

ON WEAK APPROXIMATION IN HOMOGENEOUS SPACES
OF SIMPLY CONNECTED ALGEBRAIC GROUPS

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Introduction. Let K be an algebraic number field and let S be a finite set of its places. An algebraic variety X over K is said to satisfy *the condition of weak approximation with respect to S* if

$$(WA_S) \quad X(K) \text{ is dense in } \prod_{v \in S} X(K_v),$$

K_v denoting the completion of K at v . We say that X satisfies *the condition of weak approximation* if

$$(WA) \quad X \text{ satisfies } (WA_S) \text{ for any finite } S.$$

Set $K_S = \prod_{v \in S} K_v$; then $X(K_S) = \prod_{v \in S} X(K_v)$. Let $X(K)_S^\wedge$ denote the closure of $X(K)$ in $X(K_S)$.

Let T be an algebraic K -torus. Set $A_S(T) = T(K_S)/T(K)_S^\wedge$. The finite abelian group $A_S(T)$ is the defect of (WA_S) for T ; in other words, it is the measure of failure of (WA_S) for T . In particular, T satisfies (WA_S) if and only if $A_S(T) = 0$. The group $A_S(T)$ was studied by Voskresenskii (cf. [Vo1], [Vo2]). He related $A_S(T)$ to a certain group $H^1(K, \text{Pic } \mathbb{V}(T))$. This group is a birational invariant of T ; as Sansuc later showed, it can be computed in terms of the arithmetic Brauer group $\text{Br}_a T$ (cf. [Sa]).

For any connected algebraic K -group G one can define the set

$A_S(G) := G(K_S)/G(K)_S^{\hat{}}$. As above, $A_S(G)$ is the defect of (WA_S) for G . Sansuc [Sa] has generalized Voskresenskii's results. He proved that the subgroup $G(K)_S^{\hat{}}$ is normal in $G(K_S)$ and that the quotient group $A_S(G)$ is finite and abelian. He showed also that it is possible to compute $A_S(G)$ in terms of the arithmetic Brauer group $Br_a G$ (or in terms of $H^1(K, \text{Pic } \mathbb{V}(G))$), cf. [Sa], Propositions 8.9 and 9.8).

In this paper we consider the case of a homogeneous space. Let G be a simply connected algebraic K -group and H any connected K -subgroup of G . Set $X = H \backslash G$. We define a finite abelian group which we call the defect of (WA_S) for X . This group depends on H only; we denote it by $A_S(X) = \mathcal{U}_S(H)$. The group $\mathcal{U}_S(H)$ can be computed from the Picard group H . We can also compute $A_S(X)$ from $Br_a X$ in the same way as it was done in [Sa] for algebraic groups. Our obstruction to (WA_S) is related to the Brauer–Manin obstruction that has been constructed by Colliot–Thélène and Sansuc [CT–Sa] (see also [Sa], 8.13).

Remark 1. The author has recently obtained certain sufficient conditions for *strong* approximation for $X = H \backslash G$ (cf. [Brv1]).

Remark 2. A slightly more general case of $X = H \backslash G$ with G not necessarily simply connected (e.g. with G adjoint) is considered in [Brv3].

Remark 3. Some results concerning *the Hasse principle* for homogeneous spaces were obtained by Rapinchuk [Ra]. Note that this problem is more difficult than that of weak approximation.

Our constructions and proofs are based on the results of Kottwitz [Ko1], [Ko2], who generalized the duality theory of Tate and Nakayama from the case of tori to that of all connected reductive K -groups. We use the weak approximation theorem for simply

connected groups which is due to Kneser [Kn1], [Kn2] in all the cases except E_8 and to Harder [Ha3] in the E_8 case; see Platonov [Pl1], [Pl2] for a uniform proof. We use the Hasse principle for simply connected groups which is due to Kneser [Kn3] (see also [Kn4]) in the classical cases and to Harder [Ha1] (see also [Ha2]) in exceptional ones, except E_8 ; the proof for the E_8 case, initiated by Harder [Ha1] in 1966, has been recently completed by V.I. Chernousov [Ch]. The idea to relate weak approximation to the Brauer group is inspired by the ideas and results of Manín, Voskresenskii, Colliot–Thélène and Sansuc.

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Notation.

