Extended Picard complexes for linear algebraic groups and homogeneous spaces

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(This is a joint work with Joost van Hamel.)

Notation
Let k be a field of characteristic 0, then
$\overline{k}$ denotes a fixed algebraic closure of k.
Let X/k be a smooth algebraic variety, then
$\overline{X} = X \otimes_k \overline{k}$;
$\overline{k}[\overline{X}]$ is the ring of regular functions on $\overline{X}$;
$\overline{k}(\overline{X})$ is the field of rational functions on $\overline{X}$.

Set $U(\overline{X}) = \overline{k}[\overline{X}]^\times/\overline{k}^\times$, the group of invertible regular functions on $\overline{X}$ modulo constants. It is a Gal$(\overline{k}/k)$-module.

Proposition (Rosenlicht 1961). *Let G be a connected linear algebraic group over k. Then $U(\overline{G}) = \mathbf{X}^*(\overline{G})$.*

Here $\mathbf{X}^*$ denotes the character group. The proposition says that any invertible regular function f on $\overline{G}$, such that $f(1) = 1$, is a character.

Recall the definition of the Picard group $\mathrm{Pic}(\overline{X})$. Let $\mathrm{Div}(\overline{X})$ denote the divisor group of $\overline{X}$. For $f \in \overline{k}(\overline{X})^\times/\overline{k}^\times$, let $\mathrm{div}(f)$ denote the divisor of f. Then by definition

$$\mathrm{Pic}(\overline{X}) = \mathrm{coker}[\overline{k}(\overline{X})^\times/\overline{k}^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(\overline{X})].$$

Clearly $\mathrm{Pic}(\overline{X})$ is a Galois module.

We wish to define a complex of Galois modules of length 2 whose cohomology groups are $U(\overline{X})$ and $\mathrm{Pic}(\overline{X})$. We define

$$\mathrm{UPic}(\overline{X}) = [\overline{k}(\overline{X})^\times/\overline{k}^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(\overline{X})\rangle.$$

We say that $\mathrm{UPic}(\overline{X})$ is the *extended Picard complex* of $\overline{X}$. Here $[\quad\rangle$ show that $\overline{k}(\overline{X})^\times/\overline{k}^\times$ is in degree 0, and $\mathrm{Div}(\overline{X})$ is in degree 1. We have

$$\mathcal{H}^0(\mathrm{UPic}(\overline{X})) = \ker(\mathrm{div}) = U(\overline{X})$$
$$\mathcal{H}^1(\mathrm{UPic}(\overline{X})) = \mathrm{coker}(\mathrm{div}) = \mathrm{Pic}(\overline{X}).$$

We regard $\mathrm{UPic}(\overline{X})$ as an object of the derived category of Galois modules. We wish to compute $\mathrm{UPic}(\overline{X})$ (up to a quasi-isomorphism).

Derived categories

Let $\mathcal{A}$ be an abelian category, for example the category of Γ-modules where Γ is a group. Let $K^\bullet$ be a complex in $\mathcal{A}$, i.e. a sequence of objects and morphisms

$$\dots \xrightarrow{d^{-1}} K^0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} \dots$$

such that

$$d^{i+1} \circ d^i = 0 \text{ for all } i.$$

We write $\mathcal{H}^i(K^\bullet) = \ker d^i / \operatorname{im} d^{i-1}$ for the i-th cohomology of $K^\bullet$. A morphism of complexes $\varphi\colon K^\bullet \to L^\bullet$ induces

$$\mathcal{H}^i(\varphi)\colon \mathcal{H}^i(K^\bullet) \to \mathcal{H}^i(L^\bullet).$$

A morphism of complexes φ is called a *quasi-isomorphism* if all $\mathcal{H}^i(\varphi)$ are isomorphisms. Any isomorphism of complexes is a quasi-isomorphism, but there exist quasi-isomorphisms which are not isomorphisms.

Example 1. Let $\eta\colon A \to B$ be an epimorphism, then the obvious morphism of complexes

$$[\ker \eta \to 0\rangle \longrightarrow [A \to B\rangle$$

is a quasi-isomorphism.

Example 2. Let $\eta\colon A \to B$ be a monomorphism, then the obvious morphism of complexes

$$[A \to B\rangle \longrightarrow [0 \to \operatorname{coker} \eta\rangle$$

is a quasi-isomorphism.

