Time-Dependent Two-Dimensional Fourth-Order Problems: Optimal Convergence

J.-P. Croisille and D. Fishelov

Abstract Here we present a new approach for the analysis of high-order compact schemes for the clamped plate problem. A similar model is the Navier-Stokes equation in streamfunction formulation. In our book "Navier-Stokes Equations in Planar Domains", Imperial College Press, 2013, we have suggested fourth-order compact schemes for the Navier-Stokes equations. The same type of schemes may be applied to the clamped plate problem. For these methods the truncation error is only of first-order at near-boundary points, but is of fourth order at interior points. It is proven that the rate of convergence is actually four, thus the error tends to zero as $O(h^4)$.

1 Introduction

The 2D incompressible Navier-Stokes (NS) equations $\partial_t(\Delta\psi) + (\nabla^{\perp}\psi)\cdot\nabla(\Delta\psi) = \nu\Delta^2\psi$, where ψ is the streamfunction, play an important role in various areas of physics. In [2] we suggested fourth-order compact schemes for the NS problem, including important foundations for their error analysis.

In Section 2 we analyze the error for the two-dimensional problem $\partial_t u + \Delta^2 u = f$- the time-dependent clamped plate problem. This is related to the time dependent Navier-Stokes equations since both equations include the biharmonic operator. We prove that even though the truncation error is only $O(h)$ at near boundary points, the scheme is fourth-order accurate and the error is $O(h^4)$, where h is the mesh size. Similar situations occur also for the high-order finite difference schemes suggested in [1] and [6].

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2 The equation $\partial_t u + \Delta^2 u = f$

Consider the fourth-order partial differential problem

$$\begin{array}{l} \partial_t u + \Delta^2 u = f(x,y,t), \quad (x,y) \in (0,1)\times(0,1), \qquad t>0, \\ u(0,y,t) = u(1,y,t) = 0, \quad u_x(0,y,t) = u_x(1,y,t) = 0, \quad 0 \le y \le 1, \\ u(x,0,t) = u(x,1,t) = 0, \quad u_y(x,0,t) = u_y(x,1,t) = 0, \quad 0 \le x \le 1, \\ u(x,y,0) = g(x,y), \quad (x,y) \in [0,1]\times[0,1]. \end{array} \tag{1}$$

In order to approximate the solution of Equation (1), we lay out a uniform grid $(x_j, y_k) = \left(\frac{j}{N}, \frac{k}{N}\right)$, $j,k = 0,1,...,N$. Let $\mathfrak{f}(t)$ be the evaluation of f at the grid points. Then, we define a grid function $\mathfrak{v}_{j,k}(t)$, which serves as an approximation of $u(x_j, y_k, t)$ for $j,k = 0,...,N$, to be the solution of

$$\begin{array}{l} \partial_t \mathfrak{v}_{j,k}(t) + \tilde{\Delta}_h^2 \mathfrak{v}_{j,k}(t) = \mathfrak{f}_{j,k}(t), \quad j,k = 1,...,N-1, \\ \mathfrak{v}_{0,k}(t) = \mathfrak{v}_{N,k}(t) = 0, \quad (\mathfrak{v}_x)_{0,k}(t) = (\mathfrak{v}_x)_{N,k}(t) = 0, \quad k = 0,...,N, \\ \mathfrak{v}_{j,0}(t) = \mathfrak{v}_{j,N}(t) = 0, \quad (\mathfrak{v}_y)_{j,0}(t) = (\mathfrak{v}_y)_{j,N}(t) = 0, \quad j = 0,...,N, \\ \mathfrak{v}_{j,k}(0) = g_{j,k}, \quad j,k = 0,...,N. \end{array} \tag{2}$$

Here

$$\tilde{\Delta}_h^2 = \delta_x^4 + \delta_y^4 + 2[\delta_x^2\delta_y^2 - \frac{h^2}{12}(\delta_x^4\delta_y^2 + \delta_y^4\delta_x^2)], \tag{3}$$

where, for $j,\ k = 1,...,N-1$,

$$\begin{array}{l} (\delta_x^4 \mathfrak{v})_{j,k} = \frac{12}{h^2}(\delta_x \mathfrak{v}_x - \delta_x^2 \mathfrak{v})_{j,k}, \\ (\delta_y^4 \mathfrak{v})_{j,k} = \frac{12}{h^2}(\delta_y \mathfrak{v}_y - \delta_y^2 \mathfrak{v})_{j,k}, \end{array} \tag{4}$$

