^1(K, T)$. The theorem is proved.

Proposition 5.8. *Let*

$$1 \to G_1 \xrightarrow{i} G_2 \xrightarrow{j} G_3 \to 1 \tag{5.8.1}$$

be an exact sequence of connected reductive K-groups. Suppose that the maps $\mathrm{ab}^1_{G_2}$ *and* $\mathrm{ab}^1_{G_2}$ *are surjective. Then the sequence*

$$H^1(K, G_2) \xrightarrow{j_*} H^1(K, G_3) \xrightarrow{\Delta} H^2_{\mathrm{ab}}(K, G_1) \longrightarrow H^2_{\mathrm{ab}}(K, G_2) \tag{5.8.2}$$

is exact, where the connecting homomorphism Δ is the composition

$$H^1(K, G_3) \xrightarrow{\mathrm{ab}^1} H^1_{\mathrm{ab}}(K, G_3) \xrightarrow{\delta} H^2_{\mathrm{ab}}(K, G_1).$$

Proof: Consider the commutative diagram

$$\begin{array}{ccccccc}
H^1(K, G_2) & \xrightarrow{j_*} & H^1(K, G_3) & & & & \\
\Big\downarrow{\scriptstyle \mathrm{ab}_2} & & \Big\downarrow{\scriptstyle \mathrm{ab}_3} & & & & \\
H^1_{\mathrm{ab}}(K, G_2) & \xrightarrow{j_*} & H^1_{\mathrm{ab}}(K, G_3) & \xrightarrow{\delta} & H^2_{\mathrm{ab}}(K, G_1) & \xrightarrow{i_*} & H^2_{\mathrm{ab}}(K, G_2)
\end{array}$$

with exact bottom row. Since ab_3 is surjective, the sequence (5.8.2) is exact in the term $H^2_{\mathrm{ab}}(K, G_1)$. It is clear from the diagram that the composition

$$H^1(K, G_2) \xrightarrow{j_*} H^1(K, G_3) \xrightarrow{\Delta} H^2_{\mathrm{ab}}(K, G_1)$$

is trivial.

Now let $\xi_3 \in H^1(K, G_3)$ lie in the kernel of $\Delta\colon H^1(K, G_3) \to H^2_{\mathrm{ab}}(K, G_1)$. We want to prove that $\xi_3 \in \mathrm{im}\, j_*$. Since ab_2 is surjective, there exists $\xi_2 \in H^1(K, G_2)$ such that $\mathrm{ab}_3(j_*\xi_2) = \mathrm{ab}_3(\xi_3)$. Let $\psi_2 \in Z^1(K, G_2)$ be a cocycle representing ξ_2. Twisting the short exact sequence (5.8.1) by ψ_2 and applying Proposition 3.16 and Corollary 3.17, we reduce the assertion to be proved to the case $\xi_2 = 0$. Then $\mathrm{ab}_3(\xi_3) = 0$. By Corollary 3.17 (i) there exists $\eta_3 \in H^1(K, G_3^{\mathrm{sc}})$ such that $\xi_3 = \rho_*\eta_3$. Since the exact sequence of semisimple simply connected groups

$$1 \to G_1^{\mathrm{sc}} \to G_2^{\mathrm{sc}} \to G_3^{\mathrm{sc}} \to 1$$

splits, the map $H^1(K, G_2^{\mathrm{sc}}) \to H^1(K, G_3^{\mathrm{sc}})$ is surjective. Hence η_3 is the image of some cohomology class $\eta_2 \in H^1(K, G_2^{\mathrm{sc}})$. Set $\xi_2 = \rho_*\eta_2 \in H^1(K, G_2)$; then $\xi_3 = j_*\xi_2$.