Example 3. Let $\eta\colon A \to B$ be an isomorphism, then the obvious morphism of complexes

$$[0 \to 0\rangle \longrightarrow [A \to B\rangle$$

is a quasi-isomorphism.

A complex $K^\bullet$ is called *bounded* if it has only finite number of $K^i \neq 0$.

Definition. The derived category $D^b(\mathcal{A})$ is the category, whose objects are the bounded complexes in $\mathcal{A}$, and whose morphisms are the morphisms of complexes, the formal inverses of quasi-isomorphisms, and their products.

We want to compute $\mathrm{UPic}(\overline{X})$ as an object of the derived category, that is up to a quasi-isomorphism.

Remark. If $\mathrm{Pic}(\overline{X}) = 0$, then

$$\mathrm{UPic}(\overline{X}) \simeq [U(\overline{X}) \to 0\rangle = U(\overline{X})$$

(Example 1). If $U(\overline{X}) = 0$, then

$$\mathrm{UPic}(\overline{X}) \simeq [0 \to \mathrm{Pic}(\overline{X})\rangle$$

(Example 2). We see that the complex $\mathrm{UPic}(\overline{X})$ is interesting when $U(\overline{X}) \neq 0$ and $\mathrm{Pic}(\overline{X}) \neq 0$.

We compute:
- UPic($\overline{G}$), where G is a connected linear algebraic group over k;
- UPic($\overline{G/H}$), where G/H is a homogeneous space, G is a connected k-group, $H \subset G$ is a connected k-subgroup.

We start with UPic($\overline{G}$), where G is a connected k-group.

If G is a torus, then Pic($\overline{G}$) $= 0$, hence

$$\mathrm{UPic}(\overline{G}) = \mathbf{X}^*(\overline{G}).$$

If G is semisimple, then $U(\overline{G}) = \mathbf{X}^*(\overline{G}) = 0$, hence

$$\mathrm{UPic}(\overline{G}) = [0 \to \mathrm{Pic}(\overline{G})\rangle.$$

We are interested in the general case, when G is neither semisimple nor a torus.

We consider unipotent groups and simply connected semisimple groups.

(1) Let G be a unipotent k-group. We have $U(\overline{G}) = \mathbf{X}^*(\overline{G}) = 0$. As a variety G is isomorphic to an affine space, hence $\mathrm{Pic}(\overline{G}) = 0$. Thus $\mathrm{UPic}(\overline{G}) = 0$.

(2) Let G be a simply connected semisimple k-group. Then $U(\overline{G}) = \mathbf{X}^*(\overline{G}) = 0$. We have $\mathrm{Pic}(\overline{G}) = 0$ (Voskresenskiĭ 1969, Fossum and Iversen, 1973, Popov 1974). Thus $\mathrm{UPic}(\overline{G}) = 0$.

According to the Kottwitz Principle, for any connected k-group G one can compute $\mathrm{UPic}(\overline{G})$ from the algebraic fundamental group $\pi_1(G)$.

Kottwitz Principle:
If an invariant of a connected linear k-group G is trivial for unipotent groups and for simply connected semisimple groups, then it can be computed in terms of the algebraic fundamental group $\pi_1(G)$.

Here $\pi_1(G)$ is a certain finitely generated Galois module, i.e. a finitely generated abelian group with an action of Gal$(\overline{k}/k)$.

Examples (Kottwitz 1984):

- The Tamagawa number $\tau(G)$ for groups over a number field;
- The Galois cohomology $H^1(k, G)$ for groups over a p-adic field.

We define the Galois module $\pi_1(G)$.

Let G be any connected linear k-group. Write:

G^{u}: the unipotent radical of G;

$G^{\mathsf{red}} = G/G^{\mathsf{u}}$ (it is reductive);

G^{ss}: the derived group of G^{red} (it is semisimple);

G^{sc}: the universal covering of G^{ss} (it is simply connected semisimple).

Consider the composed map

$$\rho\colon G^{\mathsf{sc}} \to G^{\mathsf{ss}} \to G^{\mathsf{red}}.$$

Choose a maximal torus $T \subset G^{\mathsf{red}}$.
Set $T^{\mathsf{sc}} = \rho^{-1}(T) \subset G^{\mathsf{sc}}$, then T^{sc} is a maximal torus in G^{sc}. We have a homomorphism

$$\rho\colon T^{\mathsf{sc}} \to T.$$

Definition. $\pi_1(G) = \mathbf{X}_*(\overline{T})/\rho_*(\mathbf{X}_*(\overline{T}^{\mathsf{sc}}))$, where $\mathbf{X}_*$ denotes the cocharacter group, $\mathbf{X}_*(\overline{T}) = \mathrm{Hom}(\mathbb{G}_{m,\overline{k}}, \overline{T})$.

