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# CONVERGENCE OF A COMPACT SCHEME FOR THE PURE STREAMFUNCTION FORMULATION OF THE UNSTEADY NAVIER–STOKES SYSTEM*

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**Abstract.** This paper is devoted to the analysis of a new compact scheme for the Navier–Stokes equations in pure streamfunction formulation. Numerical results using that scheme have been reported in [M. Ben-Artzi et al., *J. Comput. Phys.*, 205 (2005), pp. 640–664]. The scheme discussed here combines the Stephenson scheme for the biharmonic operator and ideas from box-scheme methodology. Consistency and convergence are proved for the full nonlinear system. Instead of customary periodic conditions, the case of boundary conditions is addressed. It is shown that in one dimension the truncation error for the biharmonic operator is $O(h^4)$ at interior points and $O(h)$ at near-boundary points. In two dimensions the truncation error is $O(h^2)$ at interior points (due to the cross-terms) and $O(h)$ at near-boundary points. Hence the scheme is globally of order four in the one-dimensional periodic case and of order two in the two-dimensional periodic case, but of order 3/2 for one- and two-dimensional nonperiodic boundary conditions. We emphasize in particular that there is no special treatment of the boundary, thus allowing robust use of the scheme. The finite element analogy of the finite difference schemes is invoked at several stages of the proofs in order to simplify their verifications.

**Key words.** finite difference compact schemes, Stephenson scheme, box schemes, finite elements, Navier–Stokes equations, streamfunction formulation, biharmonic problem, fourth order problem

**AMS subject classifications.** 65L20, 65L70, 65M06, 65M12, 65M70, 35Q30, 76D05, 78M10, 78M20

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## 1. Introduction.

In a recent paper [3] we presented a fourth-order compact scheme for the pure streamfunction formulation of the two-dimensional (incompressible) Navier–Stokes equations. We have given there a convergence analysis for the linearized model. In this paper we prove the convergence of the nonlinear scheme, without any further assumptions. Recall that the pure streamfunction formulation of the (two-dimensional) Navier–Stokes equations is classical [15]. It has the advantage of reducing the system to a single evolution equation for the scalar streamfunction having the form

$$\frac{\partial \Delta\psi}{\partial t} + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \nu\Delta^2\psi = 0. \tag{1}$$

The velocity field is $(u, v) = \nabla^{\perp}\psi = (-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x})$, and the vorticity is $\omega = \Delta\psi$. The price paid for reducing the system to a single equation is that one must now deal with the biharmonic $\Delta^2$ operator. There are therefore two boundary conditions imposed

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on $\psi$. For the typical "no-leak no-slip" conditions (vanishing velocity on the fixed boundary) we have

$$\nabla\psi = 0 \qquad \text{on the boundary.} \tag{2}$$

Since the function $\psi$ is only determined up to a constant, condition (2) is equivalent to

$$\psi = \frac{\partial\psi}{\partial n} = 0, \tag{3}$$

which, for simplicity, will be the case treated in this paper. Clearly (2) is equivalent to the assumption $\psi \in H_0^2$, the closure of smooth compactly supported test functions in the Sobolev space of functions having square-summable derivatives up to second order.

Our scheme can be described as follows (see [3] for details). At each time step the scheme solves a time implicit version of (1). This leads to a fourth-order biharmonic problem of the form

$$\Delta\psi - \nu\Delta^2\psi = f, \tag{4}$$

subject to the boundary conditions (2).

The spatial discretization of (4) makes use of the Stephenson scheme for the the biharmonic operator introduced in [19], [12]. See also [2]. This scheme can be interpreted as a mixed scheme in $(\psi, \nabla\psi)$, similar in form to a version of a box scheme [14], [7]. More specifically, its design is obtained by a spline collocation procedure on a nine-point stencil, which we recall in section 3 below.

The streamline-vorticity formulation has been extensively used for the simulation of the two-dimensional Navier–Stokes system. As representative references we mention [17], [8], [5], [9], [13], and the references therein. One difficult point is that "...the $\psi - \omega$ system is inextricably coupled; BC's and solution methods must contend with this fact..." [10, p. 431]. Indeed, one must cope with the vorticity boundary values, resulting from the fact that the relation $\Delta\psi = \omega$ is overdetermined under condition (2). An attempt to avoid this difficulty has been made in [4], where the need to determine these values was circumvented by switching to the biharmonic equation (at each time step), exploiting the natural condition (2). The scheme presented in [3], whose convergence is proved here, has avoided all explicit mention of the vorticity by using a pure streamfunction formulation. We mention that recently in [11] a very similar algorithm has been proposed, but it deals only with the steady-state Navier–Stokes system.

The paper is organized as follows. First, we introduce in section 2 our notation and the setup for our discrete spaces. Then we establish in sections 3 and 4 the necessary analytic properties of the scheme in one and two dimensions. In particular, in analogy with the coercivity of $\Delta^2$ in $H_0^2$, we prove the coercivity of the discretized biharmonic operator in a suitable discrete analogue of $H_0^2$. We prove that the truncation error of the biharmonic scheme is of order four in one dimension and of order two in two dimensions, at all interior points and of first order at near-boundary points, giving a 3/2 order of convergence rate in the natural discrete $L^2$ norm. Note that in the periodic case all points are interior. Then in section 5, we prove that the same order of convergence extends to the spatial semidiscrete version of the full nonlinear scheme. We emphasize the fact that we do not need any special treatment of boundary points, and the boundary condition (2) is naturally incorporated here. As mentioned above, this causes a reduced (from four to one) order of local truncation error at the boundary, and is reflected in the fact that our result yields a 3/2 convergence rate in the discrete $L^2$ norm. The present convergence result can be compared to the convergence results obtained in [9], [13]. In both papers, the time evolution is performed on the vorticity, and hence a very careful treatment of the vorticity boundary conditions is required, either by "ghost-points" [9] or by replacing condition (2) on the normal derivative of the streamfunction by boundary conditions on the vorticity [13] (which, as these authors observe, amounts to an algorithm for vorticity generation on the boundary).

**2. Discrete spaces and basic inequalities.** Let $0 \leq i,j \leq N$. We denote by $(ih, jh)$ a finite difference mesh on the square $[0,1]^2$, with equal mesh size $h = 1/N$ in the $x$ and $y$ directions. We denote by $u_{i,j}$ a grid function on $[0,1]^2$, with $0 \leq i,j \leq N$. The centered and upwind derivative operators $\delta_x$, $\delta_x^{\pm}$ are defined as usual in each direction by

$$\delta_x u_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}, \quad \delta_x^+ u_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{h}, \quad \delta_x^- u_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{h}, \tag{5}$$

and similarly in the $y$ direction:

$$\delta_y u_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2h}, \quad \delta_y^+ u_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{h}, \quad \delta_y^- u_{i,j} = \frac{u_{i,j} - u_{i,j-1}}{h}. \tag{6}$$

The centered second-order derivatives are

$$\delta_x^2 u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h^2}, \quad \delta_y^2 u_{i,j} = \frac{u_{i,j+1} + u_{i,j-1} - 2u_{i,j}}{h^2}. \tag{7}$$

The five-point Laplacian is

$$\Delta_h u_{i,j} = \delta_x^2 u_{i,j} + \delta_y^2 u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2}. \tag{8}$$

The crossed derivative operators $\delta_{xy}^+$, $\delta_{xy}^-$, $\delta_{xy}$ are

$$\delta_{xy}^+ u_{i,j} = \delta_x^+ \delta_y^+ u_{i,j} = \frac{u_{i+1,j+1} - u_{i+1,j} - u_{i,j+1} + u_{i,j}}{h^2}, \tag{9}$$

$$\delta_{xy}^- u_{i,j} = \delta_x^- \delta_y^- u_{i,j} = \frac{u_{i,j} - u_{i,j-1} - u_{i-1,j} + u_{i-1,j-1}}{h^2}, \tag{10}$$

$$\delta_{xy} u_{i,j} = \delta_x \delta_y u_{i,j} = \frac{u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}}{4h^2}. \tag{11}$$

It is easy to check that

$$\delta_x^2 \delta_y^2 u_{i,j} = \delta_{xy}^+ \delta_{xy}^- u_{i,j}. \tag{12}$$

The $L_h^2$ space is the space of sequences $u_{i,j}$, $0 \leq i,j \leq N$. $L_{h,0}^2$ is the subspace of $u_{i,j}$ with zero boundary conditions $u_{i,j} = 0$ for $i \in \{0, N\}$ or $j \in \{0, N\}$. The scalar product on $L_{h,0}^2$ is

$$(u, v)_h = h^2 \sum_{i,j=1}^{N-1} u_{i,j} v_{i,j}, \tag{13}$$

with the corresponding norm

$$|u|_h = \left\{ h^2 \sum_{i,j=1}^{N-1} (u_{i,j})^2 \right\}^{1/2}. \tag{14}$$

Furthermore, we denote by $l_h^2$ the space of sequences $u_i$, $0 \le i \le N$, and by $l_{h,0}^2$ the subspace of sequences with zero boundary conditions. The scalar product and the norm on $l_{h,0}^2$ are

$$(u,v)_h = h \sum_{i=1}^{N-1} u_i v_i, \quad |u|_h^2 = \left\{ h \sum_{i=1}^{N-1} u_i^2 \right\}^{1/2}. \tag{15}$$

We also define the discrete infinity norm

$$|u|_{\infty,h} = \max_i |u_i|. \tag{16}$$

We skip the proof of the following lemma, which states the discrete integration by parts in $L_{h,0}^2$ for the operators $\delta_x^{\pm}$, $\delta_x^2$. For each grid function $u \in L_{h,0}^2$, we denote the one-dimensional column vector $u^j = [u_{1,j}, u_{2,j}, \dots, u_{N-1,j}]^T$, $1 \le j \le N-1$.

LEMMA 2.1 (discrete integration by parts). *For any $u, v \in L_{h,0}^2$, we have*

$$\text{(i)} \quad (\delta_x^+ u, v)_h = -(u, \delta_x^- v)_h; \tag{17}$$

$$\text{(ii)} \quad (\delta_x^2 u, v)_h = -(\delta_x^+ u, \delta_x^+ v)_h = -(\delta_x^- u, \delta_x^- v)_h. \tag{18}$$

Note that in (17), (18), the finite difference operators are extended to the points $i = 0$, $i = N$ by

$$\delta_x^{\pm} u_0 = \delta_x^{\pm} u_N = 0, \quad \delta_x^2 u_0 = \delta_x^2 u_N = 0. \tag{19}$$

Observe that this assumption is only for notational convenience, in order to have formally $\delta_x^{\pm} u, \delta_x^2 u \in L_{h,0}^2$. Results similar to (17), (18) in the $y$ direction are obtained by substituting the subscript $y$ to the subscript $x$. The following lemma is the counterpart of the Poincaré inequality at the discrete level.

LEMMA 2.2 (discrete Poincaré inequality). *For all $u \in L_{h,0}^2$ and any $1 \le j \le N-1$,*

$$|u^j|_h \le 2 |\delta_x^+ u^j|_h. \tag{20}$$

COROLLARY 2.1. *For all $u \in L_{h,0}^2$,*

$$|u|_h \le \sqrt{2} \left[ |\delta_x^+ u|_h^2 + |\delta_y^+ u|_h^2 \right]^{1/2}. \tag{21}$$

*Proof.* For all $u \in l_{h,0}^2$, we have

$$|u|_h^2 = h \sum_{i_0=1}^{N-1} u_{i_0}^2. \tag{22}$$

For all $1 \le i_0 \le N-1$,

$$\begin{aligned} u_{i_0}^2 &= \sum_{i=0}^{i_0-1} (u_{i+1} - u_i)(u_{i+1} + u_i) = \sum_{i=0}^{i_0-1} h \delta_x^+ u_i (u_i + (Su)_i) \\ &\le 2 |\delta_x^+ u|_h |u|_h, \end{aligned}$$

where $(Su)_j = u_{j+1}$, $j = 0, \dots, N-1$. Therefore,

$$|u|_h^2 = h \sum_{i_0=1}^{N-1} u_{i_0}^2 \le 2|\delta_x^+ u|_h |u|_h, \tag{23}$$

which gives (20).