$$\begin{array}{l} (\sigma_x \mathfrak{v}_x)_{j,k} = (\delta_x \mathfrak{v})_{j,k}, \\ (\sigma_y \mathfrak{v}_y)_{j,k} = (\delta_y \mathfrak{v})_{j,k}, \end{array} \tag{5}$$

$$\begin{array}{l} (\sigma_x \mathfrak{w})_{j,k} = \frac{1}{6}(\mathfrak{w}_{j-1,k} + 4\mathfrak{w}_{j,k} + \mathfrak{w}_{j+1,k}), \\ (\sigma_y \mathfrak{w})_{j,k} = \frac{1}{6}(\mathfrak{w}_{j,k-1} + 4\mathfrak{w}_{j,k} + \mathfrak{w}_{j,k+1}). \end{array} \tag{6}$$

Thus, the approximated solution satisfies

$$\partial_t \mathfrak{v}_{j,k}(t) + \tilde{\Delta}_h^2 \mathfrak{v}_{j,k}(t) = \mathfrak{f}_{j,k}(t), \quad j,k = 1,...,N-1. \tag{7}$$

Let $u^*(t)$ be the evaluation of u on the grid points at time t. Then,

$$\partial_t u^*_{j,k}(t) + \tilde{\Delta}_h^2 u^*_{j,k}(t) = \mathfrak{f}_{j,k}(t) - \mathfrak{r}_{j,k}(t) \quad j,k = 1,...,N-1, \tag{8}$$

where $\mathfrak{r}(t)$ is the truncation error. By Taylor expansions, if u has continuous derivatives up to order 8, the components of the truncation error $\mathfrak{r}$ for all t may be written as (see [2] Proposition 10.8)

$$\begin{aligned} \mathfrak{r}_{j,k} &= O(h^4) \quad j,k = 2,...,N-2, \\ \mathfrak{r}_{1,k} &= O(h), \quad \mathfrak{r}_{N-1,k} = O(h), \quad k = 1,...,N \\ \mathfrak{r}_{j,1} &= O(h), \quad \mathfrak{r}_{j,N-1} = O(h), \quad j = 1,...,N. \end{aligned} \tag{9}$$

Define the error $\mathfrak{e}(t) = \mathfrak{v}(t) - u^*(t)$. Then, by subtracting (8) from (7), we have

$$\partial_t \mathfrak{e}(t) + \tilde{\Delta}_h^2 \mathfrak{e}(t) = \mathfrak{r}(t). \tag{10}$$

The following Optimal Convergence Theorem holds (see [2], [5], [4]).

Theorem 1 *(One-dimensional case) Suppose that the vector $\tau \in \mathbb{R}^{(N-1)}$, containing the truncation errors, satisfies*

$$\tau_1 = O(h) \quad \tau_j = O(h^4),\ j = 2,...,N-2, \quad \tau_{N-1} = O(h). \tag{11}$$

Then, the operator δ_x^{-4}, operating on τ satisfy

$$\max_{1 \le j \le N-1} |(\delta_x^{-4}\tau)_j| \le Ch^4, \quad \textit{where } C \textit{ does not depend on } N. \tag{12}$$

We relate the grid function $\mathfrak{v}_{j,k}$, $j,k = 1,...,N-1$ with the column vector

$$V = \left[\mathfrak{v}_{1,1},...,\mathfrak{v}_{N-1,1},\mathfrak{v}_{1,2},...\mathfrak{v}_{N-1,2},...,\mathfrak{v}_{1,N-1},...,\mathfrak{v}_{N-1,N-1}\right]^T \in \mathbb{R}^{(N-1)^2}. \tag{13}$$

The bottom ordering of vector $V \in \mathbb{R}^{(N-1)^2}$ is obtained by letting the index j vary first while keeping k fixed, then vary the index k (see [3]). Then, we relate the two-dimensional finite difference operators with matrix operators of size $(N-1)\times(N-1)$ for $N \ge 2$, acting on a vector V. Most of those operators are obtained as Kronecker products of $(N-1)\times(N-1)$ matrices. Recall that the Kronecker product of the matrices $G \in \mathbb{M}_{m,n}$ and $H \in \mathbb{M}_{p,q}$ is the matrix $G \otimes H \in \mathbb{M}_{mp,nq}$ defined by

$$G \otimes H = \begin{bmatrix} g_{1,1}H & g_{1,2}H & \dots & g_{1,n}H \\ \dots & & & \\ \dots & & & \\ g_{m,1}H & g_{m,2}H & \dots & g_{m,n}H \end{bmatrix}. \tag{14}$$

