THE BRAUER–MANIN OBSTRUCTIONS FOR HOMOGENEOUS SPACES WITH CONNECTED OR ABELIAN STABILIZER

MIKHAIL BOROVOI

Introduction

In this paper we prove that for a homogeneous space of a connected algebraic group with connected stabilizer and for a homogeneous space of a simply connected group with abelian stabilizer, the Brauer–Manin obstructions to the Hasse principle and weak approximation are the only ones.

More precisely, let X be an algebraic variety over a number field k. The variety X is called a counter-example to the Hasse principle, if X has a k_v-point for any completion k_v of k, but has no k-points. Manin [Ma1], [Ma2] proposed a general method of explaining obstructions to the Hasse principle with the use of the Brauer group of X. For a k-variety X such that $X(k_v)$ is nonempty for any place v of k, Manin's method gives an obstruction for X to have a k-point. For a class $\mathcal{C}$ of k-varieties a natural question arises, whether the Brauer–Manin obstruction is the only one, i.e. is it true that if $X \in \mathcal{C}$, $X(k_v) \neq \varnothing$ for any place v of k, and there is no Brauer–Manin obstruction, then X must have a k-point.

Let S be a finite set of places of k. A k-variety X with k-rational points is said to have the weak approximation property with respect to S, if $X(k)$ is dense in $\prod_{v \in S} X(k_v)$ with respect to the diagonal embedding. We say that X has the weak approximation property, if it has the weak approximation property with respect to S for any finite set S of places of k. Using the idea of the Brauer–Manin obstruction to the Hasse principle, Colliot-Thélène and Sansuc [CTS] defined an obstruction to weak approximation which they also called the Brauer–Manin obstruction. Again, for a class $\mathcal{C}$ of k-varieties a question arises, whether the Brauer–Manin obstruction to weak approximation is the only one, i.e. whether any variety $X \in \mathcal{C}$ with rational points and without Brauer–Manin obstruction to weak approximation has the weak approximation property.

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The Brauer–Manin obstruction was proved to be nontrivial for all the known counter-examples to the Hasse principle. For many classes of rational surfaces, the Brauer–Manin obstructions to the Hasse principle and weak approximation were proved to be the only ones. The similar assertion was proved for smooth intersections of two quadrics (cf. [CTSSD]) and for pencils of Severi-Brauer varieties (cf. [CTSD]). We refer to [CT1], [CT2], [MaTs] for a discussion and relevant bibliography.

Concerning homogeneous spaces, V. E. Voskresenskii [Vo1, Vo2] proved that the Brauer–Manin obstructions to the Hasse principle and weak approximation are the only ones for principal homogeneous spaces of algebraic tori. In the beautiful paper [Sa] Sansuc proved a similar assertion for principal homogeneous spaces of all the connected algebraic groups. The present author [Bo1, Bo2, Bo3, Bo4] proved that the Hasse principle and weak approximation hold for homogeneous spaces with connected stabilizer of a simply connected group if the stabilizer has no nontrivial characters over the algebraic closure $\bar{k}$ of k.

In this paper we consider homogeneous spaces of connected (affine) k-groups with connected stabilizer, and and homogeneous spaces of simply connected k-groups with abelian stabilizer. For these classes of varieties we prove that the Brauer–Manin obstructions to the Hasse principle and weak approximation are the only ones. Actually we prove the assertion in a more general case. Namely any homogeneous space of a connected k-group G is a homogeneous space of some connected k-group G' with simply connected semisimple part; we assume that the stabilizer in G' is an extension of a group of multiplicative type by a semidirect product of a connected semisimple group and a unipotent group. We use the fibration method and a Galois-cohomological construction of a torsor over X under an induced torus. We use the fibration method in order to reduce the case of a homogeneous space of our group G' to the already treated 'extreme' cases when G' is either a torus or simply connected. This permits us to prove the desired assertion assuming that all the $\bar{k}$-characters of the stabilizer come from $\bar{k}$-characters of the group. A new ingredient which is needed to remove this restriction is a construction, for any homogeneous space X, of a torsor $Y \to X$ such that Y is a homogeneous space (of another group) satisfying the restriction.

Note that the condition on the stabilizer is essential for us. The author does not know, whether the Brauer–Manin obstruction to the Hasse principle is the only one for homogeneous spaces with nonabelian nonconnected, in particular, nonabelian finite, stabilizer.

The structure of the paper is the following. In Section 1 we define the Brauer–Manin obstructions, and in Section 2 state the main results. In Section 3 we use the fibration method to treat the special case when all the $\bar{k}$-characters of the stabilizer come from $\bar{k}$-characters of the group. In Section 4 we construct a torsor over X and prove the main results in the general case, assuming that the semisimple part of G is simply connected. In Section 5 we treat the case when the semisimple part of G is not necessarily simply connected, and the stabilizer is connected.

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Notation and conventions

k is always a field of characteristic 0, and $\bar{k}$ is an algebraic closure of k. If not otherwise stated, k is assumed to be a number field. Then $\mathcal{V} = \mathcal{V}(k)$ is the set of places of k; $\mathcal{V}_r$ and $\mathcal{V}_\infty$ are the sets of real places and of infinite places of k, respectively; we write k_v for the completion of k at a place v.

If S is a finite set of places of k, then $k_S = \prod_{v\in S} k_v$; we write $k_\infty = k_{\mathcal{V}_\infty}$.

By a k-variety we mean a geometrically irreducible algebraic variety X over a field k; we assume X to be nonsingular. Then $\operatorname{Br} X = H^2_{\text{ét}}(X, \mathbb{G}_m)$, the cohomological Brauer group; $\operatorname{Br}_0 X = \operatorname{im}[\operatorname{Br} k \to \operatorname{Br} X]$; $\operatorname{Br}_1 X = \ker[\operatorname{Br} X \to \operatorname{Br} X_{\bar{k}}]$; $\operatorname{Br}_{\mathrm{a}} X = \operatorname{Br}_1 X / \operatorname{Br}_0 X$, the arithmetic Brauer group of X.