Now for all $u \in L^2_{h,0}$, we have

$$\begin{aligned} |u|_h^2 = h \sum_{j_0=1}^{N-1} |u^{j_0}|_h^2 &\le 2h \sum_{j_0=1}^{N-1} |\delta_x^+ u^{j_0}|_h |u^{j_0}|_h \\ &\le 2\Bigg(\sum_{j_0=1}^{N-1} h|\delta_x^+ u^{j_0}|^2\Bigg)^{1/2} \Bigg(\sum_{j_0=1}^{N-1} h|u^{j_0}|^2\Bigg)^{1/2} \\ &\le 2|\delta_x^+ u|_h |u|_h. \end{aligned} \tag{24}$$

In a similar way, we obtain in the $y$ direction

$$|u|_h^2 \le 2|\delta_y^+ u|_h |u|_h. \tag{25}$$

Summing (24) and (25), we obtain (21). $\square$

## 3. The Stephenson scheme in one dimension.

**3.1. Design by collocation.** Consider the one-dimensional biharmonic equation

$$\begin{cases} u^{(4)}(x) = f(x), \quad 0 < x < 1, \\ u(0) = u(1) = u_x(0) = u_x(1) = 0. \end{cases} \tag{26}$$

Suppose that at each node $x_j = jh$, $0 \le j \le N$, of a finite difference grid, there are two unknowns $u_j$ and $u_{x,j}$ approximating, respectively, $u(x_j)$ and $u_x(x_j)$, which is referred to as a "mixed scheme." The values $u_j$, $u_{x,j}$ are solutions of the linear system, designed by the following Galerkin collocation method. At each interior node $j$, $1 \le j \le N-1$, we consider a fourth-order polynomial, with domain $[x_{j-1}, x_{j+1}]$

$$Q(x) = a_0 + a_1(x - x_j) + a_2(x - x_j)^2 + a_3(x - x_j)^3 + a_4(x - x_j)^4. \tag{27}$$

The five coefficients $a_k$, $k \in \{0, 1, 2, 3, 4\}$, are defined by the five collocation conditions on the compact stencil $\{x_{j-1}, x_j, x_{j+1}\}$ (see Figure 1):

$$\begin{cases} Q(x_{j-1}) = u_{j-1}, \quad Q(x_j) = u_j, \quad Q(x_{j+1}) = u_{j+1}, \\ Q'(x_{j-1}) = u_{x,j-1}, \quad Q'(x_{j+1}) = u_{x,j+1}. \end{cases} \tag{28}$$

The five coefficients of the unique polynomial (27), solution of (28), are given by

$$\begin{cases} a_0 = u_j, \\ a_1 = \dfrac{3}{2}\delta_x u_j - \dfrac{1}{4}(u_{x,j+1} + u_{x,j-1}), \\ a_2 = \delta_x^2 u_j - \dfrac{1}{2}(\delta_x u_x)_j, \\ a_3 = \dfrac{1}{h^2}(\delta_x u_j - u_{x,j}) = \dfrac{1}{6}(\delta_x^2 u_x)_j, \\ a_4 = \dfrac{1}{2h^2}\big[(\delta_x u_x)_j - \delta_x^2 u_j\big]. \end{cases} \tag{29}$$

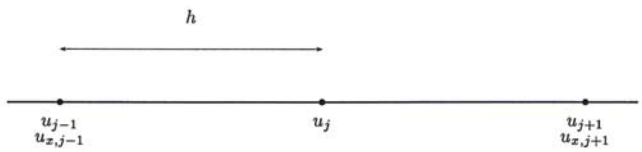


FIG. 1. *Stephenson's scheme for $u^{(4)} = f$: The finite difference operator $\delta_x^4 u_j$ at point $j$ is $Q^{(4)}(x_j)$, where $Q(x) \in P^4[x_{j-1}, x_{j+1}]$ is defined by the five collocated values for $u_{j-1}$, $u_j$, $u_{j+1}$, $u_{x,j-1}$, $u_{x,j+1}$.*

Now, since $Q'(x_j) = a_1$ and $Q''''(x_j) = 24a_4$, it is natural to define the following compact scheme: find $[u_0, u_1, \dots, u_{N-1}, u_N]$, $[u_{x,0}, u_{x,1}, \dots, u_{x,N-1}, u_{x,N}] \in l^2_{h,0}$, which solve

$$\begin{cases} \text{(a)} \ \ (P_x u_x)_j = \delta_x u_j, \quad 1 \le j \le N-1, \\ \text{(b)} \ \ \delta_x^4 u_j = f(x_j), \quad 1 \le j \le N-1, \\ \text{(c)} \ \ u_0 = u_1 = u_{x,0} = u_{x,N} = 0, \end{cases} \tag{30}$$

where the operators $P_x$, $\delta_x^4$ are, respectively, defined in (31), (34).

For $u \in l^2_{h,0}$, the operator $P_x$ is defined by

$$(P_x u)_j = \frac{1}{6} u_{j-1} + \frac{2}{3} u_j + \frac{1}{6} u_{j+1}, \quad 1 \le j \le N-1. \tag{31}$$

$P_x$ will be referred to as the *Simpson operator* in the $x$ direction, because the coefficients in (30) are those of the Simpson quadrature formula over $[x_{j-1}, x_{j+1}]$. Note also that

$$P_x = I + \frac{h^2}{6} \delta_x^2. \tag{32}$$

We also note that the connection (30)(a) is already given in the classical book by Collatz [6, Chap. III, Eq. 2.9]. We call $\mathcal{S}$ the discrete space of grid functions $(u, u_x) \in l^2_{h,0} \times l^2_{h,0}$,

$$\mathcal{S} = \big\{ (u, u_x) \in l^2_{h,0} \text{ such that } P_x u_x = \delta_x u \big\}. \tag{33}$$

In (30), we define the *Stephenson discrete biharmonic* to be the compact difference operator given on $\mathcal{S}$ by

$$\delta_x^4 u_j = \frac{12}{h^2} \big\{ (\delta_x u_x)_j - \delta_x^2 u_j \big\}, \quad 1 \le j \le N-1. \tag{34}$$

This is a one-dimensional version of the original scheme proposed by Stephenson in [19]. Note that for simplicity, we will refer in what follows to a grid function in $\mathcal{S}$ by $u \in \mathcal{S}$, meaning that it is the first component of a pair $(u, u_x) \in \mathcal{S}$.

*Remark*. We note that the implicit scheme (30)(a) defining the grid function $u_x$ as a function of $u$ is exactly the one obtained in the piecewise cubic spline interpolation; see, e.g., [18]. The classical question that occurs in spline interpolation about fixing the two degrees of freedom $u_{x,0}$, $u_{x,N}$ at end points is here pointless, since they are precisely given in (30)(c).

**3.2. Consistency.** On a periodic grid, the order of consistency can be obtained by a simple Taylor expansion at point $x_j$. Equivalently, one can compute the symbol of the operators. Recall that in the context of finite difference operators, we have to use the semidiscrete Fourier transform; see, e.g., [20]. In practice, if the values of the periodic grid function $(u_j)$ are represented by $e^{ij\xi h}$, then the symbol of the linear operator $L_h$ is $l_h(\xi)$ defined by

$$L_h u_j = l_h(\xi) u_j. \tag{35}$$

Furthermore, if $l(\xi)$ is the symbol of $L$, then the order of consistency is given by the greatest value $p > 0$ such that (see [20])

$$l_h(\xi) - l(\xi) = O(h^p). \tag{36}$$

Doing so, it is quite easy to verify that the Stephenson gradient is fourth-order accurate and that the biharmonic operator (34) is as well. Indeed, we verify the following:

- The symbol of the discrete operator $u_x$ in (30)(a) is

$$g_h(\xi) = i\xi - \frac{1}{180} i\xi^5 h^4 + O(h^6), \tag{37}$$

so that the order of accuracy with respect to the operator $\partial_x$, whose symbol is $i\xi$, is

$$g_h(\xi) - i\xi = O(h^4). \tag{38}$$

- The symbol of the discrete operator $\delta_x^4 u$ in (34) is

$$d_h(\xi) = \xi^4 - \frac{1}{720}\xi^8 h^4 + O(h^6), \tag{39}$$

so that the order of accuracy with respect to $\partial_x^4$ is

$$d_h(\xi) - (i\xi)^4 = O(h^4). \tag{40}$$

On a finite grid with homogeneous boundary conditions at the two ends, we have to perform a more careful analysis, because the symbolic computation no longer holds in this case.

LEMMA 3.1. *Suppose that $u(x)$ is a regular function on $[0,1]$. Then the finite difference gradient $u_x$ defined from the values $u(x_j)$, $0 \le j \le N$, by $(P_x u_x)_j = \delta_x u(x_j)$ has a truncation error $(u_x)_j - u'(x_j)$ of order four at each point $x_j$. More precisely,*

$$|(u_x)_j - u'(x_j)| \le C h^4 |u^{(5)}|_{\infty,[0,1]}. \tag{41}$$

*Proof.* The Stephenson gradient $u_x$ is defined in the space $l^2_{h,0}$ by

$$(P_x u_x)_j = (\delta_x u)_j, \;\; 1 \le j \le N-1, \tag{42}$$

where $P_x$ is the $N-1 \times N-1$ matrix-operator acting on $l^2_{h,0}$ as defined in (31), that is,

$$P_x = \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & 0 & \dots & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \ddots & \\ \vdots & & \ddots & \ddots & \\ 0 & \dots & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & \dots & \dots & \frac{1}{6} & \frac{2}{3} \end{pmatrix}. \tag{43}$$

Consider a regular function $u(x)$, differentiable as much as needed, and denote by $u'$, $u''$, $\dots$, $u^{(p)}$, its derivatives. At each point $x_j$, $1 \le j \le N-1$, the Taylor formula gives (we note $u_j^{(m)} = u^{(m)}(x_j)$)

$$(\delta_x u)(x_j) = u_j' + \frac{h^2}{6}u_j^{(3)} + \frac{h^4}{2\,5!}\big[u^{(5)}(\xi_{1,j}^-) + u^{(5)}(\xi_{1,j}^+)\big], \tag{44}$$

where $\xi_{1,j}^- \in \,]x_{j-1}, x_j[$ and $\xi_{1,j}^+ \in \,]x_j, x_{j+1}[$. Similarly, there exist $\xi_{2,j}^- \in \,]x_{j-1}, x_j[$, $\xi_{2,j}^+ \in \,]x_j, x_{j+1}[$ such that

$$(\delta_x^2 u)(x_j) = u_j'' + \frac{h^2}{4!}\big[u^{(4)}(\xi_{2,j}^-) + u^{(4)}(\xi_{2,j}^+)\big]. \tag{45}$$

We deduce that, applying (45) to $u'$,

$$\begin{aligned}
\delta_x u(x_j) - P_x u'(x_j) &= \delta_x u(x_j) - \left[u'(x_j) + \frac{h^2}{6}\delta_x^2 u'(x_j)\right] \\
&= u_j' + \frac{h^2}{6}u_j^{(3)} + \frac{h^4}{2.5!}\left(u^{(5)}(\xi_{1,j}^-) + u^{(5)}(\xi_{1,j}^+)\right) \\
&\quad -\left[u_j' + \frac{h^2}{6}\left(u_j^{(3)} + \frac{h^2}{4!}\big[u^{(5)}(\xi_{2,j}^-) + u^{(5)}(\xi_{2,j}^+)\big]\right)\right] \\
&= h^4 v_j,
\end{aligned}$$

where the grid function $v_j$ is defined by

$$v_j = \frac{1}{2.5!}\big(u^{(5)}(\xi_{1,j}^+) + u^{(5)}(\xi_{1,j}^-)\big) - \frac{1}{6.4!}\big(u^{(5)}(\xi_{2,j}^-) + u^{(5)}(\xi_{2,j}^+)\big). \tag{46}$$