K is an algebraic number field, \bar{K} is an algebraic closure of K . \mathcal{V} (resp. \mathcal{V}_∞ , \mathcal{V}_f) is the set of places (resp. of infinite places, of finite places) of K . For any finite $S \subset \mathcal{V}$ we set $K_S = \prod_{v \in S} K_v$. We let A denote the adèle ring of K . For an algebraic K –group H we set

$$H^1(K_S, H) = \prod_{v \in S} H^1(K_v, H), \quad H^1(A, H) = \bigoplus_{v \in \mathcal{V}} H^1(K_v, H),$$

where \bigoplus denotes the subset of the direct product, consisting of the families (ξ_v) (where $\xi_v \in H^1(K_v, H)$) such that $\xi_v = 0$ for all v outside some finite set. We sometimes use the additive notation instead of the multiplicative one and write 0 instead of 1.

For a finite abelian group A we denote by A^\sim the dual group $\text{Hom}(A, \mathbb{Q}/\mathbb{Z})$.

For a connected algebraic K –group H let H^u denote its unipotent radical. Set

$H^{\text{red}} = H/H^u$; it is a reductive group. Note that H is simply connected if and only if H^{red} is a simply connected semisimple group. We let H^{ss} denote the derived group of H^{red} , and set $H^{\text{tor}} = H^{\text{red}}/H^{\text{ss}}$. Let H^{sc} denote the universal covering of H^{ss} . We have a canonical homomorphism $\rho : H^{\text{sc}} \longrightarrow H^{\text{ss}} \longrightarrow H^{\text{red}}$.

For any K -variety we write \bar{X} for $X_{\bar{K}}$. For a K -torus T we denote by $X_*(T)$ the cocharacter group $\text{Hom}(G_{mK}, T)$.

1. Main results

We are going to construct the finite abelian group $\mathcal{U}_S(H)$. Set $B(H) = (\text{Pic } H)^{\sim}$. For $v \in \mathcal{V}$ set $B_v(H) = B(H_{K_v})$. The canonical morphism $H_{K_v} \longrightarrow H$ induces a homomorphism $\lambda_v : B_v(H) \longrightarrow B(H)$. Let $B^{(S)}(H)$ denote the subgroup of $B(H)$ generated by the groups $\text{im } \lambda_v$ for $v \notin S$, and set $B'(H) = B^{(\emptyset)}(H)$. Now we set $\mathcal{U}_S(H) = B'(H)/B^{(S)}(H)$.

Theorem 1.1. Let G be a simply connected group over a number field K , and let H be a connected K -subgroup of G . Set $X = H \backslash G$. Then there exists a canonical surjective map $X(K_S) \longrightarrow \mathcal{U}_S(H)$ whose kernel is $X(K)_S^{\wedge}$.

We see that the condition (WA_S) for X is equivalent to the condition $\mathcal{U}_S(H) = 0$. In a sense $\mathcal{U}_S(H)$ is the defect of (WA_S) for $X = H \backslash G$.

Theorem 1.2. There is a canonical exact sequence

$$H^1(K, H) \xrightarrow{j_S} \prod_{v \in S} H^1(K_v, H) \xrightarrow{\chi_S} \mathcal{U}_S(H) \longrightarrow 0,$$

where j_S is the localization map.

Remark 1.3. One can show that the above exact sequence is functorial in H .

Theorem 1.4. $\mathcal{U}_S(H) = \mathcal{U}_S(H^{\text{tor}})$.

Now set $B^{(\omega)}(H) = \bigcap_S B^{(S)}(H)$, where S runs over all the finite subsets of \mathcal{V} . We set $\mathcal{U}_\omega(H) = B'(H)/B^{(\omega)}(H)$. It is clear that $\mathcal{U}_\omega(H) = 0$ if and only if $\mathcal{U}_S(H) = 0$ for any finite $S \subset \mathcal{V}$.

Corollary 1.5. $H \backslash G$ satisfies (WA) if and only if $\mathcal{U}_\omega(H) = 0$.

Corollary 1.6. Set $T = H^{\text{tor}}$. Let L/K be a Galois extension splitting T . Let $S_0 \subset \mathcal{V}$ be the finite set of places (ramified in L) with non-cyclic decomposition groups. Then $\mathcal{U}_S(H) = \mathcal{U}_{S \cap S_0}(H)$. In particular, if $S \cap S_0 = \emptyset$ then $H \backslash G$ satisfies (WA_S) .