If k is a number field and S a finite subset of $\mathcal{V}(k)$, then $\text{Б}(X)$ (resp. $\text{Б}_S(X)$) is the subgroup of $\operatorname{Br}_{\mathrm{a}} X$ consisting of elements b whose localizations $\operatorname{loc}_v b$ in $\operatorname{Br}_{\mathrm{a}} X_{k_v}$ are trivial for all places v of k (resp. for all $v \notin S$). We set $\text{Б}_\omega(X) = \bigcup_S \text{Б}_S(X)$.

By a k-group we mean an affine algebraic group defined over k. If H is a k-group, then H° is the connected component of H; H^{u} is the unipotent radical of H°; $H^{\mathrm{red}} = H^\circ/H^{\mathrm{u}}$ (it is a connected reductive group); H^{ss} is the derived group of H^{red} (it is semisimple); $H^{\mathrm{tor}} = H^{\mathrm{red}}/H^{\mathrm{ss}}$ (it is a torus); $H^{\mathrm{ssu}} = \ker[H^\circ \to H^{\mathrm{tor}}]$ (it is an extension of H^{ss} by H^{u}).

Recall that an algebraic group is of multiplicative type if it is abelian and its connected component is a torus. We always assume that the group $H/H^{\rm ssu}$ is abelian, and set $H^{\rm mult} = H/H^{\rm ssu}$ (being abelian, this group is of multiplicative type).

For any abelian group A we set $A^D = \operatorname{Hom}(A, \mathbf{Q}/\mathbf{Z})$ (the dual group).

1. The Brauer–Manin obstructions

For our purposes the Brauer–Manin obstructions coming from the subgroup $\operatorname{Br}_1 X$ of $\operatorname{Br} X$ will suffice. Following [Sa], we define the Brauer–Manin obstructions in terms of the groups Б(X) and Б$_S(X)/$Б(X), see Notation and conventions for the definitions.

Let X be a variety over a field k. Consider the pairing

$$X(k) \times \operatorname{Br} X \to \operatorname{Br} k, \qquad (x, b) \mapsto b(x)$$

where $b(x)$ denotes the restriction of $b \in \operatorname{Br} X$ to $x \in X(k)$.

If k is a local field, then the above pairing gives us a pairing

$$X(k) \times \operatorname{Br} X \to \mathbf{Q}/\mathbf{Z}, \qquad (x, b) \mapsto \operatorname{inv}(b(x))$$

where $\operatorname{inv}\colon \operatorname{Br} k \to \mathbf{Q}/\mathbf{Z}$ is the the homomorphism given by the local class field theory. This pairing is continuous in x.

Let k be a number field. Assume that $X(k_v)$ is nonempty for any $v \in \mathcal{V}$. We define a pairing

$$\langle\, ,\, \rangle\colon \prod_{v\in\mathcal{V}} X(k_v) \times \text{Б}_\omega(X) \to \mathbf{Q}/\mathbf{Z}, \qquad \langle (x_v)_{v\in\mathcal{V}}, b\rangle = \sum_{v\in\mathcal{V}} \operatorname{inv}_v(\tilde{b}(x_v))$$

where $\tilde{b} \in \operatorname{Br}_1 X$ is a representative of $b \in \text{Б}_\omega(X) \subset \operatorname{Br}_{\rm a} X$, and $\operatorname{inv}_v\colon \operatorname{Br} k_v \to \mathbf{Q}/\mathbf{Z}$ is the homomorphism given by the local class field theory. See [Sa], 6.2, for a proof that this sum is actually finite and the resulting pairing is continuous in $(x_v)_{v\in\mathcal{V}}$. From the exact sequence

$$0 \to \operatorname{Br} k \to \oplus \operatorname{Br} k_v \xrightarrow{\sum \operatorname{inv}_v} \mathbf{Q}/\mathbf{Z} \tag{1.1}$$

it follows that the pairing $\langle\, ,\, \rangle$ is defined correctly (does not depend on the choice of $\tilde{b}$ representing b) and that if $(x_v)_{v\in\mathcal{V}}$ is the image of some $x \in X(k)$ under the diagonal embedding then $\langle (x_v)_{v\in\mathcal{V}}, b\rangle = 0$ (see [Sa] 6.2 for details).

Let $b \in \text{Б}(x) \subset \text{Б}_\omega(X)$ and let $\tilde{b} \in \operatorname{Br}_1 X$ be a representative of b. Then the localization $\operatorname{loc}_v \tilde{b} \in \operatorname{Br}_1 X_{k_v}$ is contained in $\operatorname{Br}_0 X_{k_v}$ for any v, hence $\tilde{b}(x_v) \in \operatorname{Br} k$ does not depend on x_v. We see that $\langle (x_v)_{v\in\mathcal{V}}, b\rangle$ does not depend on $(x_v)_{v\in\mathcal{V}}$. Thus we obtain an element

$$m_{\rm H}(X) \in \text{Б}(X)^D = \operatorname{Hom}(\text{Б}(X), \mathbf{Q}/\mathbf{Z}), \qquad b \mapsto \langle (x_v)_{v\in\mathcal{V}}, b\rangle\ .$$

If X has a k-point x, then $\langle x, b\rangle = 0$, and thus $m_{\mathrm{H}}(X) = 0$. We call $m_{\mathrm{H}}(X)$ *the Brauer–Manin obstruction to the Hasse principle for* X. If $X(k) \neq \varnothing$, then $m_{\mathrm{H}}(X) = 0$.