Let the matrix B represent the biharmonic discrete operator in one dimension and the matrix D represent $-\delta_x^2$ (or $-\delta_y^2$) in one dimension. Then, $I \otimes B$ and $B \otimes I$ represent the biharmonic operators δ_x^4 and δ_y^4, respectively. Similarly, $I \otimes D$ and $D \otimes I$ represents the operator $-\delta_x^2$ and $-\delta_y^2$, respectively. In addition,

$$R(t) = \left[\mathfrak{r}_{1,1},...,\mathfrak{r}_{N-1,1},\mathfrak{r}_{1,2},...,\mathfrak{r}_{N-1,2},...,\mathfrak{r}_{1,N-1},...,\mathfrak{r}_{N-1,N-1}\right]^T \in \mathbb{R}^{(N-1)^2} \tag{15}$$

is related to the truncation error. Therefore, inequality (12) may be written in vector notation as follows.

Corollary 1 *Let $R(t) = R^{(1)}(t) + R^{(2)}(t) \in \mathbb{R}^{(N-1)^2}$, where*

$$R^{(1)}(t) = \left[\mathfrak{r}_{1,1},0,...,0,\mathfrak{r}_{N-1,1},\mathfrak{r}_{1,2},...,\mathfrak{r}_{N-1,2},...,\mathfrak{r}_{1,N-1},0,...,0,\mathfrak{r}_{N-1,N-1}\right]^T, \tag{16}$$

$$R^{(2)}(t) = \left[0, \mathfrak{r}_{2,1}, .., \mathfrak{r}_{N-2,1}, 0, 0, ..., 0, ..., 0, ..., 0, 0, \mathfrak{r}_{2,N-1} ..., \mathfrak{r}_{N-2,N-1}, 0\right]^T. \quad (17)$$

Then,

$$\max_{1 \le m \le (N-1)^2} |((I \otimes B^{-1})R^{(1)}(t))_m| \le Ch^4, \quad 0 < t < T, \quad (18)$$

where $I \otimes B^{-1}$ represents the operator δ_x^{-4}, and

$$\max_{1 \le m \le (N-1)^2} |((B^{-1} \otimes I)R^{(2)}(t))_m| \le Ch^4, \quad 0 < t < T, \quad (19)$$

where $(B^{-1} \otimes I)$ represents the operator δ_y^{-4}.

Proof We may write (16) and (17) as $R^{(1)} = \left[R_1^{(1)}; ...; R_{N-1}^{(1)}\right]$ and $R^{(2)} = \left[R_1^{(2)}; ...; R_{N-1}^{(2)}\right]$, respectively, where

$$\begin{aligned} R_1^{(1)} &= [\mathfrak{r}_{1,1}, 0, ..., 0, \mathfrak{r}_{N-1,1}]^T, & R_1^{(2)} &= [0, \mathfrak{r}_{2,1}, ..., \mathfrak{r}_{N-2,1}, 0]^T, \\ R_j^{(1)} &= [\mathfrak{r}_{1,j}, ..., \mathfrak{r}_{N-1,j}]^T,\ j = 2, ..., N-2, & R_j^{(2)} &= [0, ..., 0]^T,\ j = 2, ..., N-2, \\ R_{N-1}^{(1)} &= [\mathfrak{r}_{1,N-1}, 0, ..., 0, \mathfrak{r}_{N-1,N-1}]^T. & R_{N-1}^{(2)} &= [0, \mathfrak{r}_{2,N-1}, ..., \mathfrak{r}_{N-2,N-1}, 0]^T. \end{aligned} \quad (20)$$

Using the definition of a Kronecker product, we have

$$I \otimes B = \begin{bmatrix} B & \underline{0} & \dots & \dots & \underline{0} \\ \underline{0} & B & \underline{0} & \dots & \underline{0} \\ \dots \\ \underline{0} & \underline{0} & \dots & \underline{0} & B \end{bmatrix}, \qquad (I \otimes B)^{-1} = \begin{bmatrix} B^{-1} & \underline{0} & \dots & \dots & \underline{0} \\ \underline{0} & B^{-1} & \underline{0} & \dots & \underline{0} \\ \dots \\ \underline{0} & \underline{0} & \dots & \underline{0} & B^{-1} \end{bmatrix}. \quad (21)$$