Therefore, the grid function $u \in l_{h,0}^2$ verifies the identity

$$\delta_x u(x_j) - P_x u'(x_j) = h^4 v_j. \tag{47}$$

On the other hand, $u_x \in l_{h,0}^2$ is defined by

$$\delta_x u - P_x u_x = 0. \tag{48}$$

Subtracting (48) from (47), we obtain the identity in $l_{h,0}^2$,

$$u' - u_x = h^4 P_x^{-1} v, \tag{49}$$

where $u' = [u'(x_1), \dots, u'(x_{N-1})]$. Writing $P_x = I + \frac{h^2}{6}\delta_x^2$, the inverse of $P_x$ is obtained by the Neumann series

$$P_x^{-1} = \sum_{k=0}^{\infty}\left(-\frac{h^2}{6}\delta_x^2\right)^k, \tag{50}$$

which gives the estimate of $|P_x^{-1}|_{\infty,h}$,

$$|P_x^{-1}|_{\infty,h} \le \sum_{k=0}^{\infty}\frac{h^{2k}}{6^k}|\delta_x^2|_{\infty,h}^k \le \sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = 3. \tag{51}$$

Observe that the matrix-operator $\delta_x^2$ above is defined at the near-boundary points $j=1$, $j=N-1$ by

$$\delta_x^2 u_1 = \frac{u_2 - 2u_1}{h^2}, \quad \delta_x^2 u_{N-1} = \frac{u_{N-2} - 2u_{N-1}}{h^2}. \tag{52}$$

We deduce now from (49) and (51) that

$$|u' - u_x|_{\infty,h} \le h^4 |P_x^{-1}|_{\infty,h} |v|_{\infty,h} \le C h^4 |u^{(5)}|_{\infty,[0,1]}. \tag{53}$$ □

LEMMA 3.2. *Suppose that* $u(x)$ *is a regular function on* $[0,1]$. *Then the Stephenson biharmonic operator* $\delta_x^4$ *defined by* (34) *has a truncation error* $\delta_x^4 u - u^{(4)}$ *of order* $3/2$ *in the* $l^2_{h,0}$ *norm,*

$$|\delta_x^4 u - u^{(4)}|_h \le C h^{3/2} \big( |u^{(6)}|_{\infty,[0,1]} + |u^{(5)}|_{\infty,[0,1]} \big), \tag{54}$$

*where the notation* $u^{(4)}$ *stands for*

$$u^{(4)} = [u^{(4)}(x_1), \dots, u^{(4)}(x_{N-1})] \in l^2_{h,0}. \tag{55}$$

*Remark*. The difference in accuracy between the periodic case and the nonperiodic case is only due to the near-boundary points 1 and $N-1$.

*Proof*. Recall that the finite difference biharmonic operator $\delta_x^4$ is the three-points compact operator, expressed in terms of $u$ and $u_x$ by

$$\delta_x^4 u_j = \frac{12}{h^2} \big[ \delta_x u_x - \delta_x^2 u \big]. \tag{56}$$

Here, we handle the finite difference operators acting on one-dimensional grid functions $u = [u_1, \dots, u_{N-1}]$, as $N-1 \times N-1$ matrices; see [3]. We can rewrite (30)(a) as

$$P_x u_x = \frac{1}{2h} K u = \delta_x u \in l^2_{h,0}, \tag{57}$$

where the antisymmetric matrix $K = \{K_{i,m}\}_{1 \le i,m \le N-1}$ is given by

$$K_{i,m} = \begin{cases} \operatorname{sgn}(m-i), & |m-i| = 1, \\ 0, & |m-i| \ne 1, \end{cases} \tag{58}$$

and the operator $\delta_x$ is expressed as

$$\delta_x = \frac{1}{2h} K. \tag{59}$$

In matrix form, (57) is simply written as

$$P_x u_x = \delta_x u \quad \text{or} \quad u_x = P_x^{-1} \delta_x u. \tag{60}$$

Using (34), the operator $\delta_x^4$ can be rewritten in matrix form

$$\begin{aligned} \delta_x^4 &= \frac{12}{h^2} \big[ \delta_x P_x^{-1} \delta_x - \delta_x^2 \big] \\ &= \frac{12}{h^2} \big[ P_x^{-1} (\delta_x)^2 + \big[ \delta_x P_x^{-1} - P_x^{-1} \delta_x \big] \delta_x - \delta_x^2 u \big]. \end{aligned}$$

Applying the operator $P_x$, we obtain, for all $u \in l^2_{h,0}$,

$$P_x\big[\delta_x^4 u - u^{(4)}\big] = \frac{12}{h^2}\big[(\delta_x)^2 u + [P_x\delta_x - \delta_x P_x]P_x^{-1}\delta_x u - P_x\delta_x^2 u\big] - P_x u^{(4)} := v. \tag{61}$$

Note that in (60)–(61), we refer to $P_x$ as the symmetric positive definite matrix (see (32)–(43)),

$$(P_x)_{i,m} = \begin{cases} \frac{2}{3}, & m = i, \\ \frac{1}{6}, & |m-i| = 1, \\ 0, & |m-i| \geq 2. \end{cases} \tag{62}$$

Clearly the commutator $[P_x, K] = P_x K - K P_x$ is

$$(P_x K - K P_x)_{i,j} = \begin{cases} -\frac{1}{3}, & i = j = 1, \\ \frac{1}{3}, & i = j = N-1, \\ 0 & \text{otherwise}, \end{cases} \tag{63}$$

so that the commutator $[P_x, \delta_x] = \frac{1}{2h}[P_x, K]$ is

$$P_x\delta_x - \delta_x P_x = \begin{cases} -\frac{1}{6h}, & i = j = 1, \\ \frac{1}{6h}, & i = j = N-1, \\ 0 & \text{otherwise}. \end{cases} \tag{64}$$

This means that the operators $P_x$ and $\delta_x$ do not commute and that the nonzero commutator values are restricted to points $j = 1$ and $j = N-1$.

Let us first evaluate (61) at points $j = 2, 3, \ldots, N-2$.

$$\begin{aligned} \frac{12}{h^2}\big[(\delta_x)^2 u_j - P_x\delta_x^2 u_j\big] - P_x u_j^{(4)} = \frac{12}{h^2}\Bigg\{&(\delta_x)^2 u_j \\ &-\left[\frac{2}{3}\delta_x^2 u_j + \frac{1}{6}\delta_x^2 u_{j+1} + \frac{1}{6}\delta_x^2 u_{j-1}\right]\Bigg\} \\ &-\left[\frac{2}{3}u_j^{(4)} + \frac{1}{6}u_{j-1}^{(4)} + \frac{1}{6}u_{j+1}^{(4)}\right]. \end{aligned} \tag{65}$$

The first term on the right-hand side of (65) is

$$(\delta_x)^2 u_j = u_j'' + \frac{h^2}{3}u_j^{(4)} + \frac{32}{6!}h^4 u_j^{(6)} + \frac{128}{8!}h^6 u_j^{(8)} + Ch^8 u^{(10)}(\xi_j). \tag{66}$$

Using (45) for evaluating $\delta_x^2 u_m$ at $m = j-1, j, j+1$, we find that $P_x\delta_x^2 u_j$ in (65) is

$$\frac{2}{3}\delta_x^2 u_j + \frac{1}{6}\delta_x^2 u_{j+1} + \frac{1}{6}\delta_x^2 u_{j-1} = u_j'' + \frac{1}{4}h^2 u_j^{(4)} + \frac{22}{6!}h^4 u_j^{(6)} + \frac{86}{8!}h^6 u_j^{(8)} + h^8 w_j, \tag{67}$$

where $|w_j| \leq C|u^{(10)}|_{\infty,[0,1]}$. In addition, we have that the third line of the right-hand side in (65) is

$$\left[\frac{2}{3}u_j^{(4)} + \frac{1}{6}u_{j-1}^{(4)} + \frac{1}{6}u_{j+1}^{(4)}\right] = u_j^{(4)} + \frac{1}{6}h^2 u_j^{(6)} + Ch^4 z_j, \tag{68}$$

where $|z_j| \leq C|u^{(8)}|_{\infty,[0,1]}$. Therefore, we have, for $2 \leq j \leq N-2$,

$$\left|\frac{12}{h^2}\left[(\delta_x)^2 u - P_x\delta_x^2 u_j\right] - P_x u_j^{(4)}\right| \leq Ch^4|u^{(8)}|_{\infty,[0,1]}, \tag{69}$$

and this order is optimal. Consider now the truncation term for $j = 1$ (the computation is the same for $j = N-1$). We have

$$(\delta_x^4 u)_1 = \frac{12}{h^2}\left[(\delta_x u_x)_1 - \delta_x^2 u_1\right]. \tag{70}$$

Since $|u_{x,j} - u'_j| \leq Ch^4|u^{(5)}|_{\infty,[0,1]}$, we have

$$\begin{aligned}(\delta_x u_x)_1 &= \frac{u_{x,2}}{2h} = \frac{u_{x,2} - u_{x,0}}{2h}\\ &= \frac{u'(x_2) - u'(x_0)}{2h} + \tilde{v}\\ &= u''(x_1) + \frac{h^2}{6}u^{(4)}(x_1) + \tilde{v},\end{aligned} \tag{71}$$

where $\tilde{v}$ stands for a generic term such that $|\tilde{v}| \leq Ch^3|u^{(5)}|_{\infty,[0,1]}$. In addition, we have

$$(\delta_x^2 u)_1 = u''(x_1) + \frac{h^2}{12}u^{(4)}(x_1) + w, \tag{72}$$

where

$$|w| \leq Ch^4|u^{(6)}|_{\infty,[0,1]}. \tag{73}$$

Therefore (71), (73) show that the truncation error at the near-boundary point $x_1$ is

$$\frac{12}{h^2}\left[(\delta_x u_x)_1 - (\delta_x^2 u)_1\right] - u^{(4)}(x_1) = t_1, \text{ with } |t_1| \leq Ch|u^{(5)}|_{\infty,[0,1]}. \tag{74}$$

We deduce from (61), (69), (74) that the truncation error $e = \delta_x^4 u - u^{(4)}$ is the solution of the linear system

$$\overline{P}_x e = v, \quad v \in l^2_{h,0}, e \in l^2_{h,0}, \tag{75}$$

where $\overline{P}_x$ is the matrix

$$\overline{P}_x = \begin{bmatrix} 1 & 0 & 0 & \dots & 0\\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \dots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & \dots & \frac{1}{6} & \frac{2}{3} & \frac{1}{6}\\ 0 & \dots & 0 & 0 & 1\end{bmatrix}, \tag{76}$$

and $v$ is such that

$$|v_1|, |v_{N-1}| \leq Ch|u^{(5)}|_{\infty,[0,1]}; \quad |v_j| \leq Ch^4|u^{(8)}|_{\infty,[0,1]}, \quad j = 2, \dots, N-2. \tag{77}$$

By Gerschgorin's theorem, $\overline{P}_x^{-1}$ is a bounded matrix independent of $h$; therefore $e = \overline{P}_x^{-1}v$ is such that

$$|e|_h \leq C|v|_h, \tag{78}$$

where

$$|v|_h^2 \leq Ch\left(2h^2 + \sum_{j=2}^{N-2} h^8\right) \leq Ch^3. \tag{79}$$

Taking the square root in (79), we obtain (54) (using the weaker estimate $|v_j| \leq Ch^2|u^{(6)}|_{\infty,[0,1]}$ at interior points). ∎

*Remark.* Note that the error at the interior points is fourth order and that the $h^{3/2}$ error is fully due to the loss of accuracy at the two boundary points $j = 1$, $j = N-1$.