Using Proposition 5.8 we can compute the fibers of the connecting map

$$\Delta\colon H^1(K, G_3) \to H^2_{\mathrm{ab}}(K, G_1).$$

Corollary 5.9. *With the assumptions and notation of Proposition* 5.8, *for any* $\psi \in Z^1(K, G_3)$ *we have*

$$\Delta^{-1}(\Delta(\mathrm{Cl}(\psi)) = t_\psi(im\,[{}_\psi j_*\colon H^1(K, {}_\psi G_2)) \to H^1(K, {}_\psi G_3)])$$

Proof: We apply twisting by ψ.

Applying Proposition 5.8 to the case of local and number fields, we obtain

Corollary 5.10. *If K is a local or a number field, then the sequence* (5.8.2) *of Proposition* 5.8 *is exact.*

Proof: The assertion follows from Theorems 5.4 and 5.7.

Recall that if $K = \mathbf{R}$ then $H^2_{\mathrm{ab}}(K, G) = \widehat{H}^0(\mathbf{R}, \pi_1(\bar{G}))$. If K is a number field, $H^2_{\mathrm{ab}}(K, G)$ is computed in Proposition 4.11.

When proving Theorem 5.7 we have actually proved that any $h \in H^1_{\mathrm{ab}}(K, G)$ comes from some torus $T \subset G$. We will prove that a similar result holds for usual, non-abelian cohomology $H^1(K, G)$.

Theorem 5.11. *Let G be a reductive group over a number field K. For any finite set $\Xi \subset H^1(K, G)$ there exists a torus $T \overset{j}{\hookrightarrow} G$ such that $\Xi \subset j_*H^1(K, T)$.*

Remark 5.11.1. Steinberg ([St1]) proved for arbitrary field K that if G is quasi-split and $\xi \in H^1(K, G)$, then there is a torus $j\colon T \hookrightarrow G$ such that $\xi \in j_*H^1(K, G)$. Theorem 5.11 shows that for a *number* field a similar (and even more stronger) assertion holds for *any* group, not necessarily quasi-split. Of course we use Steinberg's theorem when we use the Hasse principle for simply connected groups.

Proof of Theorem 5.11. Since Ξ is finite, there exists by Corollary 4.6 a finite set Σ of places of K such that $\mathrm{loc}_v(\mathrm{ab}^1(\Xi)) = 0$ for any $\xi \in \Xi$ and any $v \notin \Sigma$. We construct a maximal torus $T' \subset G^{\mathrm{sc}}$ as in Lemma 5.6.5. We set $T = \rho(T') \cdot Z(G)^0$; then $T^{(\mathrm{sc})} = T'$. We denote by j the inclusion $T \hookrightarrow G$. We will prove that $j_*(H^1(K,T)) \supset \Xi$.

Let $\xi \in \Xi$. Set $h = \mathrm{ab}^1(\xi) \in H^1_{\mathrm{ab}}(G)$. When proving Theorem 5.7 we have proved that there exists $\eta \in H^1(K,T)$ such that h is the image of η, i.e. $\mathrm{ab}^1(j_*(\eta)) = h = \mathrm{ab}^1(\xi)$. Thus $j_*(\eta)$ and ξ lie in the same fiber of ab^1.

Choose a cocycle $\psi \in Z^1(K,T)$ representing η. By Corollary 3.9 ξ "differs" from $j_*(\eta)$ by a certain cohomology class coming from $H^1(K, {}_\psi G^{\mathrm{sc}})$. Since ψ comes from T, we have an embedding ${}_\psi j\colon T \hookrightarrow {}_\psi G$. For any $v \in \mathcal{V}_\infty$ the torus $T^{(\mathrm{sc})}_{K_v}$ is fundamental in ${}_\psi G^{\mathrm{sc}}_{K_v}$ as well. By Lemma 5.6.4 the map $H^1(K, T^{(\mathrm{sc})}) \to H^1(K, {}_\psi G^{\mathrm{sc}})$ is surjective. Thus there excists an element $\zeta \in H^1(K, T^{(\mathrm{sc})})$ such that the image of the cohomology class $\eta + \rho_*(\zeta) \in H^1(K,T)$ in $H^1(K,G)$ is ξ. The theorem is proved.