Therefore, $(I \otimes B^{-1})R(t) = \left[B^{-1}R_1^{(1)}(t), B^{-1}R_2^{(1)}(t), ..., B^{-1}R_{N-2}^{(1)}(t), B^{-1}R_{N-1}^{(1)}(t)\right]^T$. By the optimal convergence theorem

$$\max_{1 \le m \le (N-1)^2} |((I \otimes B^{-1})R^{(1)}(t))_m| \le Ch^4, \quad 0 < t < T. \quad (22)$$

Hence (18) holds. By a similar proof (19) holds. □

Theorem 2 *Suppose the solution $u(x, y, t)$ to the system (1) has derivatives up to order 8 with respect to x and y, then the error $\mathfrak{e}(t)$ is bounded by*

$$|\mathfrak{e}(t)|_h \le Ch^4, \quad 0 < t < T, \quad (23)$$

where $|\mathfrak{e}(t)|_h = \sqrt{\sum_{j=1}^{N-1}\sum_{k=1}^{N-1} h^2 |\mathfrak{e}_{j,k}(t)|^2}$ and C depends only on $u_0(x, y)$ and T.

Proof Define $E(t)$ as the vector containing the components of the error at time t

$$E = \left[\mathfrak{e}_{1,1}, ... \mathfrak{e}_{N-1,1}, \mathfrak{e}_{1,2}, ... \mathfrak{e}_{N-1,2}, ..., \mathfrak{e}_{1,N-1}, ... \mathfrak{e}_{N-1,N-1}\right]^T \in \mathbb{R}^{(N-1)^2}. \quad (24)$$

The operator $\tilde{\Delta}_h^2$ may be represented by the matrix A of size $(N-1)^2 \times (N-1)^2$ (see [3]), where

$$A = I \otimes B + B \otimes I + 2\big[(I \otimes D)(D \otimes I) + \frac{h^2}{12}(I \otimes D)(B \otimes I) + \frac{h^2}{12}(D \otimes I)(I \otimes B)\big]. \quad (25)$$

Hence, A is a symmetric positive definite matrix. In vector notation Equation (10) may be written as $\partial_t E(t) + A\,E(t) = R(t)$. Multiplying both sides of the last equation by e^{At}, we have $\partial_t(e^{At}\ E(t)) = e^{At}\ R(t)$. Integrating the last equation for ρ from 0 to t and multiplying by e^{-At}, we have

$$E(t) = \int_0^t e^{-A(t-\rho)} R(\rho) d\rho. \quad (26)$$

Multiplying $R(\rho)$ from the left by AA^{-1} yields

$$E(t) = \int_0^t [e^{-A(t-\rho)}\ A]\ [A^{-1}R(\rho)]d\rho = \int_0^t [e^{-A(t-\rho)}\ A]\ [A^{-1}(R^{(1)}(\rho) + R^{(2)}(\rho))]d\rho, \quad (27)$$

where $R^{(1)}$ and $R^{(2)}$ are defined in (16) and (17) (see also (20)). We decompose $E(t)$ in the sum $E(t) = E^{(1)}(t) + E^{(2)}(t)$, where

$$E^{(1)} = \int_0^t [e^{-A(t-\rho)}\ A]\ [A^{-1}\ R^{(1)}(\rho)]d\rho,\ E^{(2)} = \int_0^t [e^{-A(t-\rho)}\ A]\ [A^{-1}\ R^{(2)}(\rho)]d\rho. \quad (28)$$

We show that $\|E^{(1)}\|_2 \le Ch^3$ and $\|E^{(2)}\|_2 \le Ch^3$. Using (25), then for the term $E^{(1)}$ we decompose A as follows. $A = (I \otimes B)Q_1$, where Q_1 is defined by

$$Q_1 = I{\otimes}I{+}(I{\otimes}B)^{-1}(B{\otimes}I){+}2(I{\otimes}B)^{-1}\big[(I{\otimes}D)(D{\otimes}I){+}\frac{h^2}{12}(I{\otimes}D)(B{\otimes}I){+}\frac{h^2}{12}(D{\otimes}I)(I{\otimes}B)\big]. \quad (29)$$