Now assume that X has a k-point x. Let $S \subset \mathcal{V}$ be a finite subset. We write $k_S = \prod_{v\in S} k_v$, then $X(k_S) = \prod_{v\in S} X(k_v)$. Consider a pairing

$$\langle\, ,\, \rangle_S \colon X(k_S)\times Б_S(X) \to \mathbf{Q}/\mathbf{Z}, \qquad \langle (x_v)_{v\in S}, b\rangle_S = \sum_{v\in S} \big(\mathrm{inv}_v(\tilde b(x_v)) - \mathrm{inv}_v(\tilde b(x))\big)$$

where $\tilde b \in \mathrm{Br}_1 X$ is a representative of b. We have

$$\sum_{v\in S} \mathrm{inv}_v(\tilde b(x)) = -\sum_{v\notin S} \mathrm{inv}_v(\tilde b(x)) \ ,$$

because $\sum_{v\in\mathcal{V}} \mathrm{inv}_v(\tilde b(x)) = 0$ by (1.1). Since $b \in Б_S(X)$, we have $\mathrm{loc}_v \tilde b \in \mathrm{Br}_0 X_{k_v}$ for any $v \notin S$, and therefore the pairing $\langle\, ,\, \rangle_S$ does not depend on the choice of $x \in X(k)$. If $b \in Б(X)$, then $\langle x_S, b\rangle_S = 0$ for any $x_S \in X(k_S)$. Thus the pairing $\langle\, ,\, \rangle_S$ induces a pairing

$$X(k_S) \times Б_S(X)/Б(X) \to \mathbf{Q}/\mathbf{Z} \ ,$$

or, which is the same, a map

$$m_{\mathrm{W,S}} \colon X(k_S) \to (Б_S(X)/Б(X))^D \ .$$

The map $m_{\mathrm{W,S}}$ is continuous because the pairing $\langle\, ,\, \rangle_S$ is continuous in x_S. Further, if $x \in X(k) \subset X(k_S)$, then $m_{\mathrm{W,S}}(x) = 0$. It follows that if x_S is contained in the closure $X(k)\hat{}_S$ of $X(k)$ in $X(k_S)$, then $m_{\mathrm{W,S}}(x_S) = 0$. In particular, if $X(k)$ is dense in $X(k_S)$, then $m_{\mathrm{W,S}}$ is identically 0.

We call $m_{\mathrm{W,S}}$ *the Brauer–Manin obstruction to weak approximation with respect to* S. If X has the weak approximation property with respect to S, then $m_{\mathrm{W,S}}$ is trivial (identically zero).

Let $\pi\colon X \to Y$ be a morphism of k-varieties. Assume that $X(k_v)$ is nonempty for any $v \in \mathcal{V}$. There is an induced homomorphism $\pi_* : Б(X)^D \to Б(Y))^D$ and one can easily see that

$$\pi_*(m_{\mathrm{H}}(X)) = m_{\mathrm{H}}(Y) \ . \tag{1.2}$$

If X has a k-point, then there is a commutative diagram

$$\begin{array}{ccc} X(k_S) & \xrightarrow{m_{\mathrm{W,S}}(X)} & (Б_S(X)/Б(X))^D \\ \pi\big\downarrow & & \pi_*\big\downarrow \\ Y(k_S) & \xrightarrow{m_{\mathrm{W,S}}(Y)} & (Б_S(Y)/Б(Y))^D \end{array} \tag{1.3}$$

where π_* is the homomorphism induced by π.

2. Main results

2.1. Let X be a homogeneous space of a connected affine algebraic group G defined over a number field k. This means that there is a right action $X \times G \to X$ of G on X defined over k, which is transitive (i.e. $G(\bar{k})$ act on $X(\bar{k})$ transitively). We do not assume that X has a k-point. Let $\bar{x} \in X(\bar{k})$ be a $\bar{k}$-point, and $\bar{H} = \mathrm{Stab}_{G_{\bar{k}}}(\bar{x})$ its stabilizer in $G_{\bar{k}}$. We will assume that

$$G^{\mathrm{ssu}} \text{ is simply connected,} \tag{2.1.1}$$

$$\bar{H}/\bar{H}^{\mathrm{ssu}} \text{ is abelian, hence of multiplicative type} \tag{2.1.2}$$

(see Notation and conventions), and write $\bar{H}^{\mathrm{mult}}$ for $\bar{H}/\bar{H}^{\mathrm{ssu}}$. Note that G^{ssu} is simply connected if and only if G^{ss} is simply connected.

Theorem 2.2. *Let X be a homogeneous space of a connected affine algebraic group G defined over a number field k. Assume that G satisfies* (2.1.1) *and the stabilizer $\bar{H}$ of a point $\bar{x} \in X(\bar{k})$ satisfies* (2.1.2). *If $X(k_v)$ is nonempty for any place v of k and the Brauer–Manin obstruction $m_{\mathrm{H}} \in \text{Б}(X)^D$ to the Hasse principle for X is trivial (equals zero), then X has a k-point.*

Now assume that X has a k-point, i.e. X is isomorphic to $H\backslash G$ where H is a connected k-subgroup of G.

Theorem 2.3. *Assume that $X = H\backslash G$, where G is a connected affine algebraic group defined over a number field k and satisfying* (2.1.1) *and $H \subset G$ a k-subgroup satisfying* (2.1.2). *Let S be a finite set of places of k. If the Brauer–Manin obstruction $m_{\mathrm{W,S}}$ to weak approximation in X with respect to S is trivial (identically zero), then X has the weak approximation property with respect to S.*

Theorem 2.3 follows from Theorem 2.4 below.

Theorem 2.4. *Let $X = H\backslash G$ be as in Theorem 2.3. For any finite subset $S \subset \mathcal{V}$, if $x_S \in X(k_S)$ and $m_{\mathrm{W,S}}(x_S) = 0$, then $x_S \in X(k)\hat{}_S$ with the notation of Section 1.*

Corollary 2.5. *The assertions of Theorems 2.2, 2.3, and 2.4 hold either when G is any connected k-group and $\bar{H}$ is connected or when G satisfies* (2.1.1) *and $\bar{H}$ is abelian.*

Theorems 2.2 and 2.4 and Corollary 2.5 will be proved in the rest of the paper.

