Non-abelian hypercohomology of a group with coefficients in a crossed module, and Galois cohomology

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Introduction

Here we develop a hypercohomology theory of a group with coefficients in a crossed module, and apply it to define abelianization maps for Galois cohomology of reductive algebraic groups.

Let $\Gamma$ be a group and let

$\begin{array}{c}
1 \\
\rightarrow \\
F \\
\alpha \\
G \\
\rightarrow \\
1
\end{array}$

be a short complex (a complex of length 2) of (in general non-abelian) groups, where the numbers $-1$ and $0$ over the letters denote the degrees: $F$ is in degree $-1$ and $G$ is in degree $0$. We assume that the group $\Gamma$ acts on $F$ and $G$, and that $\alpha$ is a homomorphism of $\Gamma$-groups.

For applications to the Galois cohomology of connected algebraic groups, we would like to be able to define the first hypercohomology set $H^1(\Gamma, F \rightarrow G)$ in a functorial way. In general this is not likely to be possible. Indeed, if we take $G$ to be $\{1\}$, then we must have $H^1(\Gamma, F \rightarrow 1) = H^2(\Gamma, F)$. However, as far as I know, there is no functorial definition of second cohomology in the non-abelian case (the second cohomology theory of Springer [Sp] and Giraud [Gi] is obviously non-functorial).

Fortunately it is possible to define the first hypercohomology in a functorial way when $F \rightarrow G$ is a crossed module. A crossed module is a group homomorphism $F \xrightarrow{\alpha} G$ with an action of $G$ on $F$ satisfying certain natural conditions (see 2.1 for the precise definition and [BHu] for a survey). The notion of a crossed module was introduced in 1946 by J.H.C. Whitehead [W1], [W2], who was motivated by topological problems.

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To write down hypercohomology exact sequences, what we need is not only $H^1$, but also $H^{-1}$ and $H^0$. In Section 1 for any short complex of $\Gamma$-groups $F \to G$ (not necessarily a crossed module), we define, in terms of cocycles, an abelian group $H^{-1}(F \to G)$ and pointed set $H^0(F \to G)$, where we write $H^i(F \to G)$ for $H^i(\Gamma, F \to G)$. These definitions were earlier given by Deligne [Del, 2.4.3], in terms of torsors. (Deligne writes $H^0$ for our $H^{-1}$, and $H^1$ for our $H^0$.)

For a crossed module $F \to G$ with $\Gamma$-action, we define in Section 2 a group structure on $H^0(\Gamma, F \to G)$. Then we define, again in cocyclic form, the first hypercohomology set $H^1(\Gamma, F \to G)$. We follow Dedecker [Ded2],[Ded3], who defined $H^i(\Gamma, F \to G)$ for a crossed module $F \to G$ with trivial $\Gamma$-action; the generalization to the case of non-trivial $\Gamma$-action is obvious. Note that Dedecker regards $H^1(\Gamma, F \to G)$ not as hypercohomology of a complex, but as a nice, functorial definition of the second cohomology $H^2(\Gamma, F)$, so the group $G$ and its action on $F$ are for him just auxiliary structures necessary to define $H^2(\Gamma, F)$. Dedecker writes $H^2(\Gamma, F \to G)$ for our $H^1(\Gamma, F \to G)$. We regard $H^1(\Gamma, F \to G)$ as hypercohomology of a complex, and write down the hypercohomology exact sequence associated to a short exact sequence of crossed modules.

In Section 3 we use hypercohomology exact sequences to prove

**Theorem** (Theorem 3.3) Let $(F_1 \to G_1) \to (F_2 \to G_2)$ be a quasi-isomorphism of crossed modules with $\Gamma$-action. Then the induced maps $H^i(F_1 \to G_1) \to H^i(F_2 \to G_2)$ ($i = -1,0,1$) are bijections.

In Section 4 we apply results of Sections 1–3 to the crossed module of algebraic groups $G^{sc} \to G$, introduced by Deligne ([Del, 2.4.7]). Here $G$ is a connected reductive algebraic group over a field $K$ of characteristic 0, $G^{sc}$ is the universal covering of the derived group $G^{ss}$ of $G$, the homomorphism $\rho$ is the composition $G^{sc} \to G^{ss} \to G$, and $G$ acts on $G^{sc}$ in the obvious way. Let $Z$ be the center of $G$ and $Z^{(sc)}$ the center of $G^{sc}$. Let $H^0(K,G)$ and $H^1(K,G)$ denote the 0-dimensional and 1-dimensional Galois cohomology of $G$. We define the abelian Galois cohomology groups of $G$ by

$$H^{1}_{ab}(K,G) = H^1(K, Z^{(sc)} \to Z) \quad (i \geq -1).$$

Using the morphism $(1 \to G) \to (G^{sc} \to G)$ and the quasi-isomorphism $(Z^{(sc)} \to Z) \to (G^{sc} \to G)$ of crossed modules of algebraic groups, we define for $i = 0, 1$ the abelianization maps

$$\text{ab}^0: H^0(K,G) \to H^0(G^{sc} \to G) \to H^1(Z^{(sc)} \to Z) = H^0_{ab}(K,G).$$

The abelianization map $\text{ab}^0$ was first defined by Deligne [Del]. The map $\text{ab}^1$ generalizes a map of Kottwitz ([Ko2], Thm. 1.2), which he defined and extensively used in the case when $K$ is a local field. Kottwitz defined the abelianization map with the help of a rather complicated method of $z$-extensions of reductive groups. The hypercohomology with coefficients in a crossed module permits us to define the maps $\text{ab}^0$ and $\text{ab}^1$ explicitly, in particular in terms of cocycles (Propositions 4.3.1 and 4.3.2).

Constructions of Section 4 are used in our forthcoming paper [Bo3] (cf. also [Bo1]), where we describe “explicitly” the first Galois cohomology of a connected reductive group.
over a number field. Such constructions are are useful in cohomological calculations related to Shimura varieties, cf. [Mi].

Note that it is also possible to define the abelianization map \( \text{ab}^2 : H^2(K, G) \to H^2_{\text{ab}}(K, G) \) (cf. [Bo2]), where \( H^2(K, G) \) is the second non-abelian Galois cohomology set of Springer [Sp] and Giraud [Gi]. If \( K \) is a local field or a number field and \( \eta \in H^2(K, G) \), then \( \text{ab}^2(\eta) = 0 \) if and only if \( \eta \) is a neutral class, i.e. it corresponds to a split extension.

**Remarks**

1. The cocyclic constructions of Sections 1–3 go through in the more general case of hypercohomology of a simplicial set with coefficients in a family of crossed modules.

2. In [Br1] Breen defines \( H^{-1}, H^0 \) and \( H^1 \) in a uniform way for a sheaf of crossed modules \( F \to G \) on a site, and constructs the hypercohomology exact sequence (0.1.1). Breen uses the machinery of homotopical algebra. In the particular case of the site of \( \Gamma \)-sets his definitions appear to be equivalent to ours. Our results were obtained independently (before the paper [Br1] appeared).

3. Breen ([Br1], 6.2) proves that \( H^1(F \to G) \) can be identified with the set of equivalence classes of torsors under the Picard category associated to the crossed module \( F \to G \). Deligne noticed (private communication) that our Theorem 3.3 follows from this description of \( H^1(F \to G) \), because quasi-isomorphic crossed modules define equivalent Picard categories.

4. We claim no originality. Most of the results of Sections 1–3 are known (except the construction of the connecting map \( H^1 \to H^2 \) in 2.17 – 2.22), see Remarks (2–3) above. We need however this cocyclic exposition for Section 4, where we write down explicit cocyclic formulas for \( \text{ab}^0 \) and \( \text{ab}^1 \).