Corollary 1.7. If H^{tor} splits over a cyclic extension of K (in particular, if $H^{\text{tor}} = 1$, i.e. H is semisimple), then $H \backslash G$ satisfies (WA_S) .

Corollary 1.8. (Real approximation). If $S \subset \mathcal{V}_\infty$ then $H \backslash G$ satisfies (WA_S) for any connected K -subgroup $H \subset G$.

Corollary 1.5 follows immediately from Theorem 1.1. Corollaries 1.7 and 1.8 follows from Corollary 1.6. To prove Corollary 1.6 note that $\mathcal{U}_S(H) = \mathcal{U}_S(T)$ and $\mathcal{U}_{S \cap S_0}(H) = \mathcal{U}_{S \cap S_0}(T)$ because of Theorem 1.4. By Theorem 1.2

$$\mathcal{U}_S(T) = \text{coker} [j_S: H^1(K, T) \longrightarrow \prod_{v \in S} H^1(K_v, T)] .$$

This cokernel is investigated in [Sa], where the notation \mathcal{U}_S is introduced. By Lemmas 1.5 and 1.8 of [Sa] $\mathcal{U}_S(\mathbb{T}) = \mathcal{U}_{S \cap S_0}(\mathbb{T})$. Hence $\mathcal{U}_S(\mathbb{H}) = \mathcal{U}_{S \cap S_0}(\mathbb{H})$. q.e.d.

Consider the arithmetic Brauer group $\text{Br}_a X$ (see [Sa]). Set $\mathcal{B}_S(X) = \ker [\text{Br}_a X \longrightarrow \prod_{v \notin S} \text{Br}_a(X_{K_v})]$, $\mathcal{B}(X) = \mathcal{B}_\emptyset(X)$.

Theorem 1.9. $\mathcal{U}_S(\mathbb{H}) = (\mathcal{B}_S(X)/\mathcal{B}(X))^\sim$.

Remark 1.10. One can show (cf. [Brv3]) that the group $A_S(X) := (\mathcal{B}_S(X)/\mathcal{B}(X))^\sim$ is the defect of (WA_S) for $X = \mathbb{H} \setminus G$ for any pair of connected K -groups $\mathbb{H} \subset G$ such that $A_S(G) = \text{III}(G) = 0$, where III denotes the Shafarevich-Tate group.

2. Results of Kottwitz

The group $B(\mathbb{H})$ is computed in [Kot1] in terms of a connected Langlands dual group for \mathbb{H} . We prefer to describe $B(\mathbb{H})$ in terms of the algebraic fundamental group $\pi_1(\mathbb{H})$ (cf. [Brv2], [Brv3]).

Let temporarily K be any field of characteristic 0, and let \mathbb{H} be a connected K -group. Choose a maximal torus $\mathbb{T} \subset \mathbb{H}^{\text{red}}$ (defined over K). We write $\mathbb{T}^{(\text{sc})}$ for $\rho^{-1}(\mathbb{T}) \subset \mathbb{H}^{\text{sc}}$ (see Notation).

Definition 2.1. $\pi_1(\mathbb{H}) = X_*(\mathbb{T})/\rho_*X_*(\mathbb{T}^{(\text{sc})})$. If $\mathbb{T}' \subset \mathbb{H}^{\text{red}}$ is another maximal torus, then there is a canonical isomorphism

$$X_*(\mathbb{T}')/\rho_*X_*(\mathbb{T}'^{(\text{sc})}) \xrightarrow{\sim} X_*(\mathbb{T})/\rho_*X_*(\mathbb{T}^{(\text{sc})})$$

of Galois modules (cf. [Brv2]). Thus the definition of the algebraic fundamental group $\pi_1(\mathbb{H})$ is correct.

One can easily see that π_1 is an exact functor from the category of connected K -groups to the category of $\text{Gal}(\overline{K}/K)$ -modules finitely generated over \mathbb{Z} . If $K = \mathbb{C}$ then $\pi_1(\mathbb{H})$ is the usual topological fundamental group $\pi_1^{\text{top}}(G(\mathbb{C}))$ (cf. [Brv2]); this justifies the term. For any K there is a canonical isomorphism of Galois modules $\pi_1(\mathbb{H}) = X^*(Z(\hat{\mathbb{H}}))$; where $Z(\hat{\mathbb{H}})$ is the center of a connected Langlands dual group for \mathbb{H} and $X^*(Z(\hat{\mathbb{H}}))$ is the character group of $Z(\hat{\mathbb{H}})$ (cf. [Brv2], [Brv3]).