3. A special case treated by the fibration method

In order to apply the fibration method we need the following lemma.

Lemma 3.1. *Let X be a homogeneous space of an affine k-group G, and N a normal subgroup of G. Then there exists a quotient $Y = X/N$, i.e. a homogeneous space Y of G/N and a smooth G-equivariant map $\varphi : X \to Y$ such that φ is a quotient map in the sense of* [Brl]*,*II-6.3. *In particular, φ is surjective and its geometric fibers are orbits of N.*

Proof. Let $\bar{x} \in X(\bar{k})$ be a point, and $\bar{H} \subset G_{\bar{k}}$ its stabilizer. Set $\bar{H}' = \bar{H} \cdot N_{\bar{k}}$, it is an algebraic subgroup of $G_{\bar{k}}$ because N is normal. We identify $X_{\bar{k}}$ with $\bar{H}\backslash G_{\bar{k}}$ and set $\bar{Y} = \bar{H}'\backslash G_{\bar{k}}$. Let $\bar{\varphi}\colon X_{\bar{k}} = \bar{H}\backslash G_{\bar{k}} \to \bar{H}'\backslash G_{\bar{k}} = \bar{Y}$ be the canonical morphism making the diagram

$$\begin{array}{ccc} G_{\bar{k}} & \xrightarrow{\mathrm{can}_{\bar{H}}} & \bar{H}\backslash G_{\bar{k}} \\ & \searrow^{\mathrm{can}_{\bar{H}'}} & \downarrow \bar{\varphi} \\ & & \bar{H}'\backslash G_{\bar{k}} \end{array}$$

commutative. The maps $\mathrm{can}_{\bar{H}}$ and $\mathrm{can}_{\bar{H}'}$ are smooth and surjective, cf. [Brl], II-6.8, hence $\bar{\varphi}$ is smooth and surjective. One can easily check that the geometric fibers of $\bar{\varphi}$ are orbits of $N_{\bar{k}}$. The action of the Galois group $\mathrm{Gal}(\bar{k}/k)$ on $X_{\bar{k}}$ induces an action on $\bar{Y}$, and one can check that this action yields descent data defining a k-form (Y, φ) of $(\bar{Y}, \bar{\varphi})$. By [Brl], II-6.2, φ is a quotient map. □

Lemma 3.2. *Let X be a homogeneous space of a unipotent group U over a perfect field k. Then*

(i) *X has a k-point;*
(ii) *if k is a number field and S is a finite set of places of k, then $X(k)$ is dense in $X(k_S)$.*

Proof. We prove (i). Choose $\bar{x} \in X(\bar{k})$ and set $\bar{H} = \mathrm{Stab}(\bar{x})$. Since $\bar{H}$ is a subgroup of a unipotent group $U_{\bar{k}}$, it is unipotent. The homogeneous space X of U with stabilizer $\bar{H}$ defines a class $\eta \in H^2(k, \bar{H}, \kappa)$ (see [Sp], 1.20 or [Bo4], 7.7 for the statement and a definition of κ). The second nonabelian cohomology class η is the obstruction to the existence of of a pair (P, α) where P is a principal homogeneous space of U and $\alpha\colon P \to X$ a U-equivariant morphism, both defined over k. Since $\bar{H}$ is unipotent, the cohomology class η is neutral, cf. [Dou], IV-1.3 or [Bo4], 4.2. It follows that there exists such a a pair (P, α). Any principal homogeneous space of the unipotent k-group U has a k-point, cf. [Sa], 1.13. Let p be a k-point of P. Then $x = \alpha(p)$ is a k-point of X, which proves (i).

We prove (ii). Since U is unipotent, we have the exponential map $\operatorname{Lie} U \to U$ which is a biregular isomorphism. It follows that $U(k)$ is dense in $U(k_S)$. Choose $x \in X(k)$ and set $H = \operatorname{Stab}_U(x)$. Then H is a unipotent group, hence $H^1(K, H) = 1$ for any extension K of k, cf. [Sa], 1.13. It follows that $U(k_S)$ acts on $X(k_S)$ transitively, and therefore k-points of X are dense in $X(k_S)$. □

Proposition 3.3. *Let X be a homogeneous space of G where G is a torus. Then*

(i) *the assertions of Theorem 2.2 is true;*
(ii) *the assertion of Theorem 2.4 is true;*
(iii) *$X(k)$ is dense in $X(k_\infty)$.*

Proof. Since G is abelian, any two points $\bar{x}_1, \bar{x}_2 \in X(\bar{k})$ have the same stabilizer, and therefore the stabilizer $\bar{H}$ of a point $\bar{x} \in X(\bar{k})$ is defined over k. Let H denote the corresponding k-group. Then X is a principal homogeneous space of the group G/H which is a torus. For principal homogeneous spaces of tori, Theorem 2.2 was proved by Voskresenskii [Vo1],[Vo2], see also Sansuc [Sa], 8.7; this proves (i). Theorem 2.4 for principal homogeneous space of tori was proved by Voskresenskii [Vo1], [Vo2] (see also Sansuc [Sa], 8.12), but only for the case when S is sufficiently large (e.g. contains all the places of k which are ramified in the splitting field of the torus). One obtains a proof for any S by substituting S for ω in the statements and proofs of Lemma 8.10 and Proposition 8.12 of [Sa]. This proves (ii). The assertion (iii) (due to Serre) follows from (ii), cf. [Sa], 3.5(iii). □

Proposition 3.4. *Let X be a homogeneous space of a simply connected k-group G with connected stabilizer $\bar{H}$ such that $\bar{H} = \bar{H}^{\mathrm{ssu}}$. Then*