$\pi_1(G)$ is a Galois module, it does not depend on T.

Examples: $\pi_1(T) = \mathbf{X}_*(\overline{T})$, $\pi_1(\mathsf{PGL}_n) = \mathbf{Z}/n\mathbf{Z}$ (not μ_n).

If $k = \mathbf{C}$, then $\pi_1(G) \simeq \pi_1^{\mathsf{top}}(G(\mathbf{C}))$.

Theorem 1. *Let G be a connected k-group. Then*

$$\mathsf{UPic}(\overline{G}) \simeq [\mathbf{X}^*(\overline{T}) \to \mathbf{X}^*(\overline{T}^{sc})\rangle$$

canonically and functorially.

The relation of Theorem 1 to $\pi_1(G)$ and to the Kottwitz Principle

Let A be an abelian group. How can we define a dual object A^D? If A is torsion free, we set

$$A^D := \mathrm{Hom}(A, \mathbf{Z}).$$

For a finite group this formula is stupid. So we take a torsion free resolution, i.e a quasi-isomorphism $K^\bullet \to A$, where $K^\bullet$ is a complex of torsion free abelian groups, and set

$$A^D := \mathrm{Hom}(K^\bullet, \mathbf{Z}).$$

We have a torsion free resolution for $\pi_1(G)$:

$$0 \to \mathbf{X}_*(\overline{T}^{\mathrm{sc}}) \to \mathbf{X}_*(\overline{T}) \to \pi_1(G) \to 0,$$

hence a quasi-isomorphism

$$\langle \mathbf{X}_*(\overline{T}^{\mathrm{sc}}) \to \mathbf{X}_*(\overline{T})] \longrightarrow \pi_1(G)$$

and we see that

$$\begin{aligned}\pi_1(G)^D &= \mathrm{Hom}(\langle \mathbf{X}_*(\overline{T}^{\mathrm{sc}}) \to \mathbf{X}_*(\overline{T})], \mathbf{Z}) \\ &= [\mathbf{X}^*(\overline{T}) \to \mathbf{X}^*(\overline{T}^{\mathrm{sc}})\rangle.\end{aligned}$$

Thus we can state Theorem 1 as follows:

Theorem 1'. *Let G be a connected k-group. Then*

$$\mathrm{UPic}(\overline{G}) \simeq \pi_1(G)^D.$$

canonically and functorially.

Homogeneous spaces
Let $X = G/H$ be a homogeneous space, G a connected k-group, $H \subset G$ a connected k-subgroup. We assume that G^{ss} is simply connected. If not, then we can write $X = G'/H'$, where G', H' are connected and $(G')^{\mathrm{ss}}$ is simply connected.

Theorem 2. *Let G be a connected k-group, $H \subset G$ a connected k-subgroup, and assume that G^{ss} is simply connected. Then*

$$\mathrm{UPic}(\overline{G/H}) \simeq [\mathbf{X}^*(\overline{G}) \to \mathbf{X}^*(\overline{H})\rangle.$$

We can deduce Theorem 1 from Theorem 2. Let G be a connected k-group. There exists short exact sequence

$$1 \to Z \to G' \to G \to 1$$

such that Z is a k-torus (hence connected) and $(G')^{\mathsf{ss}}$ is simply connected. Then $G \simeq G'/Z$, and by Theorem 2

$$\mathsf{UPic}(G) \simeq [\mathbf{X}^*(\overline{G'}) \to \mathbf{X}^*(\overline{Z})\rangle.$$

One can show that this formula is equivalent to that of Theorem 1.

Hypercohomology

Let Γ be a group, $K^\bullet$ a complex of Γ-modules. We can define the hypercohomology groups $H^i(\Gamma, K^\bullet)$. If $K^\bullet = [A \to B\rangle$ (a complex of length 2), then we have a long exact sequence

$$\ldots \to H^{i-1}(\Gamma, A) \to H^{i-1}(\Gamma, B) \to H^i(\Gamma, [A \to B\rangle) \\ \to H^i(\Gamma, A) \to H^i(\Gamma, B) \to \ldots$$

We see that $H^i(\Gamma, [A \to B\rangle)$ is a “mixture” of $H^{i-1}(\Gamma, B)$ and $H^i(\Gamma, A)$.