We write Γ for $\text{Gal}(\overline{K}/K)$. Consider the torsion subgroup $(\pi_1(\mathbb{H})_\Gamma)_{\text{tors}}$ of the group of coinvariants $\pi_1(\mathbb{H})_\Gamma$.

Proposition 2.2. ([Kol] 2.4.1). $B(\mathbb{H}) = (\pi_1(\mathbb{H})_\Gamma)_{\text{tors}}$.

Remark 2.2.1. Kottwitz states and proves Proposition 2.2 in terms of $Z(\hat{\mathbb{H}})$.

Hereafter K is again a number field. Consider the homomorphism $\lambda_v : B_v(\mathbb{H}) \longrightarrow B(\mathbb{H})$. Set $M = \pi_1(\mathbb{H})$. One can show (cf. [Ko2], 2.5, or [Brv2], Sect. 5) that in terms of $\pi_1(\mathbb{H})$ the homomorphism λ_v is the obvious map $(M_{\Gamma_v})_{\text{tors}} \longrightarrow (M_\Gamma)_{\text{tors}}$, where Γ_v is a decomposition group of v in \overline{K} (defined up to conjugation). Let Δ be the image of Γ in $\text{Aut } \pi_1(\mathbb{H})$, and let L/K be the Galois extension of K in \overline{K} corresponding to $\ker[\Gamma \longrightarrow \Delta]$. Let Δ_v be the image of Γ_v in Δ . Then $\text{Gal}(L/K) = \Delta$, and Δ_v is a decomposition group of v in L .

Lemma 2.3. The subgroup $\text{im } \lambda_v \subset B(\mathbb{H})$ depends only on the conjugacy class of a decomposition group of v in L .

Proof. The homomorphism λ_v is the corestriction (i.e. obvious map)

$$(M_{\Gamma_v})_{\text{tors}} = (M_{\Delta_v})_{\text{tors}} \longrightarrow (M_{\Delta})_{\text{tors}} = (M_{\Gamma})_{\text{tors}}$$

where $M = \pi_1(\mathbb{H})$. Thus if $v, w \in \mathcal{V}$ and the groups Δ_v and Δ_w are equal (up to conjugation), then $\text{im } \lambda_v = \text{im } \lambda_w$. q.e.d.

Lemma 2.4. If $H^{\text{tor}} = 1$, then $\lambda_v : B_v(\mathbb{H}) \longrightarrow B(\mathbb{H})$ is surjective for any $v \in \mathcal{V}$.

Proof. Set $M = \pi_1(\mathbb{H})$. If $H^{\text{tor}} = 1$, then H^{red} is semisimple and M is finite. Hence the homomorphism

$$\lambda_v : B_v(\mathbb{H}) = (M_{\Gamma_v})_{\text{tors}} = M_{\Gamma_v} \longrightarrow M_{\Gamma} = (M_{\Gamma})_{\text{tors}} = B(\mathbb{H})$$

is surjective.

q.e.d.

In [Ko2], Theorem 1.2, Kottwitz constructs canonical maps $\beta_v : H^1(K_v, \mathbb{H}) \longrightarrow B_v(\mathbb{H})$ for $v \in \mathcal{V}$. He proves

Proposition 2.5. (Local non-archimedean Kottwitz theorem, [Ko1], 6.4, [Ko2], 1.2). If $v \in \mathcal{V}_f$ then β_v is bijective.

Set $\mu_v = \lambda_v \circ \beta_v : H^1(K_v, \mathbb{H}) \longrightarrow B_v(\mathbb{H}) \longrightarrow B(\mathbb{H})$. Define

$$\mu = \sum \mu_v : H^1(A, H) = \bigoplus H^1(K_v, H) \longrightarrow B(H)$$

$$\mu((\xi_v)) = \sum_v \mu_v(\xi_v) .$$

Consider the localization map $H^1(K, H) \longrightarrow \prod_{v \in \mathcal{V}} H^1(K_v, H)$. Since H is connected, one can easily show that the image of this map lies in $H^1(A, H) = \bigoplus H^1(K_v, H)$ (one can use Lang's theorem and Hensel's lemma). We denote the map $H^1(K, H) \longrightarrow H^1(A, H)$ by j .