(i) *if X has a k_v-point for any archimedean place v of k, then X has a k-point;*
(ii) *if X has a k-point, then $X(k)$ is dense in $X(k_S)$ for any finite set $S \subset \mathcal{V}(k)$.*

Proof. A connected k-group G is simply connected if and only if the group $G^{\mathrm{red}} := G/G^{\mathrm{u}}$ (see Notation and conventions) is semisimple simply connected. In the case when G is semisimple simply connected, the assertion (i) was proved in [Bo4]. We will reduce the general case to this one. Let $U = G^{\mathrm{u}}$, then by Lemma 3.1 there exists a quotient $Y = X/U$ which is a homogeneous space of $G^{\mathrm{red}} = G/U$. Let $\varphi\colon X \to Y$ denote the canonical map. Assume that there exists a point $x_\infty \in X(k_\infty)$, and write $y_\infty = \varphi(x_\infty) \in Y(k_\infty)$. The variety Y is a homogeneous space of the semisimple simply connected group $G^{\mathrm{red}} = G^{\mathrm{ss}}$ with connected stabilizer having no nontrivial $\bar{k}$-characters. By [Bo4], 7.3(vi), since $Y(k_\infty) \neq \varnothing$, the variety Y has a k-point y.

The subvariety $X_y := \varphi^{-1}(y)$ of X is a homogeneous space of U, and by Lemma 3.2(i) it has a k-point, which proves (i). The assertion (ii) is proved in [Bo2], Theorems 1.1 and 1.4. □

Recall that we always assume that that (2.1.2) is satisfied, and therefore the notation $\bar{H}^{\mathrm{mult}}$ makes sense.

Proposition 3.5. *Let X be a homogeneous space of a connected k-group G with stabilizer $\bar{H} \subset G_{\bar{k}}$. Assume that G^{ssu} is simply connected and*

(*) *the homomorphism $\bar{H}^{\mathrm{mult}} \to G^{\mathrm{tor}}_{\bar{k}}$ induced by the inclusion $\bar{H} \subset G_{\bar{k}}$ is injective.*

Then

(i) *the assertion of Theorem 2.2 is true;*
(ii) *the assertion of Theorem 2.4 is true.*

Proof. We use the fibration method in order to reduce the assertion to the cases considered in Propositions 3.3 and 3.4.

Set $Y = X/G^{\mathrm{ssu}}$ (this quotient exists by Lemma 3.1), and let $\varphi : X \to Y$ be the canonical map. The base Y of the fibering $\varphi\colon X \to Y$ is a homogeneous space of the torus $G/G^{\mathrm{ssu}} = G^{\mathrm{tor}}$, and the fibers are homogeneous spaces of the simply connected group G^{ssu}.

We prove (i). Assume that $X(k_v) \neq \varnothing$ for any $v \in \mathcal{V}$ and $m_{\mathrm{H}}(X) = 0$. Then $Y(k_v) \neq \varnothing$ for any $v \in \mathcal{V}$. By (1.2) we have $m_{\mathrm{H}}(Y) = \varphi_*(m_{\mathrm{H}}(X)) = 0$. By Proposition 3.3(i) Y has a k-point.

The morphism φ is smooth, and therefore $\varphi(X(k_\infty))$ is open in $Y(k_\infty)$. Since Y is a principal homogeneous space of a torus and has a k-point, by Proposition 3.3(iii) the set $Y(k)$ is dense in $Y(k_\infty)$. It follows that there exists a k-point $y \in Y(k) \cap \varphi(X(k_\infty))$.

Consider the fiber $X_y = \varphi^{-1}(y)$ of φ over y, it is a homogeneous space of G^{ssu} with stabilizer $\bar{H}' = \bar{H} \cap G^{\mathrm{ssu}}$. Since the map $\bar{H}^{\mathrm{mult}} \to G^{\mathrm{tor}}_{\bar{k}}$ is injective, we have $\bar{H} \cap G^{\mathrm{ssu}} = \bar{H}^{\mathrm{ssu}}$, hence $\bar{H}' = \bar{H}^{\mathrm{ssu}}$. It follows that $\bar{H}'$ is connected and $(\bar{H}')^{\mathrm{mult}} = 1$. By assumption G^{ssu} is simply connected and by construction $X_y(k_\infty) \neq \varnothing$. Now it follows from Proposition 3.4(i) that X_y has a k-point. Thus X has a k-point, which proves (i).

We prove (ii). Assume that X has k-points. Let $x_S \in X(k_S)$. Assume that $m_{\mathrm{W,S}}(x_S) = 0$. Set $\Sigma = S \cup \mathcal{V}_r$, and let $x_\Sigma \in X(k_\Sigma)$ be a point projecting to x_S. It suffices to prove that $x_\Sigma \in \widehat{X(k)}_\Sigma$. Let $\mathcal{U}_X \subset X(k_\Sigma)$ be an open neighborhood of x_Σ. We will prove that $\mathcal{U}_X$ contains k-points.

Set $y_\Sigma = \varphi(x_\Sigma) \in Y(k_\Sigma)$, $y_S = \varphi(x_S) \in Y(k_S)$; then y_Σ projects to y_S. By hypothesis $m_{\mathrm{W,S}}(X)(x_S) = 0$. By (1.3) $m_{\mathrm{W,S}}(Y)(y_S) = \pi_*(m_{\mathrm{W,S}}(X)(x_S) = 0$. By Proposition 3.3(ii) $y_S \in Y(k)\hat{}_S$. By [Sa] (3.3.3) then $y_\Sigma \in Y(k)\hat{}_\Sigma$, because Σ and S differ only by real places.