Using (25), then for the term $E^{(2)}$ we decompose A as follows. $A = (B \otimes I)Q_2$, where Q_2 is defined by

$$Q_2 = I{\otimes}I{+}(B{\otimes}I)^{-1}(I{\otimes}B){+}2(B{\otimes}I)^{-1}\big[(I{\otimes}D)(D{\otimes}I){+}\frac{h^2}{12}(I{\otimes}D)(B{\otimes}I){+}\frac{h^2}{12}(D{\otimes}I)(I{\otimes}B)\big]. \quad (30)$$

Therefore,

$$\begin{aligned} E^{(1)}(t) &= \int_0^t [e^{-A(t-\rho)}\ A]\ \ Q_1^{-1}\ [(I \otimes B)^{-1}R^{(1)}(\rho)]d\rho \\ E^{(2)}(t) &= \int_0^t [e^{-A(t-\rho)}\ A]\ \ Q_2^{-1}\ [(I \otimes B)^{-1}R^{(2)}(\rho)]d\rho. \end{aligned} \quad (31)$$

First we consider $\|E^{(1)}(t)\|_2$. Expanding on $Q_1^{-1}\ [(I \otimes B)^{-1}R^{(1)}(\rho)]$, we prove that the norm of Q_1^{-1} is bounded from above. Note that (since Q_1^{-1} and Q_1 are not necessarily symmetric matrices),

$$\|Q_1^{-1}\|_2 = \sqrt{\max_{1 \le k \le (N-1)^2} |\lambda_k((Q_1^{-1})^T Q_1^{-1})|}. \quad (32)$$

We show that the eigenvalues of $(Q_1^{-1})^T Q_1^{-1}$ are positive and bounded from above by 1. Alternatively, we show that eigenvalues of $Q_1^T Q_1$ are bounded from below by 1. We may decompose Q_1 as a sum $Q_1 = K_1 + K_2$, where

$$\begin{aligned} K_1 &= I \otimes I + (I \otimes B)^{-1}(B \otimes I) \\ K_2 &= 2(I \otimes B)^{-1}\left[(I \otimes D)(D \otimes I) + \tfrac{h^2}{2}(I \otimes D)(B \otimes I) + \tfrac{h^2}{2}(D \otimes I)(I \otimes B)\right]. \end{aligned} \tag{33}$$

Thus, $Q_1^T\, Q_1 = (K_1 + K_2)^T(K_1 + K_2) = K_1^T K_1 + (K_1^T K_2 + K_2^T K_1) + K_2^T K_2$. The matrix K_1 is decomposed as a sum of the two positive definite matrices $K_1 = P_1 + P_2$, where $P_1 = I \otimes I, \quad P_2 = (I \otimes B)^{-1}(B \otimes I)$. Note that P_1 and P_2 are symmetric positive-definite matrices. Therefore, the matrix $K_1^T K_1$ may be written as

$$K_1^T K_1 = I \otimes I + 2P_2 + P_2^2. \tag{34}$$

Thus, $K_1^T K_1$ is a sum of a symmetric positive definite matrix $I \otimes I$ and a symmetric positive definite matrix $2P_2 + P_2^2$. Since all the eigenvalues of $I \otimes I$ are 1, then all the eigenvalues of $K_1^T K_1$ are greater than 1. Now we consider the matrix $K_1^T K_2 + K_2^T K_1$, which is a symmetric matrix. We show that its eigenvalues are positive. First, the matrix K_1 is symmetric positive definite. Next, the matrix K_2 is a product of two symmetric positive definite matrices S and T, where

$$S = 2(I \otimes B)^{-1},\; T = (I \otimes D)(D \otimes I) + \tfrac{h^2}{2}(I \otimes D)(B \otimes I) + \tfrac{h^2}{2}(D \otimes I)(I \otimes B). \tag{35}$$