Proposition 2.6. (Global Kottwitz theorem, [Ko2], 2.5, 2.6). $\ker \mu = \text{im } j$.

For future needs we set $\mu_S = \sum \mu_v : H^1(K_S, H) \longrightarrow B(H)$. The image of μ_S is contained in $B'(H)$. Set

$$\chi_S = \mu_S \text{ mod } B^{(S)} : H^1(K_S, H) \longrightarrow \mathcal{U}_S(H) .$$

3. The group $\mathcal{U}_S(H)$ and the Brauer group.

Proof of Theorem 1.4. First suppose that $H^{\text{tor}} = 1$. Then by Lemma 2.4 the homomorphism $\lambda_v : B_v(H) \longrightarrow B(H)$ is surjective for any $v \in \mathcal{V}$. Hence $B(H) = B'(H) = B^{(S)}(H)$, and therefore $\mathcal{U}_S(H) = 0$.

In the general case let H^{nt} be the kernel of $H \longrightarrow H^{\text{tor}}$. We have the exact sequence

$$X(H^{nt}) \longrightarrow \text{Pic } H^{\text{tor}} \longrightarrow \text{Pic } H \longrightarrow \text{Pic } H^{nt}$$

(cf. [Sa], 6.11), where the character group $X(H^{nt})$ is trivial. We obtain the exact sequence

$$B(H^{nt}) \longrightarrow B(H) \longrightarrow B(H^{\text{tor}}) \longrightarrow 0$$

and similar exact sequences for the groups B_v . We see that in the commutative diagram

$$\begin{array}{ccccccc} B(H^{nt}) = B^{(S)}(H^{nt}) & \longrightarrow & B^{(S)}(H) & \longrightarrow & B^{(S)}(H^{\text{tor}}) & \longrightarrow & 0 \\ & & \parallel & & \cap & & \cap \\ B(H^{nt}) = B'(H^{nt}) & \longrightarrow & B'(H) & \longrightarrow & B'(H^{\text{tor}}) & \longrightarrow & 0 \end{array}$$

the rows are exact, whence we derive the desired assertion.

To prove Theorem 1.9. we need to recall the definition of the arithmetic Brauer group (cf. [Sa]). Let $\text{Br } X$ denote the Brauer group of X , formed of the equivalence classes of Azumaya algebras on X . Set $\text{Br}_1 X = \ker[\text{Br } X \longrightarrow \text{Br } \bar{X}]$, $\text{Br}_a X = \text{Br}_1 X / \text{im}[\text{Br } K \longrightarrow \text{Br } X]$, where $\bar{X} = X_{\bar{K}}$. For $x \in X(K)$ set $\text{Br}_x X = \ker [x^* : \text{Br}_1 X \longrightarrow \text{Br } K]$. We have the canonical splitting $\text{Br}_1 X = \text{Br}_x X \oplus \text{Br } K$, whence $\text{Br}_x X \simeq \text{Br}_a X$. Now let $\text{Br}' X$ be the cohomological Brauer group, i.e. $\text{Br}' X = H_{\text{et}}^2(X, \mathbb{G}_m)$. Set $\text{Br}'_1 X = \ker[\text{Br}' X \longrightarrow \text{Br}' \bar{X}]$. Then $\text{Br}'_1 X = \text{Br}_1 X$ (cf. [Sa], (6.1.1)).

For a morphism $f : X \longrightarrow Y$ let

$$\text{Br}_1 f : \text{Br}_1 Y \longrightarrow \text{Br}_1 X, \quad \text{Br}_x f : \text{Br}_{f(x)} Y \longrightarrow \text{Br}_x X$$

be the canonical homomorphisms. Then

$$\text{Br}_1 f = \text{Br}_X f \oplus 1 : \text{Br}_{f(x)} Y \oplus \text{Br } K \longrightarrow \text{Br}_X X \oplus \text{Br } K .$$

We set $\text{Pic}_1 X = \ker [\text{Pic } X \longrightarrow \text{Pic } \bar{X}]$. For a connected K -group H set $B_1(H) = (\text{Pic}_1 H)^\sim$ and define $B_{1V}(H)$, $B_{1V}^{(S)}(H)$, $B'_1(H)$ and $\mathcal{U}_{1S}(H)$ like $B_V(H) \dots \mathcal{U}_S(H)$ but with $B_1(H)$ instead of $B(H)$. Let $\bar{\lambda} : B(\bar{H}) \longrightarrow B(H)$ be the homomorphism induced by the canonical morphism $\bar{H} \longrightarrow H$.