**3.3. Interpretation with finite elements.** In this section, we establish the finite element counterpart of scheme (30). This allows us to obtain in a simple way the stability of the Stephenson finite difference operator $\delta_x^4$. To each grid function $v \in l_{h,0}^2$, we match the function $v_h(x)$ defined by $v_h(x_j) = v_j$, in the finite element space $P_{c,0}^1$, the space of continuous functions, piecewise linear in each interval $[x_j, x_{j+1}]$, $j = 0, \ldots, N-1$, and such that $v_h(x_0) = v_h(x_N) = 0$. Clearly, it is an isomorphism between $l_{h,0}^2$ and $P_{c,0}^1$. In addition, starting with $v \in l_{h,0}^2$, we introduce the two piecewise constant functions $\bar{v}_h$ and $v_{h,x}$, defined in each interval $K_{j+1/2} = ]x_j, x_{j+1}[$ by

$$\bar{v}_{h,j+1/2} = \frac{v_j + v_{j+1}}{2}, \quad v_{h,x,j+1/2} = \frac{v_{j+1} - v_j}{h}. \tag{80}$$

An important aspect of using $P_{c,0}^1$ in the study of finite difference schemes is that it allows one to streamline analytic operations like integration by parts or averaged quantities over intervals $K_{j+1/2} = [x_j, x_{j+1}]$. The $L^2[0,1]$ scalar product is denoted by

$$(\varphi, \psi) = \int_0^1 \varphi(x)\psi(x)dx. \tag{81}$$

Writing the representation of $u_h(x)$ in $K_{j+1/2}$ as $(x_{j+1/2} = \frac{1}{2}(x_{j+1} + x_j))$,

$$u_h(x)_{|K_{j+1/2}} = \bar{u}_{h,j+1/2} + u_{h,x,j+1/2}(x - x_{j+1/2}), \tag{82}$$

we can compare different scalar products for $(.,.)_h$ and in $L^2(0,1)$ as follows.

Lemma 3.3. *For any $u, v \in l_{h,0}^2$, let $u_h(x), v_h(x) \in P_{c,0}^1$ be the corresponding finite element functions. Then we have*

$$\text{(i)} \quad (u,v)_h = (u_h, v_h) + \frac{h^2}{6}(u_{h,x}, v_{h,x}) = (\bar{u}_h, \bar{v}_h) + \frac{h^2}{4}(u_{h,x}, v_{h,x}); \tag{83}$$

$$\text{(ii)} \quad (\delta_x u, v)_h = (u_{h,x}, v_h); \tag{84}$$

$$\text{(iii)} \quad (\delta_x^2 u, v)_h = -(\delta_x^+ u, \delta_x^+ v)_h = -(\delta_x^- u, \delta_x^- v)_h = -(u_{h,x}, v_{h,x}) \ (\textit{see}\ (18)). \tag{85}$$

*Proof.* The proof is an elementary computation resulting from the piecewise linearity of $u_h(x)$ in each $K_{j+1/2} = [x_j, x_{j+1}]$ given by (82). In fact, it clearly suffices to check that (83), (84), (85) hold for $u_h = \varphi_k$, $v_h = \varphi_m$, where $(\varphi_k)$ is a basis of $P_{c,0}^1$. ∎

Let $(u, u_x) \in \mathcal{S}$. Since $u_x \in l_{h,0}^2$, it has a matching function $p_h \in P_{c,0}^1$. On the other hand, we have the piecewise constant function $u_{h,x}$. The connection between these two functions is given by the following lemma.

LEMMA 3.4. (i) *Let $u \in \mathcal{S}$ with grid gradient $u_x \in l^2_{h,0}$. Then the finite element function $p_h(x) \in P^1_{c,0}$ corresponding to $u_x$ is the orthogonal projection of the piecewise constant function $u_{h,x}$ onto $P^1_{c,0}$. In other words, it is the unique solution $p_h \in P^1_{c,0}$ of*

$$(p_h, q_h) = (u_{h,x}, q_h) \qquad \forall q_h \in P^1_{c,0}. \tag{86}$$

*In addition, we have, with $q_h \in P^1_{c,0}$ corresponding to $q \in l^2_{h,0}$,*

$$(P_x u_x, q)_h = (p_h, q_h) = (u_x, P_x q)_h. \tag{87}$$

(ii) *Let $u, v \in \mathcal{S}$ and let $(u_h, p_h), (v_h, q_h) \in P^1_{c,0} \times P^1_{c,0}$ be the matching finite element functions. Then the bilinear form $\langle .;. \rangle_h$ defined on $\mathcal{S} \times \mathcal{S}$ by*

$$\langle u, v \rangle_h = (\delta_x^4 u, v)_h = \frac{12}{h^2}\big(u_{h,x} - p_h, v_{h,x} - q_h\big) = (u, \delta_x^4 v)_h \tag{88}$$

*is a scalar product on $\mathcal{S} \times \mathcal{S}$.*

(iii) *Translated in terms of finite difference operators,* (88) *is*

$$\begin{aligned} \langle u, v \rangle_h = &\sum_{j=0}^{N-1} h \frac{u_{x,j+1} - u_{x,j}}{h} \frac{v_{x,j+1} - v_{x,j}}{h} \\ &+ \frac{12}{h^2} \sum_{j=0}^{N-1} h \left[\frac{u_{j+1} - u_j}{h} - \frac{1}{2}(u_{x,j} + u_{x,j+1})\right] \left[\frac{v_{j+1} - v_j}{h} - \frac{1}{2}(v_{x,j} + v_{x,j+1})\right]. \end{aligned} \tag{89}$$

*Proof.* (i) The discrete gradient $u_x \in l^2_{h,0}$ is defined by

$$\big[P_x u_x\big]_j = \delta_x u_j, \quad 1 \le j \le N-1, \tag{90}$$

where $P_x$ is the Simpson operator given in (31). Equation (90) is equivalent to

$$(u_x, q)_h + \frac{1}{6} h^2 (\delta_x^2 u_x, q)_h = (\delta_x u, q)_h \qquad \forall q \in l^2_{h,0}. \tag{91}$$

Taking any $q \in l^2_{h,0}$ and the $p_h$ corresponding to $u_x \in l^2_{h,0}$, and using (83), (84), and (85), we can rewrite (91) as

$$\begin{aligned} (u_{h,x}, q_h) = (\delta_x u, q)_h &= (u_x, q)_h + \frac{h^2}{6}(\delta_x^2 u_x, q)_h \\ &= (p_h, q_h) + \frac{h^2}{6}(p_{h,x}, q_{h,x}) - \frac{h^2}{6}(p_{h,x}, q_{h,x}) \\ &= (p_h, q_h), \end{aligned}$$

which gives (86). The symmetry of $P_x$ is clear from the definition; see (31), (62). In addition, we have

$$(P_x u_x, q)_h = (\delta_x u, q)_h = (u_{h,x}, q_h) = (p_h, q_h), \tag{92}$$

which proves (87).

(ii) The Stephenson biharmonic operator is (see (34))

$$\delta_x^4 u_j = \frac{12}{h^2}\Big\{(\delta_x u_x)_j - \delta_x^2 u_j\Big\}. \tag{93}$$

We have

$$(\delta_x^4 u, v)_h = \frac{12}{h^2}\big[(p_{h,x}, v_h) + (u_{h,x}, v_{h,x})\big] = \frac{12}{h^2}\big(v_{h,x}, u_{h,x} - p_h\big). \tag{94}$$

Subtracting $(q_h, u_{h,x} - p_h) = 0$ from (94), we deduce

$$\langle u, v\rangle_h = (\delta_x^4 u, v)_h = \frac{12}{h^2}\big(u_{h,x} - p_h, v_{h,x} - q_h\big). \tag{95}$$

We verify now that $\langle u, u\rangle_h^{1/2}$ is a norm on $\mathcal{S}$. $\langle u, u\rangle_h = 0$ is equivalent to $|u_{h,x} - p_h| = 0$. Therefore the piecewise affine function $p_h \in P^1_{c,0}$ is actually piecewise constant. Since it vanishes at $x = 0$ and is continuous at any $x_j$, we have $p_h \equiv 0$, which is $u_{h,x} \equiv 0$. Therefore $u_h$ is piecewise constant as well. Since $u_h(0) = 0$ we have also $u_h \equiv 0$.

Finally, we prove (89). Recall that for any $q_h \in P^1_{c,0}$, the difference $q_h - \overline{q}_h$ is orthogonal to piecewise constant functions. Thus, replacing in (95) $p_h$, $q_h$ by $\overline{p}_h$, $\overline{q}_h$, respectively, and noting (see (83)) that

$$(p_h, q_h) = (\overline{p}_h, \overline{q}_h) + \frac{h^2}{12}(p_{h,x}, q_{h,x}), \tag{96}$$

we get

$$\langle u, v\rangle_h = (p_{h,x}, q_{h,x}) + \frac{12}{h^2}(u_{h,x} - \bar{p}_h, v_{h,x} - \bar{q}_h), \tag{97}$$

which gives (89) using (80). $\square$

*Remarks.* The result of Lemma 3.4(ii) gives the uniqueness of the discrete solution of scheme (30).

The following lemma states the discrete counterpart of the equivalence of
(i) $|u_x|$ and $\|u\|_{H_1}$ for $u \in H^1_0$;
(ii) $|u_{xx}|$ and $\|u\|_{H_2}$ for $u \in H^2_0$.

LEMMA 3.5. *There exist constants $C$, $C'$, $C''$ independent of $h$ such that for any grid function $u \in \mathcal{S}$,*

$$\text{(i) } |u_h| \le |u|_h \le C|\delta_x^+ u|_h = C|u_{h,x}| \quad (\textit{Poincaré inequality}); \tag{98}$$

$$\text{(ii) } |\delta_x^+ u|_h \le C'\langle u, u\rangle_h^{1/2}; \tag{99}$$

$$\text{(iii) } |\delta_x^+ u_x|_h \le C''\langle u, u\rangle_h^{1/2}. \tag{100}$$

*Proof.* Inequality (i) is simply the Poincaré inequality (21) in the one-dimensional setting, reformulated with the finite element notation. Inequality (iii) follows directly from (97) since $\delta_x^+ u_x = p_{h,x}$ as piecewise constant functions.

For (ii), we use the notation $p$ for the grid function $u_x$ and, as before, denote by $u_h$, $p_h$ the $P^1_{c,0}$ functions associated with $u, p$, respectively. In view of (86), we have

$$\begin{aligned} |\delta_x^+ u|_h^2 = |u_{h,x}|^2 &= (u_{h,x} - p_h, u_{h,x} - p_h) + (p_h, p_h) \\ &= \frac{h^2}{12}\langle u, u\rangle_h + |p_h|^2, \end{aligned} \tag{101}$$

where in the second equality we have used (95). Now, applying the Poincaré inequality (98) to $p$ instead of $u$, we get

$$|p_h|^2 \leq C^2|\delta_x^+ p|_h^2 \leq C^2(C'')^2\langle u, u\rangle_h, \tag{102}$$

where in the last inequality we have used (100). Inserting this inequality in (101), we obtain (99) with $C' = CC''$. ∎

*Remarks.* 1. We know that $|u_{xx}|_{0,[0,1]}$ is a norm on the Sobolev space $H_0^2$. We may wonder if, at the discrete level, $|\delta_x^+ u_x|_h = |p_{h,x}|_{0,[0,1]}$ is a norm on $\mathcal{S}$. Actually it is a norm only if the number of points $N$ is an even integer. We have that $p_{h,x} = 0$ implies $p_h = 0$. But the relation $P_x u_x = \delta_x u$ implies only $\delta_x u = 0$, which gives $u = 0$ only if $N$ is an even integer.

2. For other finite difference schemes for the biharmonic problem and their link with the finite element method, we refer to the book by Li, Chen, and Wu [16].