Thus,

$$K_2 = ST = ST^{1/2}T^{1/2} = T^{-1/2}T^{1/2}ST^{1/2}T^{1/2} = T^{-1/2}(T^{1/2}ST^{1/2})T^{1/2}. \tag{36}$$

Therefore, K_2 is similar to a positive definite matrix, thus its eigenvalues are positive. Since K_1^T and K_2 are positive definite matrices, then by a similar argument as in (35)-(36), the eigenvalues of $K_1^T K_2$ are positive. Similarly, the eigenvalues of $K_2^T K_1$ are also positive. Therefore, the matrix $K_1^T K_2 + K_2^T K_1$ is symmetric, having positive eigenvalues. Consider now the symmetric matrix $K_2^T K_2$. We have shown that the eigenvalues of K_2 are positive, therefore so are the eigenvalues of $K_2^T K_2$. Hence, all the eigenvalues of $Q_1^T Q_1$ are greater than 1. As a result, all the eigenvalues of $(Q_1^{-1})^T Q_1^{-1}$ are smaller than 1. Hence,

$$\|Q_1^{-1}\|_2 = \sqrt{\max_{1 \le k \le (N-1)^2} |\lambda_k\left((Q_1^{-1})^T Q_1^{-1}\right)|} \le 1. \tag{37}$$

are symmetric positive definite matrices. Similarly, $\|Q_2^{-1}\|_2 \le 1$. We continue with bounding $E^{(1)}(t)$. The matrix $e^{-A(t-\rho)}A$ may be diagonalized by a unitary matrix Z, which is independent of $t - \rho$ containing the normalized eigenvectors of the symmetric matrix A. Thus,

$$e^{-A(t-\rho)}A = Z\; \Lambda(t-\rho)\; Z^T, \tag{38}$$

where $\Lambda(\rho) = \text{diag}(e^{-\lambda_1(t-\rho)}\lambda_1, ..., e^{-\lambda_{(N-1)^2}(t-\rho)}\lambda_{(N-1)^2})$ and $\lambda_1, ..., \lambda_{(N-1)^2}$ are the eigenvalues of A. Since Z is independent of $t-\rho$, we obtain from (31) and (38)

$$E^{(1)}(t) = Z\int_0^t \Lambda(t-\rho)Z^T\, Q_1^{-1}\,[(I\otimes B)^{-1}R^{(1)}(\rho)]d\rho. \quad (39)$$

We consider now the component i (for $i = 1, ..., (N-1)^2$) of the vector $E^{(1)}(t)$.

$$E_i^{(1)}(t) = \sum_{k=1}^{(N-1)^2} Z_{ik}\int_0^t \Lambda_{k,k}(t-\rho)\,(Z^T\, Q_1^{-1}\,(I\otimes B)^{-1}\,R^{(1)}(\rho))_k d\rho. \quad (40)$$

Expanding on $(Z^T\, Q_1^{-1}\,(I\otimes B)^{-1}\,R^{(1)}(\rho))_k$, we have

$$\begin{aligned}(Z^T\, Q_1^{-1}\,(I\otimes B)^{-1}\,R^{(1)}(\rho))_k &= \sum_{l=1}^{(N-1)^2}(Z^T\, Q_1^{-1})_{kl}\,((I\otimes B)^{-1}\,R^{(1)}(\rho))_l\\ &= \sum_{l=1}^{(N-1)^2}(Z^T\, Q_1^{-1})_{kl}\sum_{m=1}^{(N-1)^2}(I\otimes B)^{-1}_{lm}\,R_m^{(1)}(\rho).\end{aligned} \quad (41)$$

$$E_i^{(1)}(t) = \sum_{k=1}^{(N-1)^2} Z_{ik}\sum_{l=1}^{(N-1)^2}(Z^T\, Q_1^{-1})_{kl}\sum_{m=1}^{(N-1)^2}(I\otimes B)^{-1}_{lm}\int_0^t \Lambda_{k,k}(t-\rho)\,R_m^{(1)}(\rho)d\rho. \quad (42)$$

Since $\Lambda_{k,k}(t-\rho) = e^{-\lambda_k(t-\rho)}\lambda_k$ and $e^{-\lambda_k(t-\rho)}\lambda_k \geq 0$, we have (by the extended mean-value theorem for integration)

$$\begin{aligned}E_i^{(1)}(t) &= \sum_{k=1}^{(N-1)^2} Z_{ik}\ \sum_{l=1}^{(N-1)^2}(Z^T\, Q_1^{-1})_{kl}\ \sum_{m=1}^{(N-1)^2}(I\otimes B)^{-1}_{lm}\Big[\int_0^t e^{-\lambda_k(t-\rho)}\,\lambda_k d\rho\Big]\,R_m^{(1)}(\rho_{m,k})\\ &= \sum_{k=1}^{(N-1)^2} Z_{ik}\big[1-e^{-\lambda_k t}\big]\ \sum_{l=1}^{(N-1)^2}(Z^T\, Q_1^{-1})_{kl}\ \sum_{m=1}^{(N-1)^2}(I\otimes B)^{-1}_{lm}\,R_m^{(1)}(\rho_{m,k}),\end{aligned} \quad (43)$$

where $0 \leq \rho_{m,k} \leq t$.