Lemma 3.1. $\mathcal{U}_{1S}(H) = \mathcal{U}_S(H)$.

Proof. Consider the morphisms $\bar{H} \xrightarrow{\lambda} H_{K_V} \longrightarrow H$. We see that $\text{im}[\lambda_V : B_V(H) \longrightarrow B(H)] \supset \text{im } \bar{\lambda}$. Hence $B^{(S)}(H) \supset \text{im } \bar{\lambda}$ and $B'(H) \supset \text{im } \bar{\lambda}$. We have $B_1(H) = B(H)/\text{im } \bar{\lambda}$. Hence $\mathcal{U}_{1S}(H) = B'_1(H)/B_1^{(S)}(H) \cong B'(H)/B^{(S)}(H) = \mathcal{U}_S(H)$.
q.e.d.

Proof of Theorem 1.9. The torsor $f : G \longrightarrow X$ under H gives rise to the exact sequence

$$\text{Pic } G \longrightarrow \text{Pic } H \longrightarrow \text{Br}' X \longrightarrow \text{Br}' G$$

(cf. [Sa], (6.10.1). Since G is simply connected, $\text{Pic } G = 0$ (cf. [Sa], 6.9). From the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Pic } H & \longrightarrow & \text{Br}' X & \longrightarrow & \text{Br}' G \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Pic } \bar{H} & \longrightarrow & \text{Br}' \bar{X} & \longrightarrow & \text{Br}' \bar{G} \end{array}$$

with exact rows, we get the exact sequence

$$0 \longrightarrow \text{Pic}_1 H \longrightarrow \text{Br}'_1 X \xrightarrow{\text{Br}_1 f} \text{Br}_1 G .$$

Let $x \in X(K)$ be the image of the neutral element $e \in G(K)$. Then

$$\text{Br}_1 f = \text{Br}_e f \oplus 1 : \text{Br}_x X \oplus \text{Br} K \longrightarrow \text{Br}_e G \oplus \text{Br} K .$$

Thus the sequence

$$0 \longrightarrow \text{Pic}_1 H \xrightarrow{\sim} \text{Br}_x X \xrightarrow{\text{Br}_e f} \text{Br}_e G$$

is exact. But $\text{Br}_e G = \text{Br}_a G = 0$ (cf. [Sa], 6.9), hence $\text{Br}_a X = \text{Br}_x X = \text{Pic}_1 H$.

Now we see that

$$\begin{aligned} B_1^{(S)}(H) &= \text{coker} \left[\bigoplus_{v \notin S} B_{1v}(H) \longrightarrow B_1(H) \right] \\ &= \text{coker} \left[\bigoplus_{v \notin S} (\text{Pic}_1 H_{K_v})^\sim \longrightarrow (\text{Pic}_1 H)^\sim \right] \\ &= \text{ker} \left[\text{Pic}_1 H \longrightarrow \prod_{v \notin S} \text{Pic}_1 H_{K_v} \right]^\sim \\ &= \text{ker} \left[\text{Br}_a X \longrightarrow \prod_{v \notin S} \text{Br}_a X_{K_v} \right]^\sim \\ &= \mathfrak{B}_S(X)^\sim \end{aligned}$$

Since $B'_1(H) = B_1^{(\emptyset)}(H)$ and $\mathfrak{B}(X) = \mathfrak{B}_\emptyset(X)$, we have $B'(H) = \mathfrak{B}(X)^\sim$. By definition $\mathcal{U}_{1S}(H) = B'_1(H)/B_1^{(S)}(H)$, hence

$$\mathcal{U}_{1S}(\mathbb{H}) = \mathcal{B}(X)^\sim / \mathcal{B}_S(X)^\sim = (\mathcal{B}_S(X) / \mathcal{B}(X))^\sim .$$

By Lemma 3.1 $\mathcal{U}_{1S}(\mathbb{H}) = \mathcal{U}_S(\mathbb{H})$. Thus $\mathcal{U}_S(\mathbb{H}) = (\mathcal{B}_S(X) / \mathcal{B}(X))^\sim$.

q.e.d.