**3.4. Convergence of the Stephenson scheme.** We derive now the following convergence result

PROPOSITION 3.1. *Let $U$ be the $P_{c,0}^1$ Lagrange interpolate of the exact solution $u(x)$ of* (26) *and $\tilde{u}$ the discrete solution of* (30). *Then the following error estimate holds in the mesh dependent norm $\langle\tilde{v}, \tilde{v}\rangle_h^{1/2}$,*

$$\langle U - \tilde{u}, U - \tilde{u}\rangle_h^{1/2} \leq Ch^{3/2}\big(|f''|_{\infty,[0,1]} + |f'|_{\infty,[0,1]}\big), \tag{103}$$

*where the constant $C$ is independent of $h$.*

*Proof.* We estimate as usual the error by the sum of the approximation error and of the consistency error. Here, we work with the discrete norm $\langle.,.\rangle_h^{1/2}$, so that there is no approximation error. We have

$$\langle U - \tilde{u}, U - \tilde{u}\rangle_h^{1/2} = \sup_{\tilde{v}\in\mathcal{S},\tilde{v}\neq 0} \frac{\langle U - \tilde{u}, \tilde{v}\rangle_h}{\langle\tilde{v}, \tilde{v}\rangle_h^{1/2}}. \tag{104}$$

For the numerator on the right-hand side of (104),

$$\langle U - \tilde{u}, \tilde{v}\rangle_h = (\delta_x^4(U - \tilde{u}), \tilde{v})_h = h\sum_{j=1}^{N-1}(\delta_x^4 U_j - f_j)\tilde{v}_j. \tag{105}$$

Therefore, in view of Lemma 3.2,

$$\begin{aligned} |\langle U - \tilde{u}, \tilde{v}\rangle_h| &\leq |\delta_x^4 U - f|_h|\tilde{v}|_h \\ &\leq Ch^{3/2}|\tilde{v}|_h\big(|f''|_{\infty,[0,1]} + |f'|_{\infty,[0,1]}\big). \end{aligned} \tag{106}$$

Using the fact that $|\tilde{v}|_h \leq C\langle\tilde{v}, \tilde{v}\rangle_h^{1/2}$ (see (99), (100)), we find that

$$|\langle U - \tilde{u}, \tilde{v}\rangle_h| \leq Ch^{3/2}\langle\tilde{v}, \tilde{v}\rangle_h^{1/2}\big(|f''|_{\infty,[0,1]} + |f'|_{\infty,[0,1]}\big), \tag{107}$$

which gives the result. ∎

## 4. The Stephenson scheme in two dimensions.

**4.1. The compact biharmonic scheme of Stephenson.** We consider in this section the biharmonic problem in a square $\Omega = ]0,1[^2$:

$$\begin{cases} \Delta^2 u(x,y) = \partial_x^4 u(x,y) + \partial_y^4 u(x,y) + 2\partial_{xy}^2 u(x,y) = f(x,y), & (x,y) \in \Omega, \\ u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{108}$$

For any $f \in L^2(\Omega)$, problem (108) has a unique solution $u \in H_0^2(\Omega)$. Its discrete version, using the Stephenson scheme, is to find a solution $u_{i,j} \in L^2_{h,0}$ to the equation

$$\begin{cases} \Delta_h^2 u_{i,j} = f(x_i, y_j), \quad 1 \le i,j \le N-1, \\ u_{i,j} = u_{x,i,j} = u_{y,i,j} = 0 \quad \text{for } \{i,j\} \in \{0, N\}. \end{cases} \tag{109}$$

The Stephenson biharmonic operator $\Delta_h^2$ is defined by

$$\Delta_h^2 u_{i,j} = \delta_x^4 u_{i,j} + \delta_y^4 u_{i,j} + 2\delta_x^2 \delta_y^2 u_{i,j}. \tag{110}$$

For any $u \in L^2_{h,0}$, the grid gradient $(u_x, u_y) \in (L^2_{h,0})^2$ is defined by

$$\begin{cases} P_x u_{x,i,j} = \delta_x u_{i,j}, \quad 1 \le i,j \le N-1, \\ P_y u_{y,i,j} = \delta_y u_{i,j}, \quad 1 \le i,j \le N-1, \end{cases} \tag{111}$$

where $P_x$, $P_y$ are the Simpson operators (see (31)),

$$\begin{cases} P_x = Id + \dfrac{1}{6} h^2 \delta_x^2, \\ P_y = Id + \dfrac{1}{6} h^2 \delta_y^2. \end{cases} \tag{112}$$

The one-dimensional operators $\delta_x^4 u_{i,j}$, $\delta_y^4 u_{i,j}$ are given as functions of $u, u_x, u_y$ by

$$\delta_x^4 u_{i,j} = \frac{12}{h^2}\big[(\delta_x u_x)_{i,j} - (\delta_x^2 u)_{i,j}\big], \quad \delta_y^4 u_{i,j} = \frac{12}{h^2}\big[(\delta_y u_y)_{i,j} - (\delta_y^2 u)_{i,j}\big]. \tag{113}$$

For the convenience of the reader, we recall briefly how the operator $\Delta_h^2$ has been originally derived by Stephenson [19]. At each point $(x_i, y_j)$ of the grid, $0 \le i,j \le N$, are attached the three unknowns $u_{i,j}$, $u_{x,i,j}$, $u_{y,i,j}$ as well as a fourth-order polynomial $P_{i,j}$, simply denoted $P(x,y)$,

$$P(x,y) = \sum_{x^l y^m \in \mathcal{V}} a_{l,m} x^l y^m, \tag{114}$$

where the monomial set $\mathcal{V}$ is

$$\mathcal{V} = \{1, x, y, x^2, y^2, xy, x^3, x^2y, xy^2, y^3, x^4, x^2y^2, y^4\}, \quad \#\mathcal{V} = 13. \tag{115}$$

The 13 coefficients $a_{l,m}$ are uniquely determined by the following collocation conditions (see Figure 2):

$$\begin{cases} \bullet \ 9 \text{ collocations for } u_{l,m} \text{ at points } (x_l, y_m) \text{ for } l \in \{i-1, i, i+1\}, \\ \quad m \in \{j-1, j, j+1\}. \\ \bullet \ 2 \text{ collocations for } u_{x,l,m} \text{ at points } (x_{i-1,j}, y_{i,j}),\ (x_{i+1,j}, y_{i,j}). \\ \bullet \ 2 \text{ collocations for } u_{y,l,m} \text{ at points } (x_{i,j}, y_{i,j+1}),\ (x_{i,j}, y_{i,j-1}). \end{cases} \tag{116}$$

The collocation system gives a $13 \times 13$ linear system which can be solved explicitly. The result is given by [19].

LEMMA 4.1. *Denoting by $\diamond$, $\square$, and $\diamond'$ the finite difference operators*

$$
\begin{cases}
\diamond u_{i,j} = u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}, \\
\square u_{i,j} = u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j-1} + u_{i-1,j+1}, \\
\diamond' u_{i,j} = u_{x,i+1,j} - u_{x,i-1,j} + u_{y,i,j+1} - u_{y,i,j-1},
\end{cases}
\tag{117}
$$

*the* 13 *coefficients $a_{l,m}$ of $P(x,y)$ at point $(x_i, y_j)$ uniquely determined by the* 13 *conditions* (116) *are*

(118)

$$
\begin{cases}
a_{0,0} = u_{i,j}, \\[2mm]
a_{1,0} = \dfrac{3}{2}\delta_x u_{i,j} - \dfrac{1}{4}(u_{x,i+1,j} + u_{x,i-1,j}), \quad a_{0,1} = \dfrac{3}{2}\delta_y u_{i,j} - \dfrac{1}{4}(u_{y,i,j+1} + u_{y,i,j-1}), \\[2mm]
a_{2,0} = \delta_x^2 u_{i,j} - \dfrac{1}{2}(\delta_x u_x)_{i,j}, \quad a_{0,2} = \delta_y^2 u_{i,j} - \dfrac{1}{2}(\delta_y u_y)_{i,j}, \quad a_{1,1} = \delta_{xy} u_{i,j}, \\[2mm]
a_{3,0} = \dfrac{1}{6}(\delta_x^2 u_x)_{i,j}, \quad a_{0,3} = \dfrac{1}{6}(\delta_y^2 u_y)_{i,j}, \\[2mm]
a_{2,1} = \dfrac{1}{2}(\delta_x^2 \delta_y u)_{i,j}, \quad a_{1,2} = \dfrac{1}{2}(\delta_y^2 \delta_x u)_{i,j}, \\[2mm]
a_{4,0} = \dfrac{1}{2h^2}\left[(\delta_x u_x)_{i,j} - \delta_x^2 u_{i,j}\right], \quad a_{0,4} = \dfrac{1}{2h^2}\left[(\delta_y u_y)_{i,j} - \delta_y^2 u_{i,j}\right], \\[2mm]
a_{2,2} = \dfrac{1}{4}(\delta_x^2 \delta_y^2 u)_{i,j}.
\end{cases}
$$

The gradient of $P(x,y)$ at $(x_i, y_j)$ is $(\partial_x P(x_i,y_j), \partial_y P(x_i,y_j)) = (a_{1,0}, a_{0,1})$. Defining $u_{x,i,j} = P_x(x_i,y_j)$, $u_{y,i,j} = P_y(x_i,y_j)$, we obtain (111). Furthermore the operators $\delta_x^4$, $\delta_y^4$ are defined by

$$
\begin{cases}
\delta_x^4 u_{i,j} = \partial_x^4 P(x_i,y_j) = 24 a_{4,0}, \\
\delta_y^4 u_{i,j} = \partial_y^4 P(x_i,y_j) = 24 a_{0,4},
\end{cases}
\tag{119}
$$

which is (113). Finally the operator $\Delta_h^2 u_{i,j}$ is defined by $\Delta_h^2 u_{i,j} = \Delta^2 P(x_i,y_j) = 24a_{4,0} + 8a_{2,2} + 24a_{0,4}$, which is (110). Furthermore, by expanding the finite difference operators, we find the following expression for the biharmonic operator $\Delta_h^2$:

$$
\begin{aligned}
\Delta_h^2 u_{i,j} = \frac{1}{h^4}\Big\{ & 56u_{i,j} - 16\left[u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1}\right] \\
& + 2\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j-1} + u_{i+1,j-1}\right] \\
& + 6h\left[(u_x)_{i+1,j} - (u_x)_{i-1,j} + (u_y)_{i,j+1} - (u_y)_{i,j-1}\right]\Big\}.
\end{aligned}
$$

For alternative schemes for (108), see [19, 1].

**4.2. Consistency and convergence for the elliptic operator.** The order of consistency is deduced from the consistency in the one-dimensional case.