1 Hypercohomology in degrees \(-1\) and \(0\)

1.1 Short complexes of groups. Let \( \Gamma \) be a pro-finite group. A discrete \( \Gamma \)-group is a group \( G \) endowed with a left action of \( \Gamma \) which is continuous with respect to the discrete topology on \( G \). Here ”continuous” means that the stabilizer of any element \( g \in G \) is open in \( \Gamma \). From now on, by a \( \Gamma \)-group we mean a discrete \( \Gamma \)-group.

Let \( \alpha : F \to G \) be a morphism of \( \Gamma \)-groups, i.e. a group homomorphism respecting the action of \( \Gamma \). We consider \( F \to G \) as a short complex

\[
1 \to F \to G \to 1
\]

where \( F \) is in degree \(-1\) and \( G \) is in degree 0.

1.2 Hypercohomology. We define hypercohomology in degree \(-1\). We set

\[
H^{-1}(F \xrightarrow{\alpha} G) = (\ker \alpha)^\Gamma
\]

where \((\ )^\Gamma\) means (the group of) invariants.
We define 0-hypercohomology. We write Maps(\(\Gamma, F\)) for the set of continuous maps \(\varphi: \Gamma \to F\) and set
\[ C^0 = \text{Maps}(\Gamma, F) \times G \] (we regard \(C^0\) as a set)
\[ Z^0 = \{(\varphi, g) \in C^0 \mid \varphi(\sigma\tau) = \varphi(\sigma) \cdot \sigma \varphi(\tau), \ \sigma g = \alpha(\varphi(\sigma)^{-1}) \cdot g, \ \sigma, \tau \in \Gamma\} \]
The group \(F\) acts on the set of 0-cocycles \(Z^0\) on the right by
\[(\varphi, g) \ast f = (\varphi', g'), \ \varphi'(\sigma) = f^{-1} \cdot \varphi(\sigma) \cdot \sigma f, \ g' = \alpha(f)^{-1} \cdot g,\]
(\text{where } f \in F), \text{ and we set }
\[ H^0(F \to G) = Z^0 / F \]
The set \(H^0(F \to G)\) has a neutral element, namely the class of (1, 1). We write \(\text{Cl}(\varphi, g)\) for the hypercohomology class of a cycle \((\varphi, g)\).

1.3 Morphisms of complexes. A morphism of (short) complexes \((F_1 \to G_1) \to (F_2 \to G_2)\) is a commutative diagram
\[
\begin{array}{ccc}
F_1 & \longrightarrow & F_2 \\
\downarrow & & \downarrow \\
G_1 & \longrightarrow & G_2
\end{array}
\]
of \(\Gamma\)-groups. Such a morphism induces a canonical homomorphism
\[ H^{-1}(F_1 \to G_1) \to H^{-1}(F_2 \to G_2) \]
and a canonical map
\[ H^0(F_1 \to G_1) \to H^0(F_2 \to G_2) \]

1.4 Examples
(1) \(H^0(1 \to G) = H^0(G) = G^\Gamma\).
(2) \(H^0(F \to 1) = H^1(F)\). To \(\text{Cl}(\varphi, 1) \in H^0(F \to 1)\) we associate \(\text{Cl}(\varphi) \in H^1(F)\).
(3) If \(\alpha: F \to G\) is injective, then the morphism of complexes \((F \to G) \to (1 \to G / \alpha(F))\) induces a canonical bijection \(H^0(F \to G) \cong H^0(\text{coker } \alpha)\).
(4) If \(\alpha: F \to G\) is surjective, then the embedding \((\ker \alpha \to 1) \hookrightarrow (F \to G)\) of complexes induces a canonical bijection \(H^1(\ker \alpha) \cong H^0(F \to G)\).

In the rest of this section we define the hypercohomology exact sequence associated to a short exact sequence of complexes of \(\Gamma\)-groups.

1.5 Exact sequences. A short exact sequence of complexes of \(\Gamma\)-groups is a sequence
\[ 1 \to (F_1 \to G_1) \xrightarrow{i} (F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3) \to 1 \]
such that the rows in the commutative diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & F_3 & \longrightarrow & 1 \\
\uparrow \alpha_1 & & \uparrow \alpha_2 & & \uparrow \alpha_3 & & \\
1 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & 1
\end{array}
\]

are exact. We regard \(F_1\) and \(G_1\) as subgroups of \(F_2\) and \(G_2\), respectively. For such an exact sequence we define the connecting map

\[
\delta_{-1}: \text{H}^{-1}(F_3 \to G_3) \to \text{H}^0(F_1 \to G_1)
\]

as follows.

Let \(f_3 \in \text{H}^{-1}(F_3 \to G_3) = (\ker \alpha_3)^\Gamma\). Choose \(f \in F_2\) such that \(f \text{ (mod } F_1) = f_3\). We define a 0-cochain \((\varphi_1, g_1) \in C^0(F_2 \to G_2)\) by

\[
\varphi_1(\sigma) = f \cdot \sigma f^{-1}, \quad g_1 = \alpha_2(f)
\]

It is easy to show that \((\varphi_1, g_1) \in Z^0(F_1 \to G_1)\).

We set \(\delta_{-1}(f_3) = \text{Cl}(\varphi_1, g_1) \in \text{H}^0(F_1 \to G_1)\). We leave to the reader to check that the map \(\delta_{-1}\) is defined correctly, i.e. \(\delta_{-1}(f_3)\) does not depend on the choice of the representative \(f \in F_2\) of \(f_3\).

1.6 Proposition. Let

\[
1 \to (F_1 \to G_1) \xrightarrow{i} (F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3) \to 1
\]

be an exact sequence of complexes of \(\Gamma\)-groups. Then the hypercohomology sequence

\[
(1.6.1) \quad 1 \to \text{H}^{-1}(F_1 \to G_1) \xrightarrow{i_*} \text{H}^{-1}(F_2 \to G_2) \xrightarrow{j_*} \text{H}^{-1}(F_3 \to G_3) \bigg\downarrow \delta_{-1} \xrightarrow{} \text{H}^0(F_1 \to G_1) \xrightarrow{i_*} \text{H}^0(F_2 \to G_2) \xrightarrow{j_*} \text{H}^0(F_3 \to G_3)
\]

is exact.

Note that exactness makes sense because \(\text{H}^{-1}(F_k \to G_k)\) is a group and \(\text{H}^0(F_k \to G_k)\) is a pointed set \((k = 1, 2, 3)\).

Proof. We prove the exactness at \(\text{H}^{-1}(F_3 \to G_3)\). It follows immediately from the definition of \(\delta_{-1}\) that \(\delta_{-1} \circ j_* = 1\). Conversely, suppose that \(f_3 \in (\ker \alpha_3)^\Gamma\) and \(\delta_{-1}(f_3) = 1\). Let \(f\) be a representative of \(f_3\) in \(F_2\). Then there exists \(f_1 \in F_1\) such that

\[
\alpha_1(f_1)^{-1} \cdot \alpha_2(f) = 1 \\
f_1^{-1} \cdot f \cdot \sigma f^{-1} \cdot \sigma f_1 = 1
\]
hence
\[ \alpha_2(f_1^{-1}f) = 1, \quad \sigma(f_1^{-1}f) = f_1^{-1}f \]

Set \( f' = f_1^{-1}f \). Then \( f'(\mod F_1) = f_3, \alpha_2(f') = 1, \) and \( \sigma f' = f' \) for any \( \sigma \in \Gamma \). Thus
\[ f' \in (\ker \alpha_2)^\Gamma = H^{-1}(F_2 \to G_2) \]
and \( f_3 = j_*(f') \). We have proved that \( f_3 \in \text{im} j_* \).

We leave the proof of the exactness at the other terms to the reader.