4. Cohomological results

To prove Theorem 1.2 we need

Lemma 4.1. (i) For $v \in \mathcal{V}_f$ we have $\text{im } \lambda_v = \text{im } \mu_v$.

(ii) For any finite $S \subset \mathcal{V}$ and any $v \in \mathcal{V}_\infty$ there exists $w \in \mathcal{V}_f - S$ such that

$$\text{im } \mu_w = \text{im } \lambda_w = \text{im } \lambda_v \supset \text{im } \mu_v .$$

(iii) $B^{(S)}(\mathbb{H}) = B^{(S \cap \mathcal{V}_f)}(\mathbb{H}) = B^{(SU \mathcal{V}_\infty)}(\mathbb{H})$; in particular $B'(\mathbb{H}) = B^{(\mathcal{V}_\infty)}(\mathbb{H})$.

(iv) $\mathcal{U}_S(\mathbb{H}) = \mathcal{U}_{S \cap \mathcal{V}_f}(\mathbb{H})$.

Proof. By proposition 2.5 the map β_v is bijective for $v \in \mathcal{V}_f$, whence (i). By Lemma 2.3 the group $\text{im } \lambda_v$ depends only on the conjugacy class of a decomposition group Δ_v of v in L , where L is defined in Section 2. For $v \in \mathcal{V}_\infty$ the group Δ_v is cyclic. Chebotarev's density theorem implies the existence of $w \in \mathcal{V}_f - S$ such that $\Delta_w = \Delta_v$ up to conjugation. Then $\text{im } \lambda_w = \text{im } \lambda_v$ by Lemma 2.3. We have $\text{im } \lambda_v \supset \text{im } \mu_v$; by the assertion (i) $\text{im } \lambda_w = \text{im } \mu_w$. The assertion (ii) is proved. It is clear that (ii) implies (iii) and (iv).

Proof of Theorem 1.2. Consider the sequence

$$H^1(K, H) \xrightarrow{j_S} H^1(K_S, H) \xrightarrow{\chi_S} \mathcal{U}_S(H) \longrightarrow 0$$

where χ_S is defined at the end of Section 2. First we prove that χ_S is surjective. By definition the group $B'(H)$ is generated by the groups $\lambda_v(B_v(H))$ for all $v \in \mathcal{V}$. However, by Lemma 4.1 (iii) $B'(H) = B^{(\mathcal{V}_\omega)}(H)$, i.e. $B'(H)$ is generated by the groups $\text{im } \lambda_v = \text{im } \mu_v$ for $v \in \mathcal{V}_f = (S \cap \mathcal{V}_f) \cup (\mathcal{V}_f - S)$. We see that

$$\text{im} [\mu_S : H^1(K_S, H) \longrightarrow B(H)] + B^{(S)}(H) \supset \text{im } \mu_{S \cap \mathcal{V}_f} + B^{(S)}(H) = B'(H) ,$$

and the desired surjectivity follows.

Proposition 2.6 implies that $\text{im}(\mu_S \circ j_S^\vee) \subset B^{(S)}(H)$, hence $\chi_S \circ j_S = 0$. We must show that $\ker \chi_S = \text{im } j_S$. Suppose that $\xi_S \in H^1(K_S, H)$, $\chi_S(\xi_S) = 0$. Then $\mu_S(\xi_S) \in B^{(S)}(H)$. By Lemma 4.1 (iii) $\mu_S(\xi_S) \in B^{(S \cup \mathcal{V}_\omega)}(H)$. Hence there are a finite set $\Sigma \subset \mathcal{V}_f - S$ and an element $\xi_\Sigma \in H^1(K_\Sigma, H)$ such that $\mu_S(\xi_S) + \mu_\Sigma(\xi_\Sigma) = 0$. Set $\xi_A = \xi_S \times \xi_\Sigma \times 0 \in H^1(A, H)$. Then $\mu(\xi_A) = 0$. By Proposition 2.6 $\xi_A = j(\xi)$ for some $\xi \in H^1(K, H)$. Clearly $\xi_S = j_S(\xi)$. q.e.d.