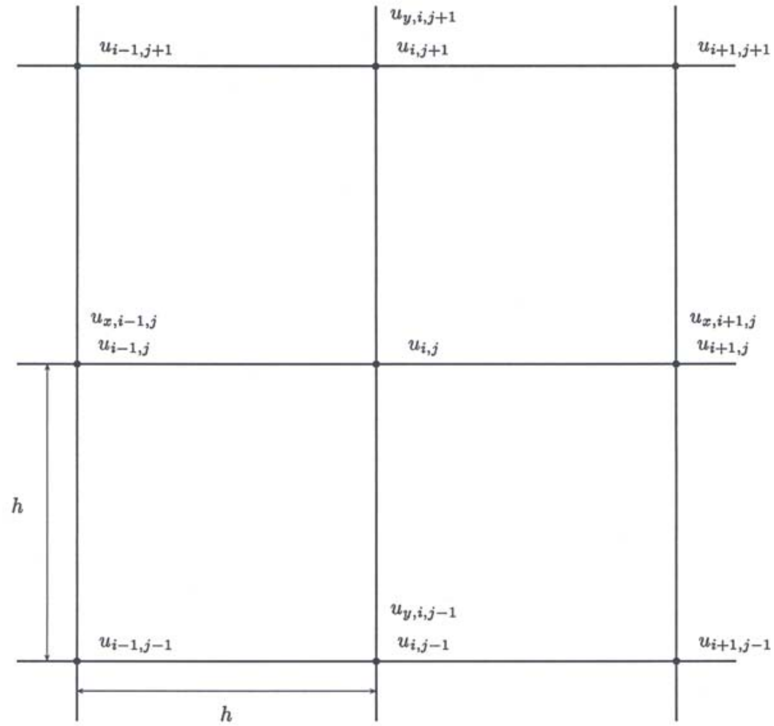


FIG. 2. *Stephenson's scheme for $\Delta^2 u = f$: The finite difference operator $\Delta_h^2 u_{i,j}$ at point $(i,j)$ is $\Delta_h^2 u_{i,j} = \Delta^2 Q(x_i, y_j)$, where $Q(x,y) \in P^{3.5}\big([x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}]\big)$ is defined by the 13 collocated values on the picture.*

LEMMA 4.2. *Let $u$ be continuously differentiable up to sixth order in $\Omega$ and suppose that it vanishes, along with its gradient on $\partial\Omega$. Then the truncation grid function $e = \Delta_h^2 u(x_i, y_j) - \Delta^2 u(x_i, y_j) \in L^2_{h,0}$ satisfies*

$$|e|_h \leq C h^{3/2} \|u\|_{6,\infty}, \tag{120}$$

*where $\|u\|_{6,\infty}$ is*

$$|u|_{6,\infty} = \sum_{0 \leq \alpha_1 + \alpha_2 \leq 6} |\partial_x^{\alpha_1} \partial_y^{\alpha_2} u|_{\infty,[0,1]^2}.$$

*Proof.* We have

$$|\Delta_h^2 u - \Delta^2 u|_h \leq |\delta_x^4 u - \partial_x^4 u|_h + |\delta_y^4 u - \partial_y^4 u|_h + 2|\delta_x^2 \delta_y^2 u - \partial_x^2 \partial_y^2 u|_h. \tag{121}$$

Using the consistency result (54) row by row and column by column we obtain

$$|\delta_x^4 u - \partial_x^4 u|_h \leq C h^{3/2} \big(|\partial_x^6 u|_{\infty,[0,1]^2} + |\partial_x^5 u|_{\infty,[0,1]^2}\big), \tag{122}$$

$$|\delta_y^4 u - \partial_y^4 u|_h \leq C h^{3/2} \big(|\partial_y^6 u|_{\infty,[0,1]^2} + |\partial_y^5 u|_{\infty,[0,1]^2}\big). \tag{123}$$

The consistency for the mixed term is deduced from (45):

$$|\delta_x^2\delta_y^2 u - \partial_x^2\partial_y^2 u|_h \le Ch^2\Bigg(\sum_{\alpha_1+\alpha_2=6}|\partial_x^{\alpha_1}\partial_y^{\alpha_2}u|_{\infty,[0,1]^2}\Bigg). \qquad \square \tag{124}$$

In order to carry out convergence analysis, we need to develop discrete analogues of the basic differential estimates, as in the one-dimensional case of section 3. We do this in the framework of a suitable "finite element" space, namely, the $Q_c^1$ space of continuous functions in $\Omega$ satisfying the following condition: In every cell $K_{i+1/2,j+1/2} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, they are linear (separately) in $x, y$. Otherwise stated, it is (in every cell) in $\text{Span}(1, x, y, xy)$. The subspace of interest to us is $Q_{c,0}^1$, consisting of functions (in $Q_c^1$) vanishing on $\partial\Omega$. It is clear how to match an element $u_h \in Q_{c,0}^1$ to a given $u \in L_{h,0}^2$: we simply take the function $a_0 + a_1x + a_2y + a_3xy$, which interpolates the four values $u_{i,j}$, $u_{i+1,j}$, $u_{i,j+1}$, $u_{i+1,j+1}$. Since $u_h(x,y)$ is linear in $x$ (resp., in $y$) for every fixed value of $y$ (resp., of $x$), we can in particular treat the function $u(x_i, y_j)$, for every fixed $j$, as a function of $x_i$ in $l_{h,0}^2$ and then associate with it the functions $u_x$ in $l_{h,0}^2$ (see (30)) and $u_h$, $p_h$, their associated $P_{c,0}^1$ functions.

Note that these functions are determined for each fixed value of $y_j$. In the same way, we define the piecewise constant in $[x_j, x_{j+1}]$ function $u_{h,x}(., y_j)$. We define also the analogous functions in the $y$ direction. Finally, $u_{h,xy}$ is the piecewise (in cells) constant function given by the coefficient $a_3$ above. We now equip $Q_{c,0}^1$ with two scalar products. Each of them corresponds to an $L^2(0,1)$ product in one direction (i.e., the function is regarded as an element of $P_{c,0}^1$ in that direction), followed by an $l_{h,0}^2$ product in the other direction. They are given by

$$\begin{cases} (u_h, v_h)^x = h\sum_{j=1}^{N-1}(u_h(., y_j), v_h(., y_j))_{L^2(0,1)}, \\ (u_h, v_h)^y = h\sum_{i=1}^{N-1}(u_h(x_i, .), v_h(x_i, .))_{L^2(0,1)}. \end{cases} \tag{125}$$

The link between the grid scalar product $(u, v)_h$ on $L_{h,0}^2$ and the two scalar products $(u_h, v_h)^x$, $(u_h, v_h)^y$ is given by (see (83))

$$(u, v)_h = (u_h, v_h)^x + \frac{h^2}{6}(u_{h,x}, v_{h,x})^x, \tag{126}$$

$$(u, v)_h = (u_h, v_h)^y + \frac{h^2}{6}(u_{h,y}, v_{h,y})^y. \tag{127}$$

As in the one-dimensional case (see (33)), we introduce here a space $\mathcal{S}$ consisting of triples $(u, u_x, u_y) \in L_{h,0}^2$, where $u_x, u_y$ are related to $u$ by (111). For brevity, we shall sometimes refer to the triple simply by $u \in \mathcal{S}$. As in the one-dimensional case (see Lemma 3.4), we have the following result.

LEMMA 4.3. *Let $u \in \mathcal{S}$. Let $p_h, q_h \in Q_{c,0}^1$ correspond to $u_x, u_y$, respectively. Then they are the projections of $u_{h,x}$, $u_{h,y}$ in the following sense:*

$$(p_h, v_h)^x = (u_{h,x}, v_h)^x,\ (q_h, v_h)^y = (u_{h,y}, v_h)^y \qquad \forall v_h \in Q_{c,0}^1. \tag{128}$$

*Proof.* For each $1 \le j_0 \le N-1$, it results from (86) that

$$\begin{aligned}(p_h, v_h)^x &= h \sum_{j=1}^{N-1} (p_h(.,y_j), v_h(.,y_j))_{L^2(0,1)} \\ &= h \sum_{j=1}^{N-1} (u_{h,x}(.,y_j), v_h(.,y_j))_{L^2(0,1)} \\ &= (u_{h,x}, v_h)^x.\end{aligned}$$

Therefore, the function $p_h \in Q^1_{c,0}$ matching $u_x \in L^2_{h,0}$ is the unique solution of

$$(p_h, v_h)^x = (u_{h,x}, v_h)^x \qquad \forall v_h \in Q^1_{c,0}. \tag{129}$$

The proof is the same for $u_{h,y}$. ◻

We summarize in the following proposition the basic properties of the discrete operator $\Delta_h^2$. As in the one-dimensional case, that operator gives rise to a *positive definite* bilinear form.

PROPOSITION 4.1. (i) *Let $(u, u_x, u_y), (v, v_x, v_y) \in \mathcal{S}$, and let $(u_h, p_h, q_h)$, $(v_h, r_h, z_h)$ be their matches, respectively, in $Q^1_{c,0}$. Then the discrete biharmonic operator $\Delta_h^2$ defined by*

$$\Delta_h^2 u_{i,j} = \delta_x^4 u_{i,j} + \delta_y^4 u_{i,j} + 2\delta_x^2\delta_y^2 u_{i,j}, \quad 1 \le i,j \le N-1, \tag{130}$$

*induces a scalar product $\langle u, v\rangle_h = (\Delta_h^2 u, v)_h$ on $\mathcal{S} \times \mathcal{S}$ defined by*

$$\begin{aligned}\langle u, v\rangle_h = (\Delta_h^2 u, v)_h = \frac{12}{h^2}(u_{h,x} - p_h, v_{h,x} - r_h)^x + \frac{12}{h^2}(u_{h,y} - q_h, v_{h,y} - z_h)^y \\ + 2(u_{h,xy}, v_{h,xy}).\end{aligned} \tag{131}$$

*In particular, the discrete operator $\Delta_h^2$ is symmetric positive definite on $\mathcal{S}$.*

(ii) *In terms of the basic finite difference operators, the product $\langle u, v\rangle_h$ is given by*

$$\begin{aligned}(\Delta_h^2 u, v)_h &= (\delta_x^+ u_x, \delta_x^+ v_x)_h + (\delta_y^+ u_y, \delta_y^+ v_y)_h + 2(\delta_x^+\delta_y^+ u, \delta_x^+\delta_y^+ v)_h \\ &+ \frac{12}{h^2}\left(\delta_x^+ u - \frac{1}{2}(u_x + u_{x,i+1,j}), \delta_x^+ v - \frac{1}{2}(v_x + v_{x,i+1,j})\right)_h \\ &+ \frac{12}{h^2}\left(\delta_y^+ v - \frac{1}{2}(u_y + u_{y,i,j+1}), \delta_y^+ v - \frac{1}{2}(v_y + v_{y,i,j+1})\right)_h.\end{aligned} \tag{132}$$

(iii) *We have the two following coercivity properties of the norm $\langle u, u\rangle_h = (\Delta_h^2 u, u)_h$:*

$$\langle u, u\rangle_h \ge C\left[|\delta_x^+ u_x|_h^2 + |\delta_y^+ u_y|_h^2 + |\delta_x^+ u_y|_h^2 + |\delta_y^+ u_x|_h^2\right] \tag{133}$$

*and*

$$\langle u, u\rangle_h^{1/2} \ge C'|u|_h, \tag{134}$$

*where $C$, $C'$ are constants independent of $h$.*

*Proof.* (i) By (130), we have

$$(\Delta_h^2 u, v)_h = \underbrace{(\delta_x^4 u, v)_h}_{\text{(I)}} + \underbrace{(\delta_y^4 u, v)_h}_{\text{(II)}} + 2\underbrace{(\delta_x^2\delta_y^2 u, v)_h}_{\text{(III)}}. \tag{135}$$

We consider separately each term (I), (II), (III). For the term (I), we have

$$\begin{aligned}(\delta_x^4 u, v)_h &= h\sum_{j=1}^{N}\Big(\delta_x^4 u(\cdot, y_j), v(\cdot, y_j)\Big)_h\\ &= h\sum_{j=1}^{N}\left\{\frac{12}{h^2}(u_{h,x}(\cdot, y_j) - p_h, v_{h,x}(\cdot, y_j) - r_h(\cdot, y_j)\right\}\\ &= \frac{12}{h^2}(u_{h,x} - p_h, v_{h,x} - r_h)^x.\end{aligned}$$

In the same way

$$(\delta_y^4 u, v)_h = \frac{12}{h^2}(u_{h,y} - q_h, v_{h,y} - z_h)^y. \tag{136}$$

For (III), we just note that

$$(\delta_x^2\delta_y^2 u, v)_h = (\delta_x^+\delta_y^+ u, \delta_x^+\delta_y^+ u)_h = (u_{h,xy}, v_{h,xy}). \tag{137}$$

Consider now the positive-definiteness of (131). Suppose that $(\Delta_h^2 u, u) = 0$. Then $p_h(., y_j)$ is constant and continuous and is zero at the end points; therefore $p_h = 0$. The same result holds for $q_h$ and $u_h$. We conclude that $\langle u, u\rangle_h^{1/2} = (\Delta_h^2 u, u)_h^{1/2}$ is a norm in $\mathcal{S}$.

(ii) Translating (131) in term of finite difference operators, we obtain (132), as in (89).