Let $L^{(k)} = [R_1(\rho_{1,k}), R_2(\rho_{2,k}), ..., R_{(N-1)^2}(\rho_{(N-1)^2,k})]^T$. Using (16), we have

$$L^{(k)} = [O(h), 0, ..., 0, O(h), O(h), O(h^4), ..., O(h^4), O(h), ..., O(h), 0, ..., 0, O(h)]^T. \quad (44)$$

Define $V^{(k)} = (I\otimes B)^{-1}L^{(k)}$. Then, Equation (43) may be written as

$$E_i(t) = \sum_{k=1}^{(N-1)^2} Z_{ik}\big[1-e^{-\lambda_k t}\big]\ \sum_{l=1}^{(N-1)^2}(Z^T\, Q_1^{-1})_{kl}\ V_l^{(k)}. \quad (45)$$

By the Corollary 1, Equation (18), we have

$$|V_l^{(k)}| = |\sum_{m=1}^{(N-1)^2}((I\otimes B)^{-1})_{lm}\ L_m^{(k)}| \leq Ch^4, \quad 0 < t < T, \quad (46)$$

where C is independent of N. Define the vector W by $W_l = \max_{k=1,...,(N-1)^2}|V_l^{(k)}|$. By Equation (46) the L_2 norm of the vector W is bounded by

$$\|W\|_2 \leq Ch^3. \quad (47)$$

Define $D_1 = \text{diag}(1 - e^{-\lambda_1 t}, ..., 1 - e^{-\lambda_{(N-1)^2} t})$. Therefore, Equation (45) yields

$$\|E^{(1)}(t)\|_2 \leq \|Z\|_2 \|D_1\|_2 \, \|Z^T\|_2 \, \|Q_1^{-1}\|_2 \, \|W\|_2. \tag{48}$$

Since $Z^T = Z^{-1}$ and by Equation (37), we have $\|Z\|_2 = \|Z^T\|_2 = 1, \quad \|Q_1^{-1}\|_2 \leq 1$. We show now that $\|D_1\|_2 \leq C$. Since the eigenvalues A are positive, we have $\|D_1\|_2 = \max_{1 \leq j \leq (N-1)^2} |1 - e^{-\lambda_j t}| \leq 1$. We conclude from (48) (47) that $\|E^{(1)}(t)\|_2 \leq Ch^3$. Similarly, $\|E^{(2)}(t)\|_2 \leq Ch^3$. Therefore, for $|\mathfrak{e}(t)|_h = \sqrt{\sum_{j=1}^{N-1} \sum_{k=1}^{N-1} h^2 |\mathfrak{e}_{j,k}|^2}$, we have $|\mathfrak{e}(t)|_h \leq Ch^4, \quad 0 < t < T$, which concludes the proof. □

3 Numerical Results

Consider the equation $u_t + \Delta^2 u = f$ with the exact solution $u = e^{-t}(1-x^2)^2(1-y^2)^2$ on $[-1, 1]$, $t > 0$, where $f(x,t)$ is chosen so that u is the solution of the differential equation above.

Table 1 Compact scheme for $u_t + \Delta^2 u = f$ with exact solution: $u = e^{-t}(1-x^2)^2(1-y^2)^2$ on $[-1, 1]$, $t > 0$. We present $|e|_h$ the error in u, and $|e_x|_h$ the error in u_x in the l_2 norm at $t = 0.25$.

Mesh	$N = 8$	Rate	$N = 16$	Rate	$N = 32$	Rate	$N = 64$
$\|e\|_h$	1.0819(-4)	3.91	7.2142(-6)	4.00	4.5152(-7)	4.00	2.8221(-8)
$\|e_x\|_h$	1.8773(-4)	3.97	1.2001(-5)	4.01	7.4422(-7)	4.00	4.6480(-8)

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