Set $\mathcal{U}_Y = \varphi(\mathcal{U}_X) \subset Y(k_\Sigma)$, it is an open subset of $Y(k_\Sigma)$, because φ is a smooth morphism. Since $\mathcal{U}_Y$ contains $y_\Sigma \in Y(k)\hat{}_\Sigma$, there exists a k-point $y \in Y(k) \cap \mathcal{U}_Y$. As above, set $X_y = \varphi^{-1}(y)$, it is a homogeneous space of the simply connected group G^{ssu} with some stabilizer $\bar{H}'$. We have remarked that $\bar{H}'$ is connected and $(\bar{H}')^{\mathrm{tor}} = 1$. Since by construction Σ contains all the real places of k, and $X_y(k_\Sigma) \neq \varnothing$, we see that $X_y(k_\infty) \neq \varnothing$. By Proposition 3.4 X_y has k-points and these k-points are dense in $X_y(k_\Sigma)$. Set $\mathcal{U}' = \mathcal{U}_X \cap X_y(k_\Sigma)$, it is a nonempty open subset in $X_y(k_\Sigma)$, hence it contains a k-point x. Thus there exists a point $x \in X(k) \cap \mathcal{U}_X$, which proves (ii). □

4. Construction of a torsor

In this section we remove the assumption (*) of Proposition 3.5, namely that the homomorphism $\bar{H}^{\mathrm{mult}} \to G^{\mathrm{tor}}_{\bar{k}}$ induced by the inclusion $\bar{H} \subset G_{\bar{k}}$ is injective.

4.1. First we define a k-form H^{m} of $\bar{H}^{\mathrm{mult}}$ depending only on G and X.

For $\sigma \in \mathrm{Gal}(\bar{k}/k)$, choose $g_\sigma \in G(\bar{k})$ such that ${}^\sigma x = x \cdot g_\sigma$. For $h \in \bar{H}(\bar{k})$ we have

$$x \cdot h = x, \quad x \cdot g_\sigma \cdot {}^\sigma h = x \cdot g_\sigma, \quad g_\sigma \cdot {}^\sigma \bar{H} \cdot g_\sigma^{-1} = \bar{H}$$

so we obtain a σ-semialgebraic automorphism (see [Bo4], 1.1 for a definition)

$$\nu_\sigma \colon \bar{H} \to \bar{H}, \quad h \mapsto g_\sigma \cdot {}^\sigma h \cdot g_\sigma^{-1} .$$

Clearly ν_σ induces a σ-semialgebraic automorphism ν_σ^{m} of $\bar{H}^{\mathrm{mult}}$. If we choose another g_σ, say $g'_\sigma = h' g_\sigma$, then we get $\nu'_\sigma = \mathrm{int}(h') \circ \nu_\sigma$. Since the inner automorphisms of $\bar{H}$ act trivially on $\bar{H}^{\mathrm{mult}}$, we see that the σ-semialgebraic automorphism ν_σ^{m} of $\bar{H}^{\mathrm{mult}}$ does not depend on the choice of g_σ. For $\sigma, \tau \in \mathrm{Gal}(\bar{k}/k)$ we have $\nu_{\sigma\tau}^{\mathrm{m}} = \nu_\sigma^{\mathrm{m}} \circ {}^\sigma\nu_\tau^{\mathrm{m}}$, hence the family $(\nu_\sigma^{\mathrm{m}})_{\sigma \in \mathrm{Gal}(\bar{k}/k)}$ is descent data defining a k-form H^{m} of $\bar{H}^{\mathrm{mult}}$.

Now let $\bar{x}_1 \in X(\bar{k})$ be another $\bar{k}$-point and $\bar{H}_1$ its stabilizer in $G_{\bar{k}}$. Let $\bar{H}_1^{\mathrm{m}}$ be the corresponding k-form of $\bar{H}_1^{\mathrm{mult}}$. Choose $g \in G(\bar{k})$ such that $\bar{x}_1 = x \cdot g$. Consider the isomorphism

$$\mu_g \colon \bar{H} \to \bar{H}_1, \quad h \mapsto g^{-1} h g .$$

One can easily check that the induced isomorphism $\mu_g^{\mathrm{m}}\colon \bar{H}^{\mathrm{mult}} \to \bar{H}_1^{\mathrm{mult}}$ does not depend on the choice of g and defines a canonical isomorphism $H^{\mathrm{m}} \xrightarrow{\sim} H_1^{\mathrm{m}}$. We can therefore identify H_1^{m} with H^{m}.

4.2. A k-torus T is called induced (or quasi-trivial) if its character group $X^*(T_{\bar{k}}) := \mathrm{Hom}(T_{\bar{k}}, \mathbb{G}_{>})$ has a $\mathrm{Gal}(\bar{k}/k)$-invariant basis. Choose an embedding $j\colon H^{\mathrm{m}} \hookrightarrow T$ of H^{m} into an induced torus and set $F = G \times T$. We wish to construct a homogeneous space Y of F and an F-equivariant morphism $\pi\colon Y \to X$.

First assume that X has a k-point x with stabilizer H. Then $H^{\mathrm{m}} = H^{\mathrm{mult}}$, and we define an embedding

$$H \to F = G \times T, \qquad h \mapsto \big(i(h), j(h)\big)$$

where $i\colon H \hookrightarrow G$ is the inclusion map. Set $Y = H\backslash F$; it is a homogeneous space of F with stabilizer H, and there is an F-equivariant map

$$\pi\colon Y \to X \qquad H\cdot(g,t) \mapsto H\cdot g\ .$$

One checks immediately that $\pi\colon Y \to X$ is a torsor under T.

In the general case we do not assume that X has a k-point. However X has a $\bar{k}$-point $\bar{x}$, whose stabilizer we denote by $\bar{H}$. We define a homogeneous space $\bar{Y} = \bar{H}\backslash F_{\bar{k}}$ of $F_{\bar{k}}$ and an $F_{\bar{k}}$-equivariant map

$$\bar{\pi}\colon \bar{Y} \to X_{\bar{k}}\ , \qquad \bar{H}\cdot(g,t) \mapsto \bar{H}\cdot g\ .$$

Then $\bar{\pi}\colon \bar{Y} \to X_{\bar{k}}$ is a torsor under $T_{\bar{k}}$. The homomorphism $\bar{H}^{\mathrm{mult}} \to F_{\bar{k}}^{\mathrm{tor}}$ is injective.