(iii) It results from (132) that

$$(\Delta_h^2 u, u)_h \geq |\delta_x^+ u_x|_h^2 + |\delta_y^+ u_y|_h^2 + 2|\delta_x^+\delta_y^+ u|_h^2. \tag{138}$$

For the mixed term $\delta_x^+\delta_y^+ u$, we will show next that

$$|\delta_x^+\delta_y^+ u|_h \geq \frac{1}{6}|\delta_x^+ u_y|_h. \tag{139}$$

Indeed

$$\delta_x^+\delta_y^+ u_{i,j} = \frac{\delta_y^+ u_{i+1,j} - \delta_y^+ u_{i,j}}{h}. \tag{140}$$

Using $\delta_y^+ u_{i,j} = \delta_y u_{i,j} + \frac{h}{2}\delta_y^2 u_{i,j}$ and the definition of $P_y$ (see (112)), we deduce

$$\begin{aligned}\delta_x^+\delta_y^+ u_{i,j} &= \frac{\delta_y u_{i+1,j} - \delta_y u_{i,j}}{h} + \frac{1}{2}\left[\delta_y^2 u_{i+1,j} - \delta_y^2 u_{i,j}\right]\\ &= \frac{1}{h}\left[u_{y,i+1,j} - u_{y,i,j}\right] + \frac{h}{6}\left[\delta_y^2 u_{y,i+1,j} - \delta_y^2 u_{y,i,j}\right] + \frac{1}{2}\left[\delta_y^2 u_{i+1,j} - \delta_y^2 u_{i,j}\right]\\ &= \delta_x^+ u_{y,i,j} + \frac{h^2}{6}\delta_y^2\delta_x^+ u_{y,i,j} + \frac{1}{2}h\delta_y^2\delta_x^+ u_{i,j}.\end{aligned}$$

In addition, using the definition of $\delta_y^2$ we have

$$|\delta_y^2 \delta_x^+ u_y| \leq \frac{4}{h^2} |\delta_x^+ u_y|_h \tag{141}$$

and

$$|\delta_y^2 \delta_x^+ u|_h \leq \frac{2}{h} |\delta_y^+ \delta_x^+ u|_h. \tag{142}$$

Therefore, we have

$$\begin{aligned} |\delta_x^+ \delta_y^+ u|_h &\geq |\delta_x^+ u_y|_h - \frac{h^2}{6} |\delta_y^2 \delta_x^+ u_y|_h - \frac{h}{2} |\delta_y^2 \delta_x^+ u|_h \\ &\geq |\delta_x^+ u_y|_h - \frac{2}{3} |\delta_x^+ u_y|_h - |\delta_x^+ \delta_y^+ u|_h, \end{aligned}$$

which gives finally $2|\delta_x^+ \delta_y^+ u|_h \geq \frac{1}{3} |\delta_x^+ u_y|_h$, or equivalently (139). We proceed in the same way in proving the symmetric estimate

$$|\delta_x^+ \delta_y^+ u|_h \geq \frac{1}{6} |\delta_y^+ u_x|_h. \tag{143}$$

Finally, the last coercivity inequality (134) is obtained starting from

$$|\delta_x^+ u|_h^2 = (|u_{h,x}|^x)^2 \tag{144}$$

and following along the same lines as in the proof of (99) in Lemma 3.5. ∎

We conclude this section with the following error estimate.

PROPOSITION 4.2. *Let $U$ be the $Q^1_{c,0}$ Lagrange interpolation of the exact solution $u(x)$ of* (108) *and $\tilde{u}$ the discrete solution of* (109). *Then there exists a constant $C$ independent of $h$ such that*

$$\langle U - \tilde{u}, U - \tilde{u} \rangle_h^{1/2} \leq C h^{3/2} \sum_{\alpha_1 + \alpha_2 \leq 6} |\partial_x^{\alpha_1} \partial_y^{\alpha_1} u|_{\infty, [0,1]^2}. \tag{145}$$

*Proof.* The proof follows along the same lines as the one of Proposition 3.1. We use in particular (134). ∎

**5. A Stephenson-based compact scheme for the streamfunction formulation of the Navier–Stokes equations.** The pure streamfunction form of the Navier–Stokes equation is

$$\partial_t \Delta \psi = -\nabla^\perp \psi \cdot \nabla(\Delta \psi) + \nu \Delta^2 \psi. \tag{146}$$

The streamfunction was introduced already by Lagrange; see [15, Chap. IV]. For simplicity, we deal only with the "no-slip" boundary condition, namely, the velocity vanishes on the boundary. This implies that we seek the streamfunction $\psi \in H^2_{h,0}$ (see [3] for a full discussion of the functional space for $\psi$). The notation is as follows. We denote by $\psi_{i,j} \in L^2_{h,0}$ a grid function and by $\psi_{x,i,j}, \psi_{y,i,j} \in L^2_{h,0}$ the Stephenson gradient defined by

$$P_x \psi_x = \delta_x \psi, \qquad P_y \psi_y = \delta_y \psi, \tag{147}$$

where the interpolation operators $P_x$, $P_y$ are (see (112))

$$(148)\quad P_x\psi_{|i,j} = \frac{1}{6}\psi_{i-1,j} + \frac{2}{3}\psi_{i,j} + \frac{1}{6}\psi_{i+1,j}, \quad P_y\psi_{|i,j} = \frac{1}{6}\psi_{i,j-1} + \frac{2}{3}\psi_{i,j} + \frac{1}{6}\psi_{i,j+1}.$$

The discrete gradient $\nabla_h\psi$ is defined as the pair of the discrete functions $(\psi_x, \psi_y)$ and the discrete velocity is defined as the discrete curl of the streamfunction in the sense

$$(149)\qquad \nabla_h^{\perp}\psi_{i,j} = U_{i,j} = \big[u_{i,j}, v_{i,j}\big] = \big[-\psi_{y,i,j}, \psi_{x,i,j}\big].$$

The discrete Laplacian is defined by the standard five-points formula

$$(150)\qquad \Delta_h\psi_{i,j} = \delta_x^2\psi_{i,j} + \delta_y^2\psi_{i,j}.$$

The discrete Stephenson biharmonic $\Delta_h^2$ introduced in (109) is

$$(151)\qquad \Delta_h^2 u_{i,j} = \delta_x^4 u_{i,j} + \delta_y^4 u_{i,j} + 2\delta_x^2\delta_y^2 u_{i,j}, \quad 1 \le i, j \le N-1.$$

$\Delta_h^2$ is a nine point operator acting at every point $(i, j)$ interior to the domain. The semidiscrete scheme associated with (146) consists in finding $\widetilde{\psi}(t) \in L^2_{h,0}$, which satisfies the evolution equation

$$(152)\qquad \partial_t\Delta_h\widetilde{\psi} = -\nabla_h^{\perp}\widetilde{\psi} \cdot (\Delta_h\nabla_h\widetilde{\psi}) + \nu\Delta_h^2\widetilde{\psi},$$

with initial condition

$$(153)\qquad \widetilde{\psi}_{i,j}(0) = (\psi_0)(x_i, y_j).$$

Note that in (152) and in what follows we use pointwise multiplication of functions in $L^2_{h,0}$, i.e., $(u \cdot v)_{i,j} = u_{i,j}v_{i,j}$. We denote by $e(t) = \widetilde{\psi}(t) - \psi(t)$ the difference between the computed and exact solutions. The exact solution verifies

$$(154)\qquad \partial_t\Delta_h\psi = -\nabla_h^{\perp}\psi \cdot \big[\Delta_h\nabla_h(\psi)\big] + \nu\Delta_h^2\psi + F,$$

where $F$ is the truncation error of the scheme depending on the regularity of the exact solution. We call $U$ and $\widetilde{U}$ the discrete velocities associated to $\psi$, $\widetilde{\psi}$ by

$$(155)\qquad U = (-\psi_y, \psi_x), \;\; \widetilde{U} = (-\widetilde{\psi}_y, \widetilde{\psi}_x).$$

Recall that in (155), the $x$ and $y$ subscripts stand for the discrete derivatives defined in (147). In particular, $\psi_x, \psi_y$ are not the values of the exact derivatives of $\psi$. The error $e(t)$ evolves according to

$$(156)\qquad \partial_t\Delta_h e - \nu\Delta_h^2 e = -\big[\widetilde{U} \cdot \Delta_h(\widetilde{\psi}_x, \widetilde{\psi}_y) - U \cdot \Delta_h(\psi_x, \psi_y)\big] - F.$$

The right-hand side of (156) is decomposed into four terms:

$$\begin{aligned}\big[(\widetilde{U} \cdot \Delta_h(\widetilde{\psi}_x, \widetilde{\psi}_y) - U \cdot \Delta_h(\psi_x, \psi_y)\big] + F = {}& (\widetilde{U} - U) \cdot \Delta_h\big[(\widetilde{\psi} - \psi)_x, (\widetilde{\psi} - \psi)_y\big]\\ &+ (\widetilde{U} - U) \cdot \Delta_h\big[(\psi_x, \psi_y\big]\\ &+ U \cdot \Delta_h\big[(\widetilde{\psi} - \psi)_x, (\widetilde{\psi} - \psi)_y\big] + F.\end{aligned}$$

Taking the $h$ scalar product with $e(t)$, we obtain

$$(157)\qquad \begin{aligned}(\partial_t\Delta_h e_h-\nu\Delta_h^2 e,e)_h=&-\Big((\widetilde{U}-U)\cdot\Delta_h\big[(\widetilde{\psi}-\psi)_x,(\widetilde{\psi}-\psi)_y\big],e\Big)_h\\&-\big((\widetilde{U}-U)\cdot\Delta_h(\psi_x,\psi_y),e\big)_h\\&-\Big(U\cdot\Delta_h\big[(\widetilde{\psi}-\psi)_x,\widetilde{\psi}-\psi)_y\big],e\Big)_h\\&-\big(F,e\big)_h.\end{aligned}$$

We denote the four terms of the right-hand side by $J_1$, $J_2$, $J_3$, $J_4$:

$$\begin{aligned}J_1&=\big((\widetilde{U}-U)\cdot\Delta_h(\widetilde{\psi}-\psi)_x,(\widetilde{\psi}-\psi)_y,e\big)_h,\\J_2&=\big((\widetilde{U}-U)\cdot\Delta_h(\psi_x,\psi_y),e\big)_h,\\J_3&=\big(U\cdot\Delta_h(\widetilde{\psi}-\psi)_x,(\widetilde{\psi}-\psi)_y,e\big)_h,\\J_4&=(F,e)_h.\end{aligned}$$

We estimate separately the four terms $J_1$, $J_2$, $J_3$, $J_4$.