### 2 Crossed modules and \( H^1 \)

To define \( H^1(F \to G) \) we need an additional structure on \( F \to G \), namely the structure of crossed module.

**2.1 Definition.** A crossed module is a short complex (homomorphism) \( \alpha: F \to G \), endowed with a left action of \( G \) on \( F \) (denoted by \( (g, f) \mapsto \tilde{g} f \tilde{g}^{-1} \)) satisfying

\[
\begin{align*}
ff'f^{-1} &= \alpha(f)f' \\
\alpha(\sigma f) &= \sigma \alpha(f) \cdot g^{-1}
\end{align*}
\]
for any \( f, f' \in F, \ g \in G \).

We say that a group \( \Gamma \) acts on a crossed module \( \alpha: F \to G \), if \( \Gamma \) acts on \( F \) and \( G \) such that
\[ \alpha(\sigma f) = \sigma \alpha(f), \quad \sigma(\sigma f) = \sigma \cdot \alpha(f) \cdot g^{-1} \quad \text{for any} \quad f \in \Gamma, \ g \in G, \ \sigma \in \Gamma. \]

**2.2 Examples of crossed modules.**

1. \( \alpha: F \to G \) where \( F \) is any (abelian) \( G \)-module, \( \alpha \) is trivial.
2. \( \alpha: F \hookrightarrow G \) where \( F \) is a normal subgroup of \( G \), \( \alpha: F \hookrightarrow G \) is the inclusion, \( \sigma f = g f g^{-1} \).
3. \( \alpha: F \to G \) where \( F \to G \) is any surjective homomorphism with central kernel. An element \( g \in G \) acts on \( F \) by \( \sigma f = \tilde{g} f \tilde{g}^{-1} \) where \( \tilde{g} \) is any lifting of \( g \) to \( F \).
4. \( F \to \text{Aut} F \) for any group \( F \), \( f \mapsto \text{int}(f) \).
5. Let \( X \) be a “nice” topological space, \( Y \subset X \) a subspace and \( x_0 \in Y \) a point. Then \( \pi_1(Y, x_0) \) acts on \( \pi_2(X, Y, x_0) \), and the complex \( \pi_2(X, Y, x_0) \xrightarrow{\partial} \pi_1(Y, x_0) \) (where \( \partial \) is the boundary homomorphism) is a crossed module.
6. Deligne’s crossed module \( \rho: G^{\text{sc}} \to G \) of algebraic groups, described in the Introduction.

**2.3 Remark.** J. H. C. Whitehead [W1], [W2], who introduced the notion of a crossed module, considered the crossed module 2.2(5). Dedecker showed in [Ded1], [Ded2] that
a crossed module $F \rightarrow G$ suits to define hypercohomology $H^1(X, F \rightarrow G)$ where $X$ is a group, a topological space and so on. For a survey on crossed modules see [BHu].

2.4. Lemma (cf. [BHu]). Let $F \xrightarrow{\alpha} G$ be a crossed module. Then

(i) the group $\ker \alpha$ is central in $F$;

(ii) $\ker \alpha$ is $G$-invariant;

(iii) $\text{im} \alpha$ is normal in $G$.

Proof. (i) follows from (2.1.1); (ii) and (iii) follow from (2.1.2).

2.5 Corollary. The action of $G$ on $F$ induces an action of $\text{coker} \alpha$ on the abelian group $\ker \alpha$.

2.6 The group structure on $H^0$. Let $F \rightarrow G$ be a crossed group with a $\Gamma$-action. We show that $C^0 = C^0(F \rightarrow G)$, $Z^0(F \rightarrow G)$ and $H^0(F \rightarrow G)$ have natural group structures.

The group $G$ acts on $\text{Maps}(\Gamma, F)$ by $(\varphi)(\sigma) = \varphi(\sigma)$ ($\varphi \in \text{Maps}(\Gamma, F)$, $\sigma \in \Gamma$). We define a group structure on $C^0$ by

$$(\varphi_1, g_1) \cdot (\varphi_2, g_2) = (g_1 \varphi_2 \cdot \varphi_1, g_1 g_2).$$

One can check that $Z^0$ is a subgroup of $C^0$ with respect to this group structure.

Consider the map $\nu: F \rightarrow Z^0$ defined by the formula $\nu(f) = (\varphi, \alpha(f))$ where $\varphi(\sigma) = f \cdot \sigma f^{-1}$. One can easily check that $\nu$ is a group homomorphism and its image is normal in $Z^0$. Moreover the right action of $F$ on $Z^0$ defined by

$$( (\varphi, g), f ) \longmapsto \nu(f^{-1}) \cdot (\varphi, g)$$

coincides with the action $\ast$ of 1.2. Thus $H^0(F \rightarrow G) = Z^0/\text{im} \nu$, and therefore $H^0(F \rightarrow G)$ has a canonical group structure. This group structure depends functorially on the crossed module $F \rightarrow G$.

2.7 Hypercohomology in degree 1. Let $F \rightarrow G$ be a crossed module with a $\Gamma$-action. Following Dedecker [Ded3] we define the first hypercohomology as follows.

Let $Z^1$ denote the set of pairs $(h, \psi) \in \text{Maps}(\Gamma \times \Gamma, F) \times \text{Maps}(\Gamma, G)$ such that for any $\sigma, \tau, \upsilon \in \Gamma$

$$\alpha(h(\sigma, \tau))^{-1} \cdot \psi(\sigma \tau) = \psi(\sigma) \cdot \sigma \psi(\tau)$$

$$h(\sigma, \tau \upsilon) \cdot \psi(\sigma) h(\tau, \upsilon) = h(\sigma \tau, \upsilon) \cdot h(\sigma, \tau).$$

We define a right action $Z^1 \times C^0 \rightarrow Z^1$ of the group of 0-cochains $C^0$ on the set of 1-cocycles $Z^1$. For $(a, g) \in C^0$ we set

$$(h, \psi) \ast (a, g) = (h', \psi')$$

where $h' = h(\sigma, \tau) \cdot (a \cdot \sigma g)$ and $\psi' = \psi(\sigma \tau) \cdot h(\sigma, \tau)$. The image of $\nu$ is a subgroup of $Z^1$, and the right action of $Z^1$ on $Z^1$ defined above coincides with the action $\ast$ of 1.1. Thus $H^1(F \rightarrow G)$ has a canonical group structure. This group structure depends functorially on the crossed module $F \rightarrow G$. 

7
\[ \psi'(\sigma) = g^{-1} \cdot \alpha(a(\sigma)) \cdot \psi(\sigma) \cdot g \]
\[ h'(\sigma, \tau) = g^{-1} \left[ a(\sigma \tau) \cdot h(\sigma, \tau) \cdot \psi(\sigma) \sigma a(\tau)^{-1} \cdot a(\sigma)^{-1} \right] \]

One can easily check that this is a group action.

Now we set
\[ H^1(F \to G) = Z^1/C_0. \]

The set \( H^1(F \to G) \) has a neutral element, namely the class of the trivial cocycle \((1, 1) \in Z^1\). We write \( \text{Cl}(h, \psi) \) for the hypercohomology class of 1-cocycle \((h, \psi)\).

### 2.8 Morphisms of crossed modules.

A morphism \( \varepsilon: (F_1 \to G_1) \to (F_2 \to G_2) \) of crossed modules is a pair of homomorphisms \((\varepsilon_0: G_1 \to G_2, \varepsilon_{-1}: F_1 \to F_2)\) such that the diagram
\[
\begin{array}{ccc}
F_1 & \xrightarrow{\varepsilon_{-1}} & F_2 \\
\downarrow{\alpha_1} & & \downarrow{\alpha_2} \\
G_1 & \xrightarrow{\varepsilon_0} & G_2
\end{array}
\]
commutes and \( \varepsilon_0(g) \varepsilon_{-1}(f) = \varepsilon_{-1}(g f) \) for any \( g \in G_1, f \in F_1 \).

