Arithmetic birational invariants of linear algebraic groups

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(This is a joint work with Boris Kunyavskiĭ.)

Manin in 1972 introduced R-equivalence.

Let X be an algebraic variety over a field k. Two points $x, y \in X(k)$ are called *elementarily related* if there exists a rational map $\phi\colon \mathbb{A}^1_k \to X$ (where $\mathbb{A}^1_k$ is the affine line over k) such that ϕ is defined in 0,1 and that $\phi(0) = x,\ \phi(1) = y$.

We say that $x, y \in X(k)$ are *R-equivalent* if there exists a finite sequence $x_0 = x,\ x_1, \dots,$ $x_n = y$ of k-points of X, such that x_i and x_{i+1} are elementarily related for all i.

We write $X(k)/R$ for the set of classes of R-equivalence.

Example. If $X = \mathbb{A}^n_k$ (an affine space), then $X(k)/R = 0$ (any two points are elementarily related).

Let G be a connected linear algebraic group over a field k of characteristic 0, then $G(k)/R$ has a group structure.

Recall that two k-varieties X_1 and X_2 are called k-birationally isomorphic if there exist open k-subvarieties $U_1 \subset X_1$ and $U_2 \subset X_2$ and a regular k-isomorphism $U_1 \xrightarrow{\sim} U_2$. A variety is called k-rational if it is birationally isomorphic to an affine space.

Lemma (Colliot-Thélène and Sansuc, 1977) *A birational isomorphism of linear algebraic groups $G_1 \to G_2$ induces a bijection $G_1(k)/R \xrightarrow{\sim} G_2(k)/R$ (not necessarily a group isomorphism).*

It follows that if G is a k-rational group, then $G(E)/R = 1$ for any field extension E/k.

Merkurjev, 1996, constructed a semisimple adjoint group G of type D_n over a number field k, such that $G(E)/R \neq 1$ for some large extension E/k. Thus he proved that there exist non-rational adjoint groups. Chernousov and Merkurjev, 2001, similarly proved existence of semisimple simply connected non-rational groups G of type D_n, using $G(E)/R$.

Problem: To compute $G(k)/R$.

Colliot-Thélène and Sansuc, 1977, computed $T(k)/R$ for k-tori T over any field. Colliot-Thélène, Gille, and Parimala, 2004, computed $G(k)/R$ for semisimple groups G over some fields k of cohomological dimension 2, in particular when k is a p-adic field or a totally imaginary number field.

We compute $G(k)/R$ when k is a p-adic field or a totally imaginary number field, for any connected k-group G, not necessarily semisimple or a torus.

Let U be a unipotent group (over any field k of characteristic 0), then

$$U(k)/R = 1$$

because U is isomorphic as a k-variety to an affine space.

If G is a semisimple simply connected k-group, where k is a p-adic field or a totally imaginary number field, then

$$G(k)/R = 1$$

(Colliot-Thélène, Gille, and Parimala, 2004).

By the Kottwitz Principle one can compute $G(k)/R$ from the algebraic fundamental group $\pi_1(G)$.

Kottwitz Principle:
If an invariant of a connected linear k-group G is trivial for unipotent groups and for semi-simple simply connected groups, then it can be computed in terms of the algebraic fundamental group $\pi_1(G)$.

Here $\pi_1(G)$ is a certain finitely generated Galois module, i.e. a finitely generated abelian group with an action of Gal$(\bar{k}/k)$, where $\bar{k}$ is a fixed algebraic closure of k.

Examples (Kottwitz):
• The Tamagawa number $\tau(G)$ for groups over a number field;
• The Galois cohomology $H^1(k, G)$ for groups over a p-adic field.

We define the Galois module $\pi_1(G)$.

Let G be any connected linear k-group. Write:

G^{u}: the unipotent radical of G;

$G^{\mathsf{red}} = G/G^{\mathsf{u}}$ (it is reductive);

G^{ss}: the derived group of G^{red} (it is semi-simple);

G^{sc}: the universal covering of G^{ss} (it is semi-simple simply connected).