We are interested whether there exists a k-form (Y,π) of the pair $(\bar{Y},\bar{\pi})$, where Y is a homogeneous space of the group F and $\pi\colon Y \to X$ is an F-equivariant map with respect to the projection $F \to G$. If there exists a k-form (Y,π) of the pair $(\bar{Y},\bar{\pi})$, then $\pi\colon\ : Y \to X$ is a torsor under T.

Lemma 4.3. *If $X(k_v) \neq \varnothing$ for any $v \in \mathcal{V}$, then there exists a k-form (Y,π) of the pair $(\bar{Y},\bar{\pi})$.*

Lemma 4.3 will be proved later.

Lemma 4.4. *Assume that there exists a k-form (Y,π) of $(\bar{Y},\bar{\pi})$. Then the map π induces isomorphisms*

$$\text{Б}_S(X) \xrightarrow{\sim} \text{Б}_S(Y)\ , \quad \text{Б}(X) \xrightarrow{\sim} \text{Б}(Y)$$

Proof. The map $\pi\colon Y \to X$ is a torsor under the induced torus T. By [Sa], (6.10.3), there is an exact sequence

$$\mathrm{Pic}\ T \to \mathrm{Br}_1 X \xrightarrow{\pi^*} \mathrm{Br}_1 Y \to \mathrm{Br_a}\, T$$

Since T is an induced torus, by [Sa], 6.9(v) we have Pic $T = 0$, hence π^* is injective. From the commutative diagram

$$\begin{array}{ccccc} \mathrm{Br}\, k & & & & \\ \downarrow & \searrow & & & \\ \mathrm{Br}_1 X & \longrightarrow & \mathrm{Br}_1 Y & \longrightarrow & \mathrm{Br_a}\, T \end{array}$$

we obtain an exact sequence

$$0 \to \mathrm{Br_a}\, X \to \mathrm{Br_a}\, Y \to \mathrm{Br_a}\, T\,.$$

We have similar exact sequences for all the completions k_v of k, whence we obtain exact sequences

$$0 \to \text{Б}_S(X) \to \text{Б}_S(Y) \to \text{Б}_S(T)$$

for any S. By [Sa], 6.9(v) we have $\text{Б}_\omega(T) = 0$, whence $\text{Б}_S(T) = 0$. It follows that the canonical homomorphisms $\text{Б}_S(X) \to \text{Б}_S(Y)$ are isomorphisms. In particular the homomorphism $\text{Б}(X) \to \text{Б}(Y)$ is an isomorphism. □

4.5. *Proof of Theorem 2.2 modulo Lemma 4.3.* Since $H^1(k_v, T) = 0$ (because T is an induced torus) and the variety X has a k_v-point for any place v of k, we see that Y has a k_v-point for any v. By Lemma 4.4 the map $\pi_*\colon \text{Б}(Y)^D \to \text{Б}(X)^D$ is an isomorphism. By (1.2) $\pi_*(m_{\mathrm{H}}(Y)) = m_{\mathrm{H}}(X)$. But $m_{\mathrm{H}}(X) = 0$, hence $m_{\mathrm{H}}(Y) = 0$. By Proposition 3.5(i) the variety Y has a k-point y. It follows that there is a k-point $x = \pi(y)$ in X, which proves Theorem 2.2. □

4.6. *Proof of Theorem 2.4 modulo Lemma 4.3.* Since $H^1(k, T) = 0$ and X has a k-point, we see that Y has a k-point. Let x_S be a point of $X(k_S)$. Since $H^1(k_v, T) = 0$ for any v, there exists a point $y_S \in Y(k_S)$ such that $\pi(y_S) = x_S$. By Lemma 4.4 the map $\pi_*\colon (\text{Б}_S(Y)/\text{Б}(Y))^D \to (\text{Б}_S(X)/\text{Б}(X))^D$ is an isomorphism. By (1.3) we have $\pi_*(m_{\mathrm{W,S}}(Y)(y_S)) = m_{\mathrm{W,S}}(X)(x_S)$. But $m_{\mathrm{W,S}}(X)(x_S) = 0$, hence $m_{\mathrm{W,S}}(Y)(y_S) = 0$. By Proposition 3.5(ii) the point y_S is contained in $Y(k)\hat{}_S$. It follows that $x_S \in X(k)\hat{}_S$, which proves Theorem 2.4. □

4.7. We will give a cohomological criterion of existence of a k-form (Y, π) of $(\bar{Y}, \bar{\pi})$ where Y is a homogeneous space of F and π is an F-equivariant map.

An element $\sigma \in \mathrm{Gal}(\bar{k}/k)$ acts on $X_{\bar{k}}$ by a σ-semialgebraic automorphism $\sigma_*\colon X_{\bar{k}} \to X_{\bar{k}}$. We can lift σ_* to a σ-semialgebraic automorphism $a_\sigma\colon \bar{Y} \to \bar{Y}$ as follows. As before, let $\bar{x} \in X(\bar{k})$ be a $\bar{k}$-point of X and $\bar{H} \subset G_{\bar{k}}$ its stabilizer. We identify $X_{\bar{k}}$ with $\bar{H}\backslash G_{\bar{k}}$. By definition

$\bar{Y} = \bar{H}\backslash F_{\bar{k}}$. Let $\sigma \in \mathrm{Gal}(\bar{k}/k)$. Choose $g_\sigma \in G(\bar{k})$ such that ${}^\sigma\bar{x} = \bar{x} \cdot g_\sigma$. We set

$$a_\sigma\big(\bar{H} \cdot (g,t)\big) = \bar{H} \cdot (g_\sigma \cdot {}^\sigma g, {}^\sigma t)\ .$$

Then $a_\sigma \colon \bar{Y} \to \bar{Y}$ is compatible with $\sigma_* \colon X_{\bar{k}} \to X_{\bar{k}}$, i.e. $\bar{\pi} \circ a_\sigma = \sigma_* \circ \bar{\pi}$.