The algebraic Brauer group $\mathsf{Br}_{\mathsf{a}}(X)$

Let X be a smooth k-variety. We write $\mathsf{Br}(X)$ for the cohomological Brauer group of X,

$$\mathsf{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m).$$

We have canonical morphisms

$$\overline{X} \xrightarrow{\alpha} X \xrightarrow{\beta} \mathsf{Spec}(k)$$

and the induced morphisms

$$\mathsf{Br}(k) \xrightarrow{\beta^*} \mathsf{Br}(X) \xrightarrow{\alpha^*} \mathsf{Br}(\overline{X}).$$

Definition. $\mathsf{Br}_{\mathsf{a}}(X) = \ker \alpha^* / \operatorname{im} \beta^*$.

Theorem 3. *If k is a number field or if X has a k-point, then* $\mathsf{Br}_{\mathsf{a}}(X) = H^2(k, \mathsf{UPic}(\overline{X}))$.

Ш^i_ω *and smooth compactifications*

Let $K^\bullet$ be a complex of $\mathrm{Gal}(\overline{k}/k)$-modules. Define

$$\text{Ш}^i_\omega(k, K^\bullet) = \ker[H^i(k, K^\bullet) \to \prod_\gamma H^i(\gamma, K^\bullet)]$$

where γ runs over all the pro-cyclic subgroups of $\mathrm{Gal}(\overline{k}/k)$.

Theorem 4. *Let X_c be a smooth compactification of a smooth k-variety X. Write $\overline{X}_c = X_c \times_k \overline{k}$. Then*

$$\text{Ш}^1_\omega(k, \mathrm{Pic}(\overline{X}_c)) = \text{Ш}^2_\omega(k, \mathrm{UPic}(\overline{X})).$$

Remark. If $X = G/H$ where G is a connected k-group and $H \subset G$ is a connected k-subgroup, and G^{ss} is simply connected, then by a difficult theorem of Colliot-Thélène and Kunyavskiĭ, 2005, $H^1(\gamma, \mathrm{Pic}(\overline{X}_c)) = 0$ for any pro-cyclic subgroup $\gamma \subset \mathrm{Gal}(\overline{k}/k)$. It follows that

$$Ш^1_\omega(k, \mathrm{Pic}(\overline{X}_c)) = H^1(k, \mathrm{Pic}(\overline{X}_c)),$$

hence

$$\begin{aligned} H^1(k, \mathrm{Pic}(\overline{X}_c)) &= Ш^2_\omega(k, \mathrm{UPic}(\overline{X})) \\ &= Ш^2_\omega(k, [\mathbf{X}^*(G) \to \mathbf{X}^*(H)\rangle) \end{aligned}$$

which gives a formula for $H^1(k, \mathrm{Pic}(\overline{X}_c))$.

It looks as if one could first compute $\mathsf{UPic}(\overline{X})$, and then obtain $\mathsf{Br_a}(X)$ and $H^1(k, \mathsf{Pic}(\overline{X}_c))$ from $\mathsf{UPic}(\overline{X})$. Actually it happened the other way around.

B. and Kunyavskiĭ, 2000, proved that

$$H^1(k, \mathsf{Pic}(\overline{G}_c)) \simeq Ш^1_\omega(k, \mathsf{Pic}(\overline{G}_c)) \simeq Ш^2_\omega(k, \pi_1(G)^D).$$

By Theorem 4 we have

$$Ш^1_\omega(k, \mathsf{Pic}(\overline{G}_c)) = Ш^2_\omega(k, \mathsf{UPic}(\overline{G})).$$

This led us to the formula of Theorem 1':

$$\mathsf{UPic}(\overline{G}) \simeq \pi_1(G)^D.$$

B., 1999, conjectured that

$$\mathsf{Br_a}(G/H) \simeq H^2(k, [\mathbf{X}^*(G) \to \mathbf{X}^*(H)\rangle).$$

when G^{ss} is simply connected. By Theorem 3 we have

$$\mathsf{Br_a}(G/H) = H^2(k, \mathsf{UPic}(\overline{G/H})).$$

This led us to the formula of Theorem 2:

$$\mathsf{UPic}(\overline{G/H}) \simeq [\mathbf{X}^*(G) \to \mathbf{X}^*(H)\rangle.$$