A morphism \( \varepsilon \) of crossed modules with \( \Gamma \)-action defines homomorphisms
\[ \varepsilon_*: H^i(F_1 \to G_1) \to H^i(F_2 \to G_2) \quad (i = -1, 0) \]
and a map \( \varepsilon_*: H^1(F_1 \to G_1) \to H^1(F_2 \to G_2) \) that takes the neutral element to the neutral element. Thus \( H^{-1}, H^0 \) and \( H^1 \) are functors.

### 2.9 Examples.

1. \( H^1(1 \to G) = H^1(G) \).
2. \( H^1(F \to 1) = H^2(F) \) (note that in this case \( F \) is abelian and therefore \( H^2(F) \) makes sense). To \( \text{Cl}(h, 1) \in H^1(F \to 1) \) we associate \( \text{Cl}(h) \in H^2(F) \).
3. If \( F \xrightarrow{\alpha} G \) is a crossed module and \( \alpha \) is injective, then the morphism of complexes \((F \to G) \to (1 \to G/\alpha(F)) \) induces a canonical bijection \( H^1(F \to G) \cong H^1(\text{coker } \alpha) \).
4. If \( \alpha \) is surjective, then the embedding \((\ker \alpha \to 1) \hookrightarrow (F \to G) \) of crossed modules induces a bijection \( H^2(\ker \alpha) \cong H^1(F \to G) \). One can check that the map \( H^i(G) \to H^i(F \to G) = H^{i+1}(\ker \alpha) \) \((i = 0, 1)\) coincides with the connecting map \( \delta_i: H^i(G) \to H^{i+1}(\ker \alpha) \) associated to the short exact sequence \( 1 \to \ker \alpha \to F \to G \to 1 \) (see \[Se\], Ch. I, §5, for the definition of \( \delta_i \)).

In the rest of this section we prolong the hypercohomology exact sequence (1.6.1).
2.10 Let 
\[ 1 \to (F_1 \to G_1) \xrightarrow{i} (F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3) \to 1 \]
be an exact sequence of complexes of groups. We identify \( F_1 \to G_1 \) with its image in \( F_2 \to G_2 \). Assume that \( F_1 \to G_1 \) and \( F_2 \to G_2 \) are endowed with structures of crossed modules such that \( i \) is a morphism of crossed modules. We assume also that

(2.10.1) \( F_1 \) is \( G_2 \)-invariant in \( F_2 \).

Then \( G_2 \) acts on \( F_3 \cong F_2/F_1 \). We do not assume that \( (F_3 \to G_3) \) is a crossed module.

We define a left action of the group \( H^0(F_2 \to G_2) \) on the set \( H^0(F_3 \to G_3) \) by

\[ \text{Cl}(\varphi_2, g_2) \cdot \text{Cl}(\varphi_3, g_3) = \text{Cl}(\varphi_2 \cdot j(\varphi_3), j(g_2) \cdot g_3) \]

One can check that this is a correctly defined group action.

2.11 The connecting map. Let a short exact sequence

\[ 1 \to (F_1 \to G_1) \to (F_2 \to G_2) \to (F_3 \to G_3) \to 1 \]

be as in 2.10. We define the connecting map

\[ \delta_0: H^0(F_3 \to G_3) \to H^1(F_1 \to G_1) \]

as follows.

Let \( \xi_3 \in H^0(F_3 \to G_3) \), \( \xi_3 = \text{Cl}(\varphi_3, g_3) \), \( (\varphi_3, g_3) \in Z^0(F_3 \to G_3) \). We lift \( (\varphi_3, g_3) \) to some \( (\varphi, g) \), \( \varphi \in \text{Maps}(\Gamma, F_2) \), \( g \in G_2 \). We set

\[ \psi_1(\sigma) = g^{-1} \cdot \alpha_2(\varphi(\sigma)) \cdot \sigma g \]
\[ h_1(\sigma, \tau) = g^{-1}[\varphi(\sigma \tau) \cdot \sigma \varphi(\tau)^{-1} \cdot \varphi(\sigma)^{-1}] \cdot g^{-1} \cdot \alpha_2(\varphi(\sigma)) \cdot \sigma \sigma g \]
\[ = h_1(\sigma, \tau)^{-1} \cdot \psi_1(\sigma \tau)^{-1} \cdot \psi_1(\sigma \tau) \cdot h_1(\sigma, \tau) \]

Then \( \psi_1(\sigma) \in G_1 \) and \( h_1(\sigma, \tau) \in F_1 \) for any \( \sigma, \tau \in \Gamma \) (we use (2.10.1)).

We show that \( (h_1(\sigma), \psi_1) \in Z^1(F_1 \to G_1) \). We have

\[ \psi_1(\sigma) \cdot \sigma \psi_1(\tau) = g^{-1} \cdot \alpha_2(\varphi(\sigma)) \cdot \sigma g \cdot g^{-1} \cdot \alpha_2(\varphi(\sigma)) \cdot \sigma \sigma g \]
\[ = g^{-1} \cdot \alpha_2(\varphi(\sigma)) \cdot \sigma \varphi(\tau) \cdot \varphi(\sigma \tau)^{-1} \cdot g \cdot g^{-1} \cdot \alpha_2(\varphi(\sigma)) \cdot \sigma \sigma g \]
\[ = \alpha_2(h_1(\sigma, \tau)^{-1}) \cdot \psi_1(\sigma \tau) \cdot h_1(\sigma, \tau) \]

Hence \( (h_1, \psi_1) \in Z^1(F_1 \to G_1) \).

We set \( \delta_0(\xi_3) = \text{Cl}(h_1, \psi_1) \in H^1(F_1 \to G_1) \). We leave to the reader to check that \( \delta_0(\xi_3) \) is defined correctly.
2.12 Proposition. Let

\[(2.12.1) \quad 1 \to (F_1 \to G_1) \overset{i}{\to} (F_2 \to G_2) \overset{j}{\to} (F_3 \to G_3) \to 1 \]

be a short exact sequence of complexes of \(\Gamma\)-groups where \(i\) is an embedding of crossed modules with \(\Gamma\)-action. We identify \((F_1 \to G_1)\) with its image in \((F_2 \to G_2)\) and assume that the subgroup \(F_1 \subset F_2\) is \(G\)-invariant. Then

(i) the sequence

\[(2.12.2) \quad 1 \to H^{-1}(F_1 \to G_1) \overset{i^*}{\to} H^{-1}(F_2 \to G_2) \overset{j^*}{\to} H^{-1}(F_3 \to G_3) \]

\[\overset{\delta_0}{\to} H^0(F_1 \to G_1) \overset{i^*}{\to} H^0(F_2 \to G_2) \overset{j^*}{\to} H^0(F_3 \to G_3) \]

is exact.

(ii) \(\delta_0\) defines a bijection

\[(2.12.3) \quad H^0(F_2 \to G_2) \setminus H^0(F_3 \to G_3) \xrightarrow{\sim} \ker[H^1(F_1 \to G_1) \to H^1(F_2 \to G_2)].\]

Proof. We leave the proof of (ii) to the reader. To prove (i) we must prove the exactness at the terms \(H^0(F_3 \to G_3)\) and \(H^1(F_1 \to G_1)\). We leave the proof for \(H^0(F_3 \to G_3)\) to the reader.

We prove the exactness at \(H^1(F_1 \to G_1)\). It is clear that \(i_* \circ \delta_0 = 1\). Indeed, the cocycle \((h_1, \psi_1) \in Z^1(F_1 \to G_1)\) constructed in 2.11 is cohomologous to \((1, 1)\) in \(Z^1(F_2 \to G_2)\).