Consider the composed map

$$\rho\colon G^{\text{sc}} \to G^{\text{ss}} \to G^{\text{red}}.$$

Choose a maximal torus $T \subset G^{\text{red}}$.
Set $T^{\text{sc}} = \rho^{-1}(T) \subset G^{\text{sc}}$, then T^{sc} is a maximal torus in G^{sc}. We have a homomorphism

$$\rho\colon T^{\text{sc}} \to T.$$

Definition. $\pi_1(G) = \mathbf{X}_*(T)/\rho_*(\mathbf{X}_*(T^{\text{sc}}))$, where $\mathbf{X}_*$ denotes the cocharacter group, $\mathbf{X}_*(T) = \text{Hom}_{\bar{k}}(\mathbb{G}_m, T)$.

$\pi_1(G)$ is a Galois module, it does not depend on T.

Examples: $\pi_1(T) = \mathbf{X}_*(T)$, $\pi_1(\text{PGL}_n) = \mathbf{Z}/n\mathbf{Z}$ (not μ_n).

If $k = \mathbf{C}$, then $\pi_1(G) \simeq \pi_1^{\text{top}}(G(\mathbf{C}))$.

Another invariant of G: Pic($\overline{G}_c$).
Here G_c is a smooth compactification of G, i.e. a smooth projective k-variety such that G is an open k-subvariety of G_c. A smooth compactification exists (in characteristic 0) by Hironaka's theorem.

We write $\overline{G}_c = G_c \times_k \bar{k}$. We write Pic($\overline{G}_c$) for the Picard group of $\overline{G}_c$, it is a finitely generated torsion free Galois module.

We wish to compute Pic($\overline{G}_c$) in terms of $\pi_1(G)$. Note that a smooth compactification G_c is not unique, but Pic($\overline{G}_c$) is in a sense unique.

Now I want to write formulas for $G(k)/R$ and $\mathrm{Pic}(\overline{G}_c)$ in terms of the Galois module $\pi_1(G)$.

Let Γ be a finite group. A Γ-module is an abelian group with an action of Γ. We consider only finitely generated Γ-modules.

Definition. A permutation module is a torsion free Γ-module P, such that P has a Γ-invariant basis (Γ permutes the elements of the basis).

Example: $\Gamma = \{1, \sigma\}$ is a group of order 2, $P = \mathbf{Z} \oplus \mathbf{Z}$, and σ permutes $(1, 0)$ and $(0, 1)$.

Lemma (well known) *If P is a permutation module, then $H^1(\Gamma, P) = 0$.*

Definition. A coflasque module is a torsion free Γ-module Q such that

$$H^1(\Gamma', Q) = 0$$

for any subgroup $\Gamma' \subseteq \Gamma$.

Every permutation module, or a direct summand of a permutation module, is coflasque.

Proposition (Endo and Miyata, 1974) *For a given finite group Γ, every coflasque Γ-module is a direct summand of a permutation module if and only if all Sylow subgroups of Γ are cyclic.*

Lemma (Colliot-Thélène and Sansuc 1987) *Every* Γ*-module* M *admits a* **coflasque resolution**, *i.e. a short exact sequence*

$$0 \to Q \to P \to M \to 0$$

where P *is a permutation module and* Q *is a coflasque module.*

Note that a cofasque resolution is not unique. However Q is unique up to addition of a permutation module. That is, if

$$0 \to Q_1 \to P_1 \to M \to 0$$
$$0 \to Q_2 \to P_2 \to M \to 0$$

are two coflasque resolutions of M, then there exist permutation modules P_1' and P_2' such that $Q_1 \oplus P_1' \simeq Q_2 \oplus P_2'$.

We say that Q_1 and Q_2 are *similar*, and write $Q_1 \sim Q_2$.