Now using Theorem 5.7 we shall compute the first non-abelian Galois cohomology in terms of abelian cohomology and real cohomology.

Theorem 5.12. *Let G be a reductive group over a number field K. Then*

(i) *the diagram*

$$(5.12.1) \quad H^1(K,G) \xhookrightarrow{\mathrm{ab}^1 \times \mathrm{loc}_\infty} H^1_{\mathrm{ab}}(K,G) \times \prod_\infty H^1(K_v,G) \longrightarrow \prod_\infty H^1_{\mathrm{ab}}(K_v,G)$$

is exact;

(ii) *both the projections* $\mathrm{loc}_\infty\colon H^1(K,G) \to \prod_\infty H^1_{\mathrm{ab}}(K_v,G)$ *and* $\mathrm{ab}^1\colon H^1(K,G) \to H^1_{\mathrm{ab}}(K,G)$ *are surjective.*

Here the exactness of the diagram (5.12.1) means that the commutative diagram

$$\begin{array}{ccc} H^1(K,G) & \xrightarrow{\mathrm{ab}^1} & H^1_{\mathrm{ab}}(K,G) \\ {\scriptstyle \mathrm{loc}_\infty}\downarrow & & \downarrow \\ \prod_\infty H^1(K_v,G) & \longrightarrow & \prod_\infty H^1_{\mathrm{ab}}(K_v,G) \end{array}$$

(in which all the maps are surjective) identifies $H^1(K\ ,\ G)$ with the fiber product of $H^1_{\mathrm{ab}}(K\ ,\ G)$ and $\prod_\infty H^1(K_v,G)$ over $\prod_\infty H^1_{\mathrm{ab}}(K_v,G)$.

Remark 5.12.2. For semisimple groups this assertion was proved by Sansuc [Sa]. In the case $G^{\mathrm{ss}} = G^{\mathrm{sc}}$ Theorem 5.12 generalizes a result of Milne and Shih ([M-Sh], 3.1) stating that then the kernel of the map

$$H^1(K,G) \to H^1(K, G^{\mathrm{tor}}) \times \prod_\infty H^1(K_v,G)$$

is trivial.

Proof of Theorem 5.12. By Theorem 5.7 the map $\mathrm{ab}^1\colon H^1(K,G) \to H^1_{\mathrm{ab}}(K,G)$ is surjective. By Corollary 4.12 the homomorphism $\mathrm{loc}_\infty\colon H^1(K,G) \to \prod_\infty H^1_{\mathrm{ab}}(K_v,G)$ is also surjective. Thus the assertion (ii) is proved.

We prove the injectivity of

$$H^1(K,G) \to H^1_{\mathrm{ab}}(K,G) \times \prod_\infty H^1(K_v,G). \tag{5.12.3}$$

From the exact sequence (3.10.1) we obtain the commutative diagram

$$\begin{tikzcd}
G^{\mathrm{sc}}(K) \arrow[r] \arrow[d] & G(K) \arrow[r] \arrow[d] & H^0_{\mathrm{ab}}(K,G) \arrow[r, "\delta"] \arrow[d] & H^1(K,G^{\mathrm{sc}}) \arrow[r, "\rho_*"] \arrow[d] & H^1(K,G) \arrow[r] \arrow[d] & H^1_{\mathrm{ab}}(K,G) \arrow[d] \\
\prod_\infty G^{\mathrm{sc}}(K_v) \arrow[r] & \prod_\infty G(K_v) \arrow[r] & \prod_\infty H^0_{\mathrm{ab}}(K_v,G) \arrow[r, "\delta"] & \prod_\infty H^1(K_v,G^{\mathrm{sc}}) \arrow[r, "\rho_*"] & \prod_\infty H^1(K_v,G) \arrow[r] & \prod_\infty H^1_{\mathrm{ab}}(K_v,G)
\end{tikzcd}$$

with exact rows. Let $\xi \in H^1(K,G)$ be such that the images of ξ in $H^1_{\mathrm{ab}}(K,G)$ and $\prod_\infty H^1(K,G)$ are trivial. Then $\xi = \rho_*(\eta)$ for some $\eta \in H^1(K,G^{\mathrm{sc}})$, and $\rho_*(\mathrm{loc}_\infty(\eta)) = 1$. Hence $\mathrm{loc}_\infty(\eta)$ must be the image of some $\zeta_\infty \in \prod_\infty H^0_{\mathrm{ab}}(K_v,G)$.