Conversely, let \(\eta_1 \in H^1(F_1 \to G_1)\), \(\eta_1 = \text{Cl}(h_1, \psi_1)\). Assume that \(i_*(\eta_1) = 1\). Then

\[
\psi_1(\sigma) = g^{-1} \cdot a(\sigma) \cdot \alpha^g
\]

\[
h_1(\sigma, \tau) = g^{-1} [a(\sigma \tau) \cdot \alpha a(\tau)^{-1} \cdot \alpha(\sigma)^{-1}]
\]

for some \(a: \Gamma \to F_2\), \(g \in G_2\). Set \(g_3 = g(\text{mod } G_1) \in G_3\), \(\varphi_3(\sigma) = a(\sigma)(\text{mod } F_1) \in F_3\). Using (2.10.1) one can easily check that \((\varphi_3, g_3) \in Z^0(F_3 \to G_3)\). Set \(\xi_3 = \text{Cl}(\varphi_3, g_3)\); then \(\eta_1 = \delta_0(\xi_3)\). Thus \(\eta_1 \in \text{im } \delta_0\), which was to be proved.

2.12.4 Remark. The hypercohomology exact sequence (2.12.2) depends on the short exact sequence (2.12.1) functorially.

2.13 Corollary. Let \(F \overset{\alpha}{\to} G\) be a crossed module with \(\Gamma\)-action.

(i) ([Br1], (4.2.2)). There is an exact sequence

\[(2.13.1) \quad 1 \to H^{-1}(F \to G) \overset{\lambda_1}{\to} H^0(F) \overset{\alpha^*}{\to} H^0(G) \overset{\kappa_0}{\to} H^0(F \to G) \]

\[\overset{\lambda_0}{\to} H^1(F) \overset{\alpha^*}{\to} H^1(G) \overset{\kappa_1}{\to} H^1(F \to G).\]
(ii) The map $\alpha_* : H^1(F) \to H^1(G)$ defines a bijection

\[
\text{H}^0(F \to G) \setminus \text{H}^1(F) \sim \ker[\kappa_1 : H^1(G) \to \text{H}^1(F \to G)]
\]

Here $\alpha_* : H^i(F) \to H^i(G)$ are the canonical maps induced by $\alpha$. The maps $\lambda_0$, $\kappa_0$, $\lambda_1$ and $\kappa_1$ can be described as follows:

\[
\lambda_1 : H^{-1}(F \to G) = (\ker \alpha)^G \hookrightarrow F^\Gamma = H^0(F), \ f \mapsto f
\]

\[
\kappa_0 : H^0(G) = G^\Gamma \to H^0(F \to G), \ g \mapsto \text{Cl}(1, g)
\]

\[
\lambda_0 : \text{Cl}(\varphi, g) \mapsto \text{Cl}(\varphi)
\]

\[
\kappa_1 : \text{Cl}(\psi) \mapsto \text{Cl}(1, \psi)
\]

2.13.3 Remark. The exact sequence (2.13.1), but without the last term, was earlier constructed by Deligne ([Del], (2.4.3.1)).

Proof. Consider the short exact sequence

\[
1 \to (1 \to G) \to (F \to G) \to (F \to 1) \to 1
\]

of complexes of $\Gamma$-groups, where $(1 \to G) \to (F \to G)$ is a morphism of crossed modules. The exact sequence (2.12.2) takes in our case the form (2.13.1), and the bijection (2.12.3) takes the form (2.13.2).

2.14 Twisting. To describe the fibers of the map $\kappa_1 : H^1(G) \to \text{H}^1(F \to G)$ we need twisting.

The group $G$ acts on the crossed module $(F \to G)$. An element $g_* \in G$ acts by

\[
f \mapsto g_* f, \ g \mapsto g_* gg_*^{-1} \ (f \in F, g \in G)
\]

Let $\psi \in Z^1(G)$. We can define the twisted crossed module $\psi(F \to G) = (\psi F \to \psi G)$, where the twisted groups $\psi F$ and $\psi G$ are the same $F$ and $G$ as abstract groups, but $\Gamma$ acts differently, namely,

\[
\sigma^* f = \psi(\sigma)^\sigma f, \ \sigma^* g = \psi(\sigma) \cdot \sigma \cdot \psi(\sigma)^{-1} \ (\sigma \in \Gamma, f \in F, g \in G).
\]

We define a map

\[
t_\psi : \text{H}^1(\psi(F \to G)) \to \text{H}^1(F \to G)
\]

taking $1 \in \text{H}^1(\psi(F \to G))$ to Cl$(1, \psi) \in \text{H}^1(F \to G)$. Let $(h', \psi') \in Z^1(\psi(F \to G))$. By definition this means that

\[
\psi'(\sigma) \cdot \psi(\sigma) \cdot \sigma \cdot \psi'(\tau) \cdot \psi(\sigma)^{-1} = \alpha(h'(\sigma, \tau))^{-1} \cdot \psi'(\sigma \tau)
\]

\[
h'(\sigma, \tau) \cdot \psi'(\sigma) \psi(\sigma)^\sigma h'(\tau, v) = h'(\sigma \tau, v) \cdot h'(\sigma, \tau).
\]
We set
\[ t_\psi(\text{Cl}(h', \psi')) = \text{Cl}(h', \psi' \psi) \]
One can easily check that the map \( t_\psi \) is defined correctly.

We can define a map \( t_\psi : H^1(\psi G) \rightarrow H^1(G) \) in a similar way. The diagram

\[
\begin{array}{ccc}
H^1(\psi G) & \xrightarrow{t_\psi} & H^1(G) \\
\downarrow_\psi \wr \omega_1 & & \downarrow_\omega_1 \\
H(\psi(F \rightarrow G)) & \xrightarrow{t_\psi} & H(F \rightarrow G)
\end{array}
\]
commutes.

\[ \textbf{2.15 Proposition.} \] \( \text{Let } (F \overset{\alpha}{\longrightarrow} G) \text{ be a crossed module with } \Gamma\text{-action. Consider the exact sequence (2.13.1). Let } \eta \in H^1(G), \eta = \text{Cl}(\psi), \psi \in Z^1(G). \text{ Then the fiber of } \omega_1 \text{ over } \omega_1(\eta) \text{ is in canonical bijection with the quotient set } H^0(\psi(F \rightarrow G)) \setminus H^1(\psi F). \)

\[ \text{Proof.} \text{ The map } t_\psi : H^1(\psi(F \rightarrow G)) \rightarrow H^1(F \rightarrow G) \text{ takes } 1 \text{ to } \text{Cl}(1, \psi) = \omega_1(\eta). \text{ Since the diagram (2.14.1) is commutative, the map } t_\psi : H^1(\psi G) \rightarrow H^1(G) \text{ takes the kernel of } \psi \omega_1 \text{ to the fiber of } \omega_1 \text{ over } \omega_1(\eta). \text{ By Corollary 2.13 (ii) the kernel of } \psi \omega_1 \text{ is in canonical bijection with } H^0(\psi(F \rightarrow G)) \setminus H^1(\psi F). \text{ This proves the proposition.} \]

\[ \textbf{2.16 Proposition } ([Br1], (5.1.3)). \] \( \text{Let } 1 \rightarrow (F_1 \rightarrow G_1) \overset{i}{\longrightarrow} (F_2 \rightarrow G_2) \overset{j}{\longrightarrow} (F_3 \rightarrow G_3) \rightarrow 1 \)
be an exact sequence of crossed modules with \( \Gamma\text{-action. Then the sequence } \)

\[
\begin{array}{c}
1 \longrightarrow H^{-1}(F_1 \rightarrow G_1) \overset{i}{\longrightarrow} H^{-1}(F_2 \rightarrow G_2) \overset{j}{\longrightarrow} H^{-1}(F_3 \rightarrow G_3) \\
\delta_1 \longrightarrow H^0(F_1 \rightarrow G_1) \overset{i}{\longrightarrow} H^0(F_2 \rightarrow G_2) \overset{j}{\longrightarrow} H^0(F_3 \rightarrow G_3) \\
\delta_0 \longrightarrow H^1(F_1 \rightarrow G_1) \overset{i}{\longrightarrow} H^1(F_2 \rightarrow G_2) \overset{j}{\longrightarrow} H^1(F_3 \rightarrow G_3)
\end{array}
\]
is defined and exact.