Results of a paper of B. and Kunyavskiĭ with an appendix by Gille (J. Algebra 2004):

Let G be a connected k-group. Let Γ be a finite quotient group of $\mathrm{Gal}(\bar{k}/k)$ acting on $\pi_1(G)$, then $\pi_1(G)$ is a Γ-module. Choose a coflasque resolution

$$0 \to Q \to P \to \pi_1(G) \to 0.$$

We write $Q = Q_G$. Then Q_G is a Γ-module, hence a $\mathrm{Gal}(\bar{k}/k)$-module.

We compute our invariants $G(k)/R$ and $\mathrm{Pic}(\overline{G}_c)$ in terms of Q_G, hence in terms of $\pi_1(G)$.

Let F_G be the k-torus such that

$$\mathbf{X}_*(F_G) = Q_G.$$

Theorem 1. *Let k be a p-adic field or a totally imaginary number field. Then*

$$G(k)/R \simeq H^1(k, F_G)$$

canonically and functorially.

$$G \mapsto \pi_1(G) \mapsto Q_G \mapsto F_G \mapsto H^1(k, F_G)$$

The Galois module Q_G is not unique, hence the k-torus F_G is not unique, but $H^1(k, F_G)$ is unique up to a canonical isomorphism.

In the case when G is a k-torus (over any field) Theorem 1 was proved by Colliot-Thélène and Sansuc, 1977. In the case when G is semi-simple, a similar result was proved by Colliot-Thélène, Gille, and Parimala, 2004.

Theorem 2. *Let k be any field of characteristic 0. Then*

$$\mathsf{Pic}(\overline{G}_c) \sim Q_G^D.$$

Here Q_G^D denotes the dual Galois module to Q_G, i.e.

$$Q_G^D := \mathsf{Hom}(Q_G, \mathbf{Z}).$$

In addition, $\sim$ means "similar", i.e. there exist permutation modules P_1' and P_2' such that

$$\mathsf{Pic}(\overline{G}_c) \oplus P_1' \simeq Q_G^D \oplus P_2'.$$

A corollary of Theorem 2:
Theorem 3. $\mathsf{Pic}(\overline{G}_c)^D$ *is a coflasque module.*

This means that

$$H^1(\Gamma', \mathsf{Pic}(\overline{G}_c)^D) = 0 \quad \forall \Gamma' \subseteq \Gamma,$$

where Γ is a finite quotient of $\mathsf{Gal}(\bar{k}/k)$ acting on $\mathsf{Pic}(\overline{G}_c)$.

In the case when G is a torus, Theorems 2 and 3 were proved by Voskresenskiĭ, 1970.

A generalization of Theorem 3:

Theorem (Colliot-Thélène and Kunyavskiĭ, Preprint, 2005) *Let* X *be a homogeneous space,* $X = G/H$*, where* G *is a connected* k*-group and* H *is a connected* k*-subgroup of* G*. Let* X_c *be a smooth compactification of* X*. Then* $\mathrm{Pic}(\overline{X}_c)^D$ *is a coflasque module.*

The defect of weak approximation for G

Let k be a number field, Σ a finite set of places of k. For $v \in \Sigma$ let k_v denote the completion of k at v. Let G be a connected k-group. Consider the diagonal embedding

$$G(k) \to \prod_{v \in \Sigma} G(k_v),$$

and let $G(k)\widehat{_\Sigma}$ denote the closure of $G(k)$ in $\prod_{v \in \Sigma} G(k_v)$.

Definition.

$$A_\Sigma(G) = \left(\prod_{v \in \Sigma} G(k_v) \right) / G(k)\widehat{_\Sigma} \,,$$

this is the defect of weak approximation for G with respect to Σ.

Theorem 4.

$$A_\Sigma(G) \simeq \text{coker}\left[H^1(k, F_G) \to \prod_{v \in \Sigma} H^1(k_v, F_G) \right].$$

The Tate-Shafarevich kernel for G.

Let G be a connected group over a number field k.

Definition.

$$Ш^1(k, G) = \ker\left[H^1(k, G) \to \prod_v H^1(k_v, G)\right]$$

where the product is taken over all the places v of k. This is the Tate-Shafarevich kernel for G.