The automorphism a_σ depends on the choice of g_σ, and in general $a_{\sigma\tau} \neq a_\sigma \cdot {}^\sigma a_\tau$ for $\sigma, \tau \in \mathrm{Gal}(\bar{k}/k)$. However one can check that $\mathrm{Aut}_{\,F,X}\bar{Y} = T(\bar{k})$, and therefore we may write

$$a_\sigma \cdot {}^\sigma a_\tau = d_{\sigma,\tau}\ a_{\sigma\tau}$$

with $d_{\sigma,\tau} \in T(\bar{k})$. One checks immediately that $d_{\sigma,\tau}$ is a cocycle of $\mathrm{Gal}(\bar{k}/k)$ with coefficients in $T(\bar{k})$ and that its class $\eta \in H^2(k,T)$ does not depend on the choice of the family $(a_\sigma)_{\sigma \in \mathrm{Gal}(\bar{k}/k)}$.

Lemma 4.8. *The cohomology class $\eta \in H^2(k,T)$ constructed in 4.7 equals zero if and only if there exists a k-form (Y,π) of the pair $(\bar{Y},\bar{\pi})$.*

Idea of proof. If there exists a k-form (Y,π) of $(\bar{Y},\bar{\pi})$, we can set $a_\sigma = \sigma_* \colon Y_{\bar{k}} \to Y_{\bar{k}}$. Then $a_{\sigma\tau} = a_\sigma \cdot {}^\sigma a_\tau$, hence $\eta = 0$. Conversely, it is easy to check that if $\eta = 0$ then we can choose $(a_\sigma)_{\sigma \in \mathrm{Gal}(\bar{k}/k)}$ such that $a_{\sigma\tau} = a_\sigma \cdot {}^\sigma a_\tau$. The family $(a_\sigma)_{\sigma \in \mathrm{Gal}(\bar{k}/k)}$ is descent data defining a k-form (Y,π) of $(\bar{Y},\bar{\pi})$. □

Note that if X has a k-point, then, as we have seen in 4.2, there exist a k-form (Y,π) of $(\bar{Y},\bar{\pi})$, hence $\eta = 0$.

4.9. *Proof of Lemma 4.3.* By hypothesis X has a k_v-point for any place v of k. It follows that the localization $\mathrm{loc}_v\eta \in H^2(k_v,T)$ equals zero for any $v \in \mathcal{V}$. This means that

$$\eta \in Ш^2(k,T) := \ker[H^2(k,T) \to \prod_{v \in \mathcal{V}} H^2(k_v,T)]\ .$$

But $Ш^2(k,T) = 0$ because T is an induced torus, cf. e.g. [Sa], (1.9.1). It follows that $\eta = 0$. By Lemma 4.8 there exist a k-form (Y,π) of $(\bar{Y},\bar{\pi})$. Lemma 4.3 is proved. □

This completes the proof of Theorems 2.2, 2.3, and 2.4.

5. Connected stabiliser

In this section we prove Corollary 2.5.

Lemma 5.1. *Let G be a connected group over any field k. Then there exists an extension*

$$1 \to Z \to G' \to G \to 1$$

such that Z is a torus and $(G')^{\mathrm{ssu}}$ is simply connected.

Proof. By Levi's theorem G is isomorphic to the semidirect product $G^{\mathrm{u}} \rtimes G^{\mathrm{red}}$. By [La], pp. 228–229, [MSh], 3.1, for the reductive group G^{red} there exists an extension

$$1 \to Z \to G'' \to G^{\mathrm{red}} \to 1$$

such that Z is a torus and $(G'')^{\mathrm{ss}}$ is simply connected. We set $G' = G^{\mathrm{u}} \rtimes G''$ □

Lemma 5.2. *Let X be a homogeneous space of a connected k-group with connected stabilizer $\bar{H} \subset G_{\bar{k}}$. Then X is a homogeneous space of another k-group G' with connected stabilizer $\bar{H}' \subset G'_{\bar{k}}$ such that the group $(G')^{\mathrm{ssu}}$ is simply connected.*

Proof. Let $G' \to G$ be the extension constructed in Lemma 5.1. Then $(G')^{\mathrm{ssu}}$ is simply connected. For $\bar{x} \in X(\bar{k})$ set $\bar{H} = \mathrm{Stab}_G(\bar{x})$, $\bar{H}' = \mathrm{Stab}_{G'}(\bar{x})$. From the exact sequence

$$1 \to Z_{\bar{k}} \to \bar{H}' \to \bar{H} \to 1$$

we see that $\bar{H}'$ is connected. □

5.3. *Proof of Corollary 2.5.* In the second case $\bar{H}/\bar{H}^{\mathrm{ssu}}$ is clearly abelian. (Note that in this case $\bar{H}$ is defined over k.) In the first case by Lemma 5.2, X is a homogeneous space of some connected k-group G' such that $(G')^{\mathrm{ssu}}$ is simply connected, with connected stabilizer $\bar{H}'$. We have $\bar{H}'/(\bar{H}')^{\mathrm{ssu}} = (\bar{H}')^{\mathrm{tor}}$, hence $\bar{H}'/(\bar{H}')^{\mathrm{ssu}}$ is abelian. We can now apply Theorems 2.2 and 2.4 to the pair (G', X). The corollary is proved. □

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Raymond and Beverly Sackler School of Mathematical Sciences, Tel Aviv University, 69978 Tel Aviv, Israel

E-mail address: borovoi@math.tau.ac.il