\[ \text{Proof.} \text{ Since } j \text{ is a morphism of crossed modules, the subgroup } F_1 \subset F_2 \text{ is } G_2\text{-invariant, and therefore the map } \delta_0 \text{ is defined. We must prove only the exactness at } H^1(F_2 \rightarrow G_2); \text{ we leave it to the reader.} \]

\[ \textbf{2.17 The case of a normal abelian submodule.} \text{ We want to prolong the exact sequence (2.16.1). We assume that the crossed submodule } (F_1 \rightarrow G_1) \subset (F_2 \rightarrow G_2) \text{ is abelian, i.e. } F_1 \text{ and } G_1 \text{ are abelian groups and } G_1 \text{ acts on } F_1 \text{ trivially. We assume also that} \]

12
(2.17.1) \( \alpha_2(F_2) \) commutes with \( G_1 \) in \( G_2 \);

(2.17.2) \( F_1 \) is central in \( F_2 \).

It follows from (2.17.1) and (2.17.2) that the group \( G_3 \) acts on the complex \((F_1 \to G_1)\) through coker \( \alpha_3 \). A cocycle \((h_3, \psi_3) \in Z^1(F_3 \to G_3)\) defines a cocycle \( \bar{\psi}_3 \in Z^1(\text{coker } \alpha_3) \), namely \( \bar{\psi}_3(\sigma) = \psi_3(\sigma)(\text{mod } \alpha_3(F_3)) \). Since coker \( \alpha_3 \) acts on the complex \((F_1 \to G_1)\), we can define the twisted complex \( \bar{\psi}_3(F_1 \to G_1) \). We write \( \psi_3(F_1 \to G_1) \) for \( \bar{\psi}_3(F_1 \to G_1) \).

We define a hypercohomology class \( \Delta_1(h_3, \psi_3) \in \mathbf{H}^2(\bar{\psi}_3(F_1 \to G_1)) \) as follows. We lift \( \psi_3 \) to some continuous map \( \psi : \Gamma \to G_2 \) and lift \( h_3 \) to some continuous map \( h : \Gamma \times \Gamma \to F_2 \). Then we set

\[
d_1(\sigma, \tau) = \psi(\sigma) \cdot \sigma \psi(\tau) \cdot \psi(\sigma \tau)^{-1} \cdot \alpha_2(h(\sigma, \tau)) \\
a_1(\sigma, \tau, \nu) = \psi(\sigma)^{\sigma} h(\tau, \nu)^{-1} \cdot h(\sigma, \tau \nu)^{-1} \cdot h(\sigma \tau, \nu) : h(\sigma, \tau)
\]

It is clear that \( d_1(\sigma, \tau) \in G_1 \), \( a_1(\sigma, \tau, \nu) \in F_1 \). We must show now that \((a_1, d_1) \in Z^2(\bar{\psi}_3(F_1 \to G_1))\), i.e.

\[
\psi(\sigma)^{\sigma} d_1(\tau, \nu)^{-1} d_1(\sigma, \tau) d_1(\sigma \tau, \nu) d_1(\sigma, \tau \nu)^{-1} = \alpha_1(a_1(\sigma, \tau, \nu)) \\
\psi(\sigma)^{\sigma} a_1(\tau, \nu, \rho) \cdot a_1(\sigma \tau, \nu, \rho)^{-1} \cdot a_1(\sigma, \tau \nu, \rho) \cdot a_1(\sigma, \tau, \nu \rho)^{-1} a_1(\sigma, \tau, \nu) = 1
\]

We skip this tedious (though not easy) calculation.

We set \( \Delta_1(h_3, \psi_3) \) = Cl\((a_1, d_1) \in \mathbf{H}^2(\bar{\psi}_3(F_1 \to G_1)) \). We must check that the cohomology class \( \delta_1(h_3, \psi_3) \) is defined correctly, i.e. it does not depend on the choice of the lifting \( (h, \psi) \) of \((h_3, \psi_3) \). We leave the check to the reader.

**2.18 Proposition.** Let

\[
1 \to (F_1 \to G_1) \xrightarrow{i} (F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3) \to 1
\]

be an exact sequence of crossed modules with \( \Gamma \)-action such that the crossed module \((F_1 \to G_1)\) is abelian and (2.17.1) and (2.17.2) hold. Let \((h_3, \psi_3) \in Z^1(F_3 \to G_3)\). Then Cl\((h_3, \psi_3) \in \text{im } j_* \) if and only if \( \Delta_1(h_3, \psi_3) = 1 \).

**Proof.** Left to the reader.

**2.19 The fibers of \( j_* \).** Let the exact sequence

\[
1 \to (F_1 \to G_1) \xrightarrow{i} (F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3) \to 1
\]

be as in Proposition 2.18. We want to describe the fibers of the map \( j_* : \mathbf{H}^1(F_2 \to G_2) \to \mathbf{H}^1(F_3 \to G_3) \).
Let
\[(h, \psi) \in Z^1(F_2 \to G_2), \eta_2 = \text{Cl}(h, \psi) \in H^1(F_2 \to G_2),\]
\[(h_3, \psi_3) = j(h, \psi), \eta_3 = j(\eta_2) = \text{Cl}(h_3, \psi_3) \in H^1(F_3 \to G_3).\]
We define a map \(t_{(h,\psi)} : H^1(\psi(F_1 \to G_1)) \to H^1(F_2 \to G_2)\) which takes 1 to \(\eta_2\). We set
\[t_{(h,\psi)}(\text{Cl}(h_1, \psi_1)) = \text{Cl}(hh_1, \psi_1).\]
One can check that \((hh_1, \psi_1) \in Z^1(F_2 \to G_2)\) and that the map \(t_{(h,\psi)}\) is defined correctly.

2.19.1 Lemma. The fiber of the map \(j_\ast : H^1(F_2 \to G_2) \to H^1(F_3 \to G_3)\) over \(\eta_3 = j_\ast(\eta_2)\) is the image of the map \(t_{(h,\psi)}\).

Proof. Easy.