5. Weak approximation

To prove Theorem 1.1 consider the orbit spaces: the set of orbits $\mathcal{O}(X, G, K)$ of $G(K)$ in $X(K)$ and the set of orbits $\mathcal{O}(X, G, K_S)$ of $G(K_S)$ in $X(K_S) = \prod_{v \in S} X(K_v)$. Any orbit of $G(K_S)$ in $X(K_S)$ is open, hence any orbit is closed. Since G is simply connected, G satisfies (WA_S) for any finite S (cf. [P11], or [P12], § 4). Thus $(x \cdot G(K))_S = x \cdot G(K)_S = x \cdot G(K_S)$ for any $x \in X(K)$. Consider the map

$i_S : \mathcal{D}(X, G, K) \longrightarrow \mathcal{D}(X, G, K_S)$ induced by the embedding $X(K) \longrightarrow X(K_S)$. We see that $X(K)_S = U \circ$ where \circ runs over $\text{im } i_S$. Thus (WA_S) for X is equivalent to the surjectivity of i_S .

The orbit spaces can be described in cohomological terms. Set

$$\begin{aligned} k &= \ker [H^1(K, H) \longrightarrow H^1(K, G)] , \\ k_S &= \ker [H^1(K_S, H) \longrightarrow H^1(K_S, G)] . \end{aligned}$$

Using the exact cohomology sequences associated with the subgroup H of G (see [Se], Ch.1, § 5.4, Cor. 1 of Prop. 36), we can identify

$$\mathcal{D}(X, G, K) = k , \quad \mathcal{D}(X, G, K_S) = k_S .$$

In these terms the map $i_S : k \longrightarrow k_S$ is the restriction of the localization map $j_S : H^1(K, H) \longrightarrow H^1(K_S, H)$ to k . Now it is clear that Theorem 1.1 follows from

Proposition 5.1. Let ν_S be the restriction of $\chi_S : H^1(K_S, H) \longrightarrow {}_S(H)$ to $k_S \subset H^1(K_S, H)$. Then the sequence

$$k \xrightarrow{i_S} k_S \xrightarrow{\nu_S} {}_S(H) \longrightarrow 0$$

is exact.

Proof of Proposition 5.1. First we prove the surjectivity of ν_S . Set $\Sigma = S \cap \mathcal{V}_f$. By Theorem 1.2 χ_Σ is surjective, i.e. $B^{(\Sigma)}(H) + \text{im } \mu_\Sigma = B'(H)$. Since $H^1(K_\Sigma, G) = 0$ by Kneser's theorem, $k_\Sigma = H^1(K_\Sigma, H)$. Hence $\mu_\Sigma(k_\Sigma) + B^{(\Sigma)}(H) = B'(H)$. By Lemma 4.1

(iii) $B^{(\Sigma)}(\mathbb{H}) = B^{(S)}(\mathbb{H})$. Since $\mu_{\Sigma}(k_{\Sigma}) \subset \mu_S(k_S)$, we see that $\mu_S(k_S) + B^{(S)}(\mathbb{H}) = B'(\mathbb{H})$, i.e. $\nu_S(k_S) = \mathcal{U}_S(\mathbb{H})$.

By Theorem 1.2 $\chi_S \circ j_S = 0$, hence $\nu_S \circ i_S = 0$. We must show that $\ker \nu_S = \text{im } i_S$. Assume that $\xi_S \in k_S$ and $\nu_S(\xi_S) = 0$ (i.e. $\chi_S(\xi_S) = 0$). We construct an element $\xi \in H^1(K, \mathbb{H})$ as in the proof of Theorem 1.2. Then $j_S(\xi) = \xi_S$. We wish to prove that $\xi \in k$. By construction we have $j_S(\xi) = \xi_A = (\xi_v^*)_{v \in \mathcal{V}} \in H^1(A, \mathbb{H})$, where $\xi_v^* = \xi_v$ for $v \in S$ and $\xi_v^* = 0$ for $v \in \mathcal{V}_{\mathfrak{o}} - S$. Hence $\xi_v^* \in k_v = \ker [H^1(K_v, \mathbb{H}) \longrightarrow H^1(K_v, G)]$ for any $v \in \mathcal{V}_{\mathfrak{o}}$. By the Hasse principle for G , we have $\xi \in k$. Proposition 5.1 is proved, and so is Theorem 1.1.

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