Theorem 5. *There is a canonical bijection*

$$Ш^1(k, G) \xrightarrow{\sim} Ш^2(k, F_G).$$

Another formula for $Ш^1(k, G)$ in terms of $\pi_1(k, G)$ was given by Kottwitz, 1984.

Remark. Recall that $G(k)/R$, $A_\Sigma(G)$, and $Ш^1(k,G)$ were computed in terms of a torus F_G, hence in terms of a Galois module Q_G. By Theorem 2

$$Q_G \sim \mathrm{Pic}(\overline{G}_c)^D.$$

But $\mathrm{Pic}(\overline{G}_c)$ is a k-birational invariant of G (up to similarity): if G_1 and G_2 are two k-birationally isomorphic groups, then $Q_{G_1} \sim Q_{G_2}$. We obtain

Corollary. *The group $G(k)/R$ (in Theorem 1), the group $A_\Sigma(G)$, and the set $Ш^1(k,G)$ are k-birational invariants of G.*

For $A_\Sigma(G)$ and $Ш^1(k,G)$ it was proved by Sansuc, 1981. For $G(k)/R$ it is new (though it was known that the *set* $G(k)/R$ is a k-birational invariant).

Recently Colliot-Thélène, 2004, found beautiful formulas for the maps

$$G(k)/R \to H^1(k, F_G),$$

$$A_\Sigma(G) \to \text{coker}\left[H^1(k, F_G) \to \prod_{v\in\Sigma} H^1(k_v, F_G)\right],$$

$$Ш^1(k, G) \to Ш^2(k, F_G).$$

Definitions.
A k-torus T is called *quasi-trivial* if $\mathbf{X}_*(T)$ is a permutation module. A k-torus F is called *flasque* if $X_*(F)$ is a coflasque module. Any quasi-trivial torus is flasque.

A k-group H is called *quasi-trivial* if H^{ss} is simply connected and the torus $G^{\text{tor}} := G^{\text{red}}/G^{\text{ss}}$ is quasi-trivial. In other words, H is quasi-trivial if and only if $\pi_1(H)$ is a permutation module.

Let G be a reductive k-group. A *flasque resolution* of G is a short exact sequence

$$1 \to F \to H \to G \to 1$$

where H is a quasi-trivial reductive k-group and F is a flasque k-torus.

Proposition (Colliot-Thélène 2004) *Any reductive k-group G admits a flasque resolution.* (For k-tori it was proved by Colliot-Thélène and Sansuc, 1977.)

From a flasque resolution of G we obtain a short exact sequence of fundamental groups

$$0 \to \mathbf{X}_*(F) \to \pi_1(H) \to \pi_1(G) \to 0$$

where $\pi_1(H)$ is a permutation module and $\mathbf{X}_*(F)$ is a coflasque module. We see that it is a coflasque resolution of $\pi_1(G)$, hence $\mathbf{X}_*(F)$ is our Q_G, and F is our F_G.

But now we have connecting (coboundary) maps

$$\delta\colon G(k) \to H^1(k, F_G)$$
$$\Delta\colon H^1(k, G) \to H^2(k, F_G)$$

(over any field), which over nice fields induce the maps which we need. For example, Colliot-Thélène proves that when k is a p-adic field or a totally imaginary number field, then the homomorphism $\delta\colon G(k) \to H^1(k, F_G)$ induces an isomorphism $G(k)/R \xrightarrow{\simeq} H^1(k, F_G)$.

Geometric two-dimensional fields

We do not use class field theory in our constructions or proofs. Our results generalize to geometric fields of cohomological dimension 2 investigated by Colliot-Thélène, Gille, and Parimala, 2004, in particular to the fields of the following types:

• the field of rational functions $\mathbf{C}(S)$, where S is an algebraic surface over the field $\mathbf{C}$ of complex numbers; in this case we assume that G has no factors of type E_8.

• the field of fractions $\mathbf{C}((X_1, X_2))$ of the ring of formal power series in two variables $\mathbf{C}[[X_1, X_2]]$.

(In these cases $H^1(k, G) = 1$ and $G(k)/R = 1$ when G is semisimple simply connected.)