2.20 Example. Let \(F \xrightarrow{\alpha} G\) be a crossed module with \(\Gamma\)-action. Consider the canonical exact sequence of crossed modules
\[(2.20.1) 1 \to (\ker \alpha \to 1) \xrightarrow{i} (F \to G) \xrightarrow{j} (F/\ker \alpha \lhd \to G) \to 1\]
The complex \((\ker \alpha \to 1)\) is abelian, and
\[
H^i(\ker \alpha \to 1) = H^{i+1}(\ker \alpha) \quad (i \geq -1),
H^i(\ker \alpha \to 1) = H^i(\ker \alpha) \quad (i \geq -1).
\]
Note that conditions (2.17.1) and (2.17.2) are satisfied. By Propositions 2.16 and 2.18, to the short exact sequence (2.20.1) we can associate the hypercohomology exact sequence
\[(2.20.2)
1 \to H^1(\ker \alpha) \xrightarrow{i_\ast} H^0(F \to G) \xrightarrow{j_\ast} (coker \alpha) \xrightarrow{\delta_0} H^2(\ker \alpha)
\to H^1(F \to G) \xrightarrow{j_\ast} H^1(coker \alpha) \to H^2(\psi(F/\ker \alpha))
\]
The arrow \(\cdots\to\) in (2.20.2) is not a map, it just indicates that if \(\eta_3 \in H^1(coker \alpha)\), \(\eta_3 = \text{Cl}(\psi_3)\), where \(\psi_3 \in Z^1(coker \alpha)\), then \(\eta_3\) comes from \(H^1(F \to G)\) if and only if \(\Delta_1(\psi_3) = 1\). The fiber \(j^{-1}_\ast(\eta_3)\) is described in Lemma 2.19.1

2.21 The case of a central submodule. Let
\[1 \to (F_1 \to G_1) \xrightarrow{i} (F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3) \to 1\]
be a short exact sequence. We identify the crossed module \((F_1 \to G_1)\) with its image in \((F_2 \to G_2)\).
We say that \((F_1 \to G_1)\) is central in \((F_2 \to G_2)\), if \(G_1\) is central in \(G_2\), \(F_1\) is central in \(F_2\), and \(G_2\) acts trivially on \(F_1\). Assume that \((F_1 \to G_1)\) is central in \((F_2 \to G_2)\). Then we can define the connecting map \(\delta_1: H^1(F_3 \to G_3) \to H^2(F_1 \to G_1)\).

Let \(\eta_3 \in H^1(F_3 \to G_3)\), \(\eta_3 = \text{Cl}(h_3, \psi_3)\). Then \(\Delta_1(h_3, \psi_3) \in H^2(F_1 \to G_1)\) (we write \(H^2(F_1 \to G_1)\) instead of \(H^2(\psi_3(F_1 \to G_1))\)) because \((F_1 \to G_1)\) is central in \((F_2 \to G_2)\).

One can check that \(\Delta_1(h_3, \psi_3)\) does not depend on the choice of the cocycle \((h_3, \psi_3)\) representing \(\eta_3\). We set \(\delta_1(\eta_3) = \Delta_1(h_3, \psi_3)\).

Propositions 2.16 and 2.18 imply

**2.22 Proposition.** Let

\[
1 \to (F_1 \to G_1) \xrightarrow{i} (F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3) \to 1
\]

be a short exact sequence of crossed modules with \(\Gamma\)-action, where the crossed submodule \((F_1 \to G_1)\) is central in \((F_2 \to G_2)\). Then the sequence

\[
H^1(F_1 \to G_1) \xrightarrow{i_*} H^1(F_2 \to G_2) \xrightarrow{j_*} H^1(F_3 \to G_3) \xrightarrow{\delta_1} H^2(F_1 \to G_1)
\]

is exact.

**3 Quasi-isomorphisms**

Let \((F_1 \xrightarrow{\alpha_1} G_1) \to (F_2 \xrightarrow{\alpha_2} G_2)\) be a morphism of crossed modules. Such a morphism induces group homomorphisms \(\text{ker}\ \alpha_1 \to \text{ker}\ \alpha_2\), \(\text{coker}\ \alpha_1 \to \text{coker}\ \alpha_2\).

**3.1 Definition.** A morphism \((F_1 \xrightarrow{\alpha_1} G_1) \to (F_2 \xrightarrow{\alpha_2} G_2)\) is called a quasi-isomorphism if the induced homomorphisms \(\text{ker}\ \alpha_1 \to \text{ker}\ \alpha_2\) and \(\text{coker}\ \alpha_1 \to \text{coker}\ \alpha_2\) are isomorphisms.

**3.2 Examples.**

(1) Let \((F \xrightarrow{\alpha} G)\) be a crossed module. If \(\alpha\) is injective, then \((F \to G) \to (1 \to \text{coker}\ \alpha)\) is a quasi-isomorphism. If \(\alpha\) is surjective, then \((\ker\ \alpha \to 1) \to (F \to G)\) is a quasi-isomorphism.

(2) The morphism of crossed modules of algebraic groups \((Z^{(\text{sc})} \to Z) \hookrightarrow (G^{\text{sc}} \to G)\), described in the introduction, is a quasi-isomorphism.

**3.3 Theorem.** Let \(\varepsilon: (F_1 \to G_1) \to (F_2 \to G_2)\) be a quasi-isomorphism of crossed modules with \(\Gamma\)-action. Then \(\varepsilon\) induces bijections

\[
\varepsilon_*: H^i(F_1 \to G_1) \to H^i(F_2 \to G_2)
\]
Proof. For $i = -1$ the assertion is obvious.

Let $i = 0$. From 2.20 we obtain a commutative diagram

$$
\begin{array}{cccccc}
1 & \rightarrow & H^1(\ker \alpha_1) & \rightarrow & H^0(F_1 \rightarrow G_1) & \rightarrow & H^2(\ker \alpha_1) \\
\downarrow \sim & & \downarrow & & \downarrow \sim & & \downarrow \sim \\
1 & \rightarrow & H^1(\ker \alpha_2) & \rightarrow & H^0(F_2 \rightarrow G_2) & \rightarrow & H^2(\ker \alpha_2)
\end{array}
$$

with exact rows. Three vertical arrows in this diagram are isomorphisms because $\varepsilon$ is a quasi-isomorphism. Then by the five-lemma the map $\varepsilon^*: H^0(F_1 \rightarrow G_1) \rightarrow H^0(F_2 \rightarrow G_2)$ is also an isomorphism, which was to be proved.

Let $i = 1$. From 2.20 we obtain a commutative diagram

$$
\begin{array}{cccccc}
(coker \alpha_1)^\Gamma & \rightarrow & H^2(\ker \alpha_1) & \rightarrow & H^1(F_1 \rightarrow G_1) & \rightarrow & H^1(\ker \alpha_1) & \rightarrow & H^3(\psi_1(\ker \alpha_1)) \\
\downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
(coker \alpha_2)^\Gamma & \rightarrow & H^2(\ker \alpha_2) & \rightarrow & H^1(F_2 \rightarrow G_2) & \rightarrow & H^1(\ker \alpha_2) & \rightarrow & H^3(\psi_2(\ker \alpha_2))
\end{array}
$$

with exact rows. Four vertical arrows in this diagram are bijections because $\varepsilon$ is a quasi-isomorphism. We prove the assertion by diagram chasing. To prove the surjectivity of the map $\varepsilon_*: H^1(F_1 \rightarrow G_1) \rightarrow H^1(F_2 \rightarrow G_2)$ we use Lemma 2.19.1.

4 Abelianization maps

Let $K$ be a field of characteristic 0, and $\bar{K}$ an algebraic closure of $K$. We set $\Gamma = \text{Gal}(\bar{K}/K)$.

The notions of a crossed module of algebraic groups and a quasi-isomorphism of crossed modules of algebraic groups are defined in the obvious way. If $F \rightarrow G$ is a crossed module of algebraic groups, then $F(\bar{K}) \rightarrow G(\bar{K})$ is a (discrete) crossed module with a $\Gamma$-action. We define the Galois hypercohomology of $F \rightarrow G$ by

$$
H^i(K, F \rightarrow G) = H^i(\Gamma, F(\bar{K}) \rightarrow G(\bar{K})) \quad (i = -1, 0, 1).
$$

We often abbreviate $H^i(K, F \rightarrow G)$ to $H^i(F \rightarrow G)$.

If $(F_1 \rightarrow G_1) \rightarrow (F_2 \rightarrow G_2)$ is a quasi-isomorphism of crossed modules of $K$-groups, then

$$(F_1(\bar{K}) \rightarrow G_1(\bar{K})) \rightarrow (F_2(\bar{K}) \rightarrow G_2(\bar{K}))$$

is a quasi-isomorphism of crossed modules with $\Gamma$-action, and by Theorem 3.3 we have a bijection $H^i(F_1 \rightarrow G_1) \cong H^i(F_2 \rightarrow G_2)$. 