The group $G(K_v)/\rho(G^{\mathrm{sc}}(K_v))$ is a subgroup of finite index of the group $H^0_{\mathrm{ab}}(K_v,G)$ (because the set $H^1(K_v,G^{\mathrm{sc}})$ is finite), hence it is open. It follows from Proposition 4.9 (i) that the image of the homomorphism $\mathrm{loc}_\infty\colon H^0_{\mathrm{ab}}(K,G) \to \prod_\infty H^0_{\mathrm{ab}}(K_v,G)$ is dense. Thus there exists an element $\zeta \in H^0_{\mathrm{ab}}(K,G)$ whose image in $\prod_\infty H^0_{\mathrm{ab}}(K_v,G)$ equals ζ_∞ modulo $\prod_\infty (G(K_v)/\rho(G^{\mathrm{sc}}(K_v))$. Then $\delta(\mathrm{loc}_\infty(\zeta)) = \mathrm{loc}_\infty(\eta)$. Hence $\mathrm{loc}_\infty(\eta) = \mathrm{loc}_\infty(\delta(\zeta))$, and by the Hasse principle for G^{sc} (Theorem 5.0.3) $\eta = \delta(\zeta)$. We conclude that $\xi = 1$.

We have proved that the kernel of (5.12.3) is trivial. Using twisting (and applying Proposition 3.16 and Corollary 3.17) we obtain the injectivity of (5.12.3).

We prove the exactness at the term $H^1_{\mathrm{ab}}(K,G) \times \prod_\infty H^1(K_v,G)$. It is clear that the image of (5.12.3) is contained in the kernel of the double arrow. Conversely, let

$$h \times \xi_\infty \in H^1_{\mathrm{ab}}(K,G) \times \prod_\infty H^1(K_v,G)$$

be in the kernel of the double arrow, i.e. $\mathrm{loc}_\infty(h) = \mathrm{ab}^1(\xi_\infty)$. We wish to show that $h \times \xi_\infty$ comes from $H^1(K,G)$.

By Theorem 5.7 $h = \mathrm{ab}^1(\eta)$ for some $\eta \in H^1(K,G)$. Then $\mathrm{ab}^1(\mathrm{loc}_\infty(\eta)) = \mathrm{ab}^1(\xi_\infty)$. Let $\psi \in Z^1(K,G)$ be a cocycle representing η. By Corollary 3.9 $\mathrm{loc}_\infty(\eta)$ and ξ_∞ "differ" by an element of the form ${}_\psi\rho_*(\zeta_\infty)$ where $\zeta_\infty \in \prod_\infty H^1(K_v, {}_\psi G^{\mathrm{sc}})$. To be more precise, $\xi_\infty = t_\psi({}_\psi\rho_*(\zeta_\infty))$. By Lemma 5.6.1 there exists a cohomology class $\zeta \in H^1(K, {}_\psi G^{\mathrm{sc}})$ such that $\mathrm{loc}_\infty(\zeta) = \zeta_\infty$. We set $\xi = t_\psi({}_\psi\rho_*(\zeta))$. Then $\mathrm{ab}^1(\xi) = \mathrm{ab}^1(\eta) = h$ and $\mathrm{loc}_\infty(\xi) = t_\psi({}_\psi\rho_*(\zeta_\infty)) = \xi_\infty$. The theorem is proved.