16
4.1 Let $G$ be a connected reductive $K$-group. Let $G^{ss}$ denote its derived group (which is semisimple), and let $G^{sc} \to G^{ss}$ be the universal covering of $G^{ss}$. Consider the composition

$$
\rho: G^{sc} \to G^{ss} \to G.
$$

Then $G$ acts on $G^{sc}$, and $G^{sc} \xrightarrow{\rho} G$ is a crossed module of $K$-groups. Let $Z$ denote the center of $G$, and $Z^{(sc)}$ the center of $G^{sc}$.

Let $T \subset G$ be a maximal torus defined over $K$. We set $T^{(sc)} = \rho^{-1}(T)$. We define the abelian Galois cohomology $H^i_{ab}(K, G)$ (which we usually abbreviate to $H^i_{ab}(G)$) by

$$
H^i_{ab}(K, G) := H^i(K, T^{(sc)} \to T) = H^i(K, Z^{(sc)} \to Z) \quad (i \geq -1),
$$

where we identify the abelian groups $H^i(K, T^{(sc)} \to T)$ and $H^i(K, Z^{(sc)} \to Z)$ using the quasi-isomorphism $(Z^{(sc)} \to Z) \to (T^{(sc)} \to T)$ of abelian complexes. Note that $H^i_{ab}(K, \cdot)$ is a functor from the category of connected reductive $K$-group to the category of abelian groups. We are interested here in $H^0_{ab}$ and $H^1_{ab}$.

4.1.1 Lemma. For $i = 0, 1$ there is a canonical and functorial in $G$ bijection $H^i(G^{sc} \to G) \xrightarrow{\sim} H^i_{ab}(G)$, which is a group isomorphism when $i = 0$.

Proof. The assertion follows from Theorem 3.3, applied to the quasi-isomorphisms

$$(Z^{(sc)} \to Z) \to (T^{(sc)} \to T) \to (G^{sc} \to G).$$

4.1.2 Remark (essentially due to L. Breen). There is another, more intrinsic explanation of the fact that $H^1(G^{sc} \to G)$ has a canonical structure of abelian group. Deligne ([De], 2.0.2) noted that the commutator morphism

$$(g_1, g_2) \mapsto g_1 g_2 g_1^{-1} g_2^{-1} : \ G \times G \to G$$

can be uniquely lifted to a morphism

$$(g_1, g_2) \mapsto \{g_1, g_2\} : \ G \times G \to G^{sc},$$

and we have $\{g_1, g_2\} = \{g_2, g_1\}$. The crossed module $G^{sc} \to G$ together with the map $\{,\}$ is a stable crossed module in the terminology of Conduché ([Co], 3.1). To the crossed module $G^{sc} \to G$ one associates a (fibered) Picard category $\mathcal{C}(G^{sc} \to G)$ (cf. [Br2], the remark after Def. 1.1.6). The map $\{,\}$ defines a commutativity constraint in $\mathcal{C}(G^{sc} \to G)$, and thus turns it into a commutative Picard category (cf. [Br3]). A commutative Picard category is a categoric analogue of an abelian group, so the set of isomorphism classes of torsors under such a category has a canonical structure of abelian group. Since $H^1(G^{sc} \to G)$ is the set of isomorphism classes of torsors under $\mathcal{C}(G^{sc} \to G)$ ([Br1], 6.2), we see that $H^1(G^{sc} \to G)$ has a canonical structure of abelian group.

4.2 For $i = 0, 1$ we define the abelianization map $\text{ab}^i$ as the composition

$$
\text{ab}^i : H^i(G) \xrightarrow{\kappa_i} H(G^{sc} \to G) \xrightarrow{\sim} H(Z^{(sc)} \to Z) = H^i_{ab}(G),
$$

where the map $\kappa_i$ is induced by the imbedding $(1 \to G) \to (G^{sc} \to G)$ of crossed modules. By Corollary 2.13(i) we have an exact sequence

$$
(4.2.1) \quad G^{sc}(K) \xrightarrow{\rho} G(K) \xrightarrow{\text{ab}^0 H^0_{ab}(K, G)} H^1(K, G^{sc}) \xrightarrow{\rho^*} H^1(K, G) \xrightarrow{\text{ab}^1} H^1_{ab}(K, G)
$$

In [Bo3] (see also [Bo1]) we prove

17
4.2.2 Proposition. If $K$ is a local field of characteristic 0 (archimedean or not) or a number field, then the map $\text{ab}^1$ is surjective.

¿From Proposition 4.2.2 we deduce here

4.2.3 Corollary. If $K$ is a non-archimedean local field of characteristic 0, then the map $\text{ab}^1$ is bijective.

Proof. Consider the exact sequence (4.2.1). By Proposition 2.15 any fiber of the map $\text{ab}^1$ comes from $H^1(K, \psi G^{sc})$ where $\psi \in Z^1(K, G)$. Since $\psi G^{sc}$ is simply connected, by Kneser’s theorem ([Kn]) we have $H^1(K, \psi G^{sc}) = 1$, so the map $\text{ab}^1$ is injective. By Proposition 4.2.2 the map $\text{ab}^1$ is surjective. We conclude that the abelianization map $\text{ab}^1$ is bijective, which was to be proved.

We see that when $K$ is a non-archimedean local field, the set $H^1(K, G)$ has a canonical and functorial structure of abelian group. (This result is due to Kottwitz [Ko1], [Ko2] in a slightly less functorial form.)

4.3 We can now describe the abelianization maps $\text{ab}^i: H^i(K, G) \to H^i_{ab}(K, G) = H^i(K, Z^{sc}(\bar{K}) \to Z)$ ($i = 0, 1$) explicitly in terms of cocycles.

4.3.1 Proposition. Let $g \in H^0(k, G) = G(K)$. Write $g = \rho(g') \cdot z$ where $g \in G^{sc}(\bar{K})$, $z \in Z(\bar{K})$. Then $\text{ab}^0(g) = \text{Cl}(\varphi, z)$ where the map $\varphi: \Gamma \to Z^{sc}(\bar{K})$ is defined by $\varphi(\sigma) = (g')^{-1} \cdot \sigma g'$.

Proof. We have $(1, g) * g' = (\varphi, z)$ with the notation of 1.2. Thus the 0-cocycles $(1, g)$ and $(\varphi, z)$ are cohomological in $H^0(K, G^{sc} \to G)$. This proves the assertion.

4.3.2 Proposition. Let $\xi \in H^1(K, G)$ be a cohomology class, $\xi = \text{Cl}(\psi)$, $\psi \in Z^1(K, G)$. Write $\psi(\sigma) = \rho(\psi'(\sigma)) \cdot z(\sigma)$ for $\sigma \in \Gamma$, where $\psi': \Gamma \to G^{sc}(\bar{K})$ and $z: \Gamma \to Z^{sc}(\bar{K})$ are continuous maps. Then $\text{ab}^1(\xi) = \text{Cl}(h, z)$, where the map $h: \Gamma \times \Gamma \to Z^{sc}(\bar{K})$ is given by

$$h(\sigma, \tau) = \psi'(\sigma) \cdot \sigma \psi'(\tau) \cdot \psi'(\sigma \tau)^{-1}.$$ 

Proof. With the notation of 2.7 we have $(1, \psi) * ((\psi')^{-1}, 1) = (h, z)$. Thus the 1-cocycles $(1, \psi)$ and $(h, z)$ are cohomological in $H^1(K, G^{sc} \to G)$. This proves the assertion.

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18
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References


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