Theorem 5.13. *Let G be a connected reductive K-group. The abelianization map* ab^1:
$H^1(K,G) \to H^1_{\mathrm{ab}}(K,G)$ *induces a canonical, functorial in G bijection of the Shafarevich-Tate kernel $\text{Ш}(G)$ onto the abelian group $\text{Ш}^1_{\mathrm{ab}}(G)$.*

Recall that by definition

$$\text{Ш}(G) = \ker\left[H^1(K,G) \to \prod_{v\in\mathcal{V}} H^1(K_v,G)\right]$$

Proof: From the commutative diagram

$$\begin{array}{ccc} H^1(K,G) & \xrightarrow{\mathrm{ab}^1} & H^1_{\mathrm{ab}}(K,G) \\ \downarrow & & \downarrow \\ H^1(K_v,G) & \xrightarrow{\mathrm{ab}^1_v} & H^1_{\mathrm{ab}}(K_v,G) \end{array} \tag{5.13.1}$$

it is clear that ab^1 takes $Ш(G)$ into $Ш^1_{\mathrm{ab}}(G)$. Write temporarily $\mathrm{ab}_{Ш}$ for the restriction of ab^1_G to $Ш(G)$.

We prove the injectivity of $\mathrm{ab}_{Ш}$. By Theorem 5.12 the map

$$\mathrm{ab}^1_G \times \mathrm{loc}_\infty\colon H^1(K,G) \to H^1_{\mathrm{ab}}(K,G) \times \prod_\infty H^1(K_v,G)$$

is injective. Since $\mathrm{loc}_\infty(Ш(G)) = 1$, we conclude that the restriction $\mathrm{ab}_{Ш}$ of ab^1_G to $Ш(G)$ is injective.

We prove the surjectivity of $\mathrm{ab}_{Ш}$. Let $h \in Ш^1_{\mathrm{ab}}(G) \subset H^1_{\mathrm{ab}}(K,G)$. Then $\mathrm{loc}_\infty(h) = 1 \in \prod_\infty H^1_{\mathrm{ab}}(K_v,G)$. Hence the element

$$h \times 1 \in H^1_{\mathrm{ab}}(K,G) \times \prod_\infty H^1(K_v,G)$$

lies in the fiber product over $\prod_\infty H^1_{\mathrm{ab}}(K_v,G)$. By Theorem 5.12 $h \times 1$ is the image of some element $\xi \in H^1(K,G)$. We will show that $\xi \in Ш(G)$.

We observe that $\mathrm{loc}_\infty(\xi) = 1$. Now let $v \in \mathcal{V}_f$; consider the element $\mathrm{loc}_v(\xi) \in H^1(K_v,G)$. Since $\xi \in H^1(K,G)$, we see from the diagram (5.13.1) that $\mathrm{ab}^1_v(\mathrm{loc}_v(\xi)) = 0$. By Corollary 5.4.1 the map $\mathrm{ab}^1_v\colon H^1(K_v,G) \to H^1_{\mathrm{ab}}(K_v,G)$ is bijective. Hence $\mathrm{loc}_v(\xi) = 1$ for any $v \in \mathcal{V}_f$. We conclude that $\xi \in Ш(G)$. The theorem is proved.

Corollary 5.14 [Ko3]**.** *With the notation of 4.13 we have a canonical, functorial in G bijection $Ш(G) \xrightarrow{\sim} c_1(K,\pi_1(\bar{G}))$.*

Remark 5.14.1. Voskresenskii [Vo1] was first to prove that $Ш(G)$ has a canonical structure of abelian group. Sansuc [Sa] showed that this abelian group structure is functorial in G. He computed $Ш(G)$ in terms of the arithmetic Brauer group $\mathrm{Br}_a G$. Our formula is equivalent to the formula (4.2.2) of [Ko2].

5.15 Corollary 5.14 shows that the kernel of the localization map

$$H^1(K,G) \to \prod_{v\in\mathcal{V}} H^1(K_v,G) \tag{5.15.1}$$

has a natural structure of an abelian group and can be computed in terms of $\pi_1(\bar{G})$. We show that a similar assertion holds for the cokernel of (5.15.1) as well.

Set $M = \pi_1(\bar{G})$. Let the groups $\mathcal{T}^{-1}(M)$, $\mathcal{T}_v^{-1}(M)$ and the corestriction map cor_v^{-1}:
$\mathcal{T}_v^{-1}(M) \to \mathcal{T}^{-1}(M)$ be as in 4.7. We define the composition

$$\mu_v\colon H^1(K_v,G) \xrightarrow{\mathrm{ab}^1} H^1_{\mathrm{ab}}(K_v,G) = \mathcal{T}_v^{-1}(M) \xrightarrow{\mathrm{cor}_v^{-1}} \mathcal{T}^{-1}(M) = (M_\Gamma)_{\mathrm{tors}}$$

Let $\bigoplus_{\mathcal{V}} H^1(K_v,G)$ denote the subset of the direct product consisting of the families $(\xi_v)_{v\in\mathcal{V}}$ such that $\xi_v = 1$ for v outside some finite set. We consider the map

$$\mu = \Sigma\mu_v\colon \bigoplus_{\mathcal{V}} H^1(K_v,G) \to (M_\Gamma)_{\mathrm{tors}}$$

The map μ is functorial in G.

Theorem 5.16 [Ko3]**.** *The sequence*

$$0 \to Ш(G) \to H^1(K,G) \to \oplus H^1(K_v,G) \xrightarrow{\mu} (\pi_1(\bar{G})_\Gamma)_{\mathrm{tors}}$$

is exact.

Proof: We have to prove only the exactness in the term $\oplus H^1(K_v,G)$. Consider the commutative diagram

$$\begin{array}{ccccc} H^1(K,G) & \longrightarrow & \oplus H^1(K_v,G) & & \\ \Big\downarrow \mathrm{ab} & & \Big\downarrow \oplus \mathrm{ab}_v & & \\ H^1_{\mathrm{ab}}(K,G) & \longrightarrow & \oplus H^1_{\mathrm{ab}}(K_v,G) & \longrightarrow & (\pi_1(\bar{G})_\Gamma)_{\mathrm{tors}} \end{array} \tag{5.16.1}$$

Set$M=\pi_1(\bar{G})$; then using Proposition 4.8 we see that the lower row of the diagram is the exact sequence (4.3.1)

$$\mathcal{H}^1(K,M;\bar{K}^\times) \to \mathcal{H}^1(K,M,\bar{\mathbf{A}}^\times) \to \mathcal{H}^1(K,M,\bar{C}),$$

hence the lower row of (5.16.1) is exact.

It is clear from the diagram that the composition

$$H^1(K,G) \to \oplus H^1(K_v,G) \to (M_\Gamma)_{\mathrm{tors}}$$

is zero. Now let $\xi_{\mathbf{A}} = \xi_\infty \times \xi_f \in \oplus H^1(K_v,G)$, where $\xi_\infty \in \prod_\infty H^1(K_v,\, G)$, $\xi_f \in \bigoplus_{\mathcal{V}_f} H^1(K_v,\, G)$. Suppose that $\mu(\xi_{\mathbf{A}}) = 0$. Let $h_{\mathbf{A}}$ be the image of $\xi_{\mathbf{A}}$ in $\oplus H^1_{\mathrm{ab}}(K_v,G)$. Then the image of $h_{\mathbf{A}}$ in $(M_\Gamma)_{\mathrm{tors}}$ is zero, hence $h_{\mathbf{A}}$ is the image of some element $h \in H^1_{\mathrm{ab}}(K,G) \times \prod_\infty H^1(K_v,G)$. It is clear that $h \times \xi_\infty$ is contained in the fiber product over $\prod_\infty H^1_{\mathrm{ab}}(K_v,G)$. By Theorem 5.12 $h \times \xi_\infty$ comes from $H^1(K,G)$. The theorem is proved.

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