**Term $J_1$.** The term $J_1$ is

$$(158)\qquad J_1=\big((\widetilde{U}-U)\cdot\Delta_h(e_x,e_y),e\big)_h.$$

We have

$$(159)\qquad \widetilde{U}-U=\big[-(\widetilde{\psi}-\psi)_y,(\widetilde{\psi}-\psi)_x\big]=(-e_y,e_x),$$

where the subscripts $x$ and $y$ are the Stephenson derivation operators. Therefore

$$\begin{aligned}J_1&=\big((\widetilde{U}-U)\cdot\Delta_h(e_x,e_y),e\big)_h=\big(-e_y(\delta_x^2e_x+\delta_y^2e_x)+e_x(\delta_x^2e_y+\delta_y^2e_y),e\big)_h\\&=\big(-e_y(\delta_x^2e_x+\delta_y^2e_x),e\big)_h+\big(e_x(\delta_x^2e_y+\delta_y^2e_y),e\big)_h\\&=-\big(\delta_x^2e_x,ee_y\big)_h-\big(\delta_y^2e_x,ee_y\big)_h+\big(\delta_x^2e_y,ee_x\big)_h+\big(\delta_y^2e_y,ee_x\big)_h\\&=\big(\delta_x^+e_x,\delta_x^+(ee_y)\big)_h+\big(\delta_y^+e_x,\delta_y^+(ee_y)\big)_h\\&\quad-\big(\delta_x^+e_y,\delta_x^+(ee_x)\big)_h-\big(\delta_y^+e_y,\delta_y^+(ee_x)\big)_h.\end{aligned}$$

In order to formulate a discrete Leibniz rule for $w,z\in L^2_{h,0}$ we use the "shift operators" $(S_xw)_{i,j}=w_{i+1,j},(S_yz)_{i,j}=z_{i,j+1}$. In terms of these operators we have

$$(160)\qquad \delta_x^+(wz)=(S_xw)_{i,j}\delta_x^+z+z\delta_x^+w,$$

which is quite easy to verify. Using (160), we expand $J_1$ in the sum of eight terms:

$$\begin{aligned}J_1=&\big(\delta_x^+e_x,(S_xe_y)_{i,j}\delta_x^+e\big)_h+\big(\delta_x^+e_x,e\delta_x^+e_y\big)_h\\&+\big(\delta_y^+e_x,(S_ye_y)_{i,j}\delta_y^+e\big)_h+\big(\delta_y^+e_x,e\delta_y^+e_y\big)_h\\&-\big(\delta_x^+e_y,(S_xe_x)_{i,j}\delta_x^+e\big)_h-\big(\delta_x^+e_y,e\delta_x^+e_x\big)_h\\&-\big(\delta_y^+e_y,(S_ye_x)_{i,j}\delta_y^+e\big)_h-\big(\delta_y^+e_y,e\delta_y^+e_x\big)_h.\end{aligned}$$

There is a cancellation of terms 2 and 6 on one hand, and 4 and 8 on the other hand, so that

$$J_1 = \big(\delta_x^+ e_x, (S_x e_y)\delta_x^+ e\big)_h + \big(\delta_y^+ e_x, (S_y e_y)\delta_y^+ e\big)_h \\ +\big(\delta_x^+ e_y, (S_x e_x)\delta_x^+ e\big)_h + \big(\delta_y^+ e_y, (S_y e_x)\delta_y^+ e\big)_h.$$

We now observe that if $\theta \in L^2_{h,0}$, then $|\theta|_{\infty,h} \le \frac{1}{h}|\theta|_h$. We can therefore estimate $J_1$ as follows:

$$\begin{aligned} |J_1| &= \big|\big((\widetilde{U} - U)\cdot \Delta_h(e_x, e_y), e\big)_h\big| \\ &\le \varepsilon\big[|\delta_x^+ e_x|_h^2 + |\delta_y^+ e_x|_h^2 + |\delta_x^+ e_y|_h^2 + |\delta_y^+ e_y|_h^2\big] + \frac{1}{4\varepsilon}\big[|(e_x, e_y)|^2_{\infty,h}\big(|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\big)\big] \\ &\le \varepsilon\big[|\delta_x^+ e_x|_h^2 + |\delta_y^+ e_x|_h^2 + |\delta_x^+ e_y|_h^2 + |\delta_y^+ e_y|_h^2\big] + \frac{C}{\varepsilon h^2}\big[|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\big]^2, \end{aligned}$$

where in the last step we have used (51) to estimate $|e_x|_{\infty,h} \le C|\delta_x^+ e|_{\infty,h}$ and $|e_y|_{\infty,h} \le C|\delta_y^+ e|_{\infty,h}$ with a constant independent of $h$. The factor $\varepsilon > 0$ will be specified later.

**Term $J_2$.** The term $J_2$ is estimated by ($C$ is a generic constant)

$$|J_2| = |\big((\widetilde{U} - U)\cdot \Delta_h(\psi_x, \psi_y), e\big)_h| \le C\big[|\widetilde{U} - U|_h^2 + |e|_h^2\big]. \tag{161}$$

We have used that $\Delta_h(\psi_x, \psi_y)$ is the discrete operator $\Delta_h$ composed by the Stephenson gradient applied to the exact solution, and is bounded if the exact solution is sufficiently regular. In addition, using the fact that $\widetilde{U} - U = \big[-(\widetilde{\psi}_y - \psi_y), \widetilde{\psi}_x - \psi_x\big]$, we have

$$|\widetilde{U} - U|_h^2 = |e_x|_h^2 + |e_y|_h^2. \tag{162}$$

Furthermore, we have, in view of (60), (78),

$$|e_x|_h \le C|\delta_x^+ e|_h, \quad |e_y|_h \le C|\delta_y^+ e|_h, \tag{163}$$

and, due to the Poincaré inequality (21), we deduce

$$|J_2| \le C\big[|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\big]. \tag{164}$$

**Term $J_3$.** We have

$$J_3 = \big[U\cdot\Delta_h(e_x, e_y), e\big]_h = \underbrace{(u\delta_x^2 e_x, e)_h}_{J_{3,1}} + \underbrace{(u\delta_y^2 e_x, e)_h}_{J_{3,2}} \\ + \underbrace{(v\delta_x^2 e_y, e)_h}_{J_{3,3}} + \underbrace{(v\delta_y^2 e_y, e)_h}_{J_{3,4}}.$$

We have

$$J_{3,1} = (u\delta_x^2 e_x, e)_h = (\delta_x^2 e_x, ue)_h = -\big[\delta_x^+ e_x, \delta_x^+(ue)\big]_h. \tag{165}$$

Using (160), the term $J_{3,1}$ is estimated by

$$\begin{aligned} |J_{3,1}| = |\big[\delta_x^+ e_x, \delta_x^+(ue)\big]_h &\le |\delta_x^+ e_x|_h |\delta_x^+(ue)|_h \\ &\le |\delta_x^+ e_x|_h\big[|(S_x u)_{i,j}\delta_x^+ e|_h + |e\delta_x^+ u|_h\big] \\ &\le |\delta_x^+ e_x|_h\big[|u|_{\infty,h}|\delta_x^+ e|_h + |\delta_x^+ u|_{\infty,h}|e|_h\big]. \end{aligned}$$

Therefore, using the Poincaré inequality (21), the term $J_{3,1}$ is estimated by

$$\begin{aligned} |J_{3,1}| &\le \max\left[|u|_{\infty,h}, |\delta_x^+ u|_{\infty,h}\right]\left[\varepsilon|\delta_x^+ e_x|_h^2 + \frac{1}{4\varepsilon}(|\delta_x^+ e|_h + |e|_h)^2\right] \\ &\le \max(|u|_{\infty,h}, |\delta_x^+ u|_{\infty,h})\left[\varepsilon|\delta_x^+ e_x|_h^2 + \frac{C}{\varepsilon}(|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2)\right]. \end{aligned}$$

Using the same principle in the $y$ direction, we obtain for the term $J_{3,2}$

(166)

$$|J_{3,2}| = |(u\delta_y^2 e_x, e)_h| \le \max(|u|_{\infty,h}, |\delta_y^+ u|_{\infty,h})\left[\varepsilon|\delta_y^+ e_x|_h^2 + \frac{C}{\varepsilon}(|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2)\right].$$

Therefore, with $m(u) = \max\left[|u|_{\infty,h}, |\delta_x^+ u|_{\infty,h}, |\delta_y^+ u|_{\infty,h}\right]$, the estimate for the term $J_{3,1} + J_{3,2}$ is

(167)

$$|J_{3,1} + J_{3,2}| \le |J_{3,1}| + |J_{3,2}| \le m(u)\left[\varepsilon\left\{|\delta_x^+ e_x|_h^2 + |\delta_y^+ e_x|_h^2\right\} + \frac{C}{\varepsilon}\left\{|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\right\}\right].$$

Treating the term $J_{3,3} + J_{3,4}$ in the same way, we obtain

(168)

$$|J_{3,3} + J_{3,4}| \le |J_{3,3}| + |J_{3,4}| \le m(v)\left[\varepsilon\{|\delta_x^+ e_y|_h^2 + |\delta_y^+ e_y|_h^2\} + \frac{C}{\varepsilon}\{|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\}\right].$$

The estimate for the term $J_3$ is finally, with $M(u,v) = \max(m(u), m(v))$,

(169)

$$|J_3| \le M(u,v)\left[\varepsilon\left\{|\delta_x^+ e_x|_h^2 + |\delta_y^+ e_x|_h^2 + |\delta_x^+ e_y|_h^2 + |\delta_y^+ e_y|_h^2\right\} + \frac{2C}{\varepsilon}\left\{|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\right\}\right].$$

**Term $J_4$.** The term $J_4$ is the truncation error and is of order 3/2 (in the $|\cdot|_h$ norm) in view of Lemmas 3.1 and 4.2. For any time $T > 0$, the term $J_4$ is estimated by

$$|J_4| \le C(T)|e|_h h^{3/2} \le C(T)\left[|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2 + h^3\right], \tag{170}$$

where $C(T)$ is a constant depending only on $T > 0$ and on the regularity of the exact solution $\psi(t)$ on $[0,T]$.

Turning back to (157), we have, on $[0, T_0]$,

$$\begin{aligned} \left(\frac{\partial}{\partial t}\Delta_h e, e\right)_h - \nu(\Delta_h^2 e, e)_h &= -\frac{1}{2}\frac{d}{dt}\{|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\} - \nu(\Delta_h^2 e, e)_h \\ &= -J_1 - J_2 - J_3 - J_4, \end{aligned}$$

or

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\{|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\} &= J_1 + J_2 + J_3 + J_4 - \nu(\Delta_h^2 e, e)_h \\ &\le |J_1| + |J_2| + |J_3| + |J_4| - \nu(\Delta_h^2 e, e)_h \\ &\le |J_1| + |J_2| + |J_3| + |J_4| \\ &\quad - C\nu\left[|\delta_x^+ e_x|_h^2 + |\delta_y^+ e_y|_h^2 + |\delta_x^+ e_y|_h^2 + |\delta_y^+ e_x|_h^2\right], \end{aligned}$$

where in the last inequality we have used the coercivity property (133). Collecting the terms of the form $|\delta_x^+ e_x|_h^2 + |\delta_y^+ e_y|_h^2 + |\delta_x^+ e_y|_h^2 + |\delta_y^+ e_x|_h^2$, which appear in the estimates for $J_1$, $J_2$, $J_3$, $J_4$, and selecting $\varepsilon > 0$ sufficiently small, we find that these terms are absorbed in the right-hand side of the last inequality. We are therefore left with the estimate

$$\text{(171)} \quad \frac{d}{dt}\left\{|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\right\} \le C\left[|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\right]\left[1 + \frac{1}{h^2}(|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2)\right] + C'h^3,$$

where $C, C'$ depend on the exact solution $\psi$ and on the viscosity coefficient $\nu$ but not on $h$.

In order to prove convergence of the approximate solution $\widetilde{\psi}$ to the exact solution $\psi$ using (171), we proceed as follows. We use the fact that at $t = 0$ the error $e = 0$ to prove an estimate for $|\delta_x^+ e|_h + |\delta_y^+ e|_h$ up to any given time $T > 0$.

THEOREM 5.1. *Let $T > 0$. Then there exist constants $C, h_0 > 0$, depending possibly on $T, \nu$, and the exact solution $\psi$, such that, for all $0 \le t \le T$,*

$$\text{(172)} \qquad |\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2 \le Ch^3, \quad 0 < h \le h_0.$$

Using Corollary 2.1, we obtain a 3/2 convergence rate in the discrete $L^2$ norm.

*Proof.* Fix some $K > 0$. Observe that at $t = 0$ we have $e = 0$; hence also $\delta_x^+ e = \delta_y^+ e = 0$ (at $t = 0$). Thus, taking $h > 0$, there exists a time $\tau > 0$ (in general depending on $h$) such that

$$\text{(173)} \qquad \sup_{0 \le t \le \tau} \left\{|\delta_x^+ e|_h + |\delta_y^+ e|_h\right\} \le Kh.$$

Inserting (173) in (171) we have for $t \le \tau$

$$\text{(174)} \qquad \frac{d}{dt}\left[|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\right] \le C(1 + K^2)\left[|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\right] + C'h^3, \quad 0 < h \le h_0;$$

hence by Gronwall's inequality (174) gives

$$\text{(175)} \qquad |\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2 \le C_1 e^{C(1+K^2)t} h^3, \quad t \le \tau,$$

with a suitable constant $C_1 > 0$. Observe that in (175) $\tau$ depends on $h$, and define $\tau_0 = \tau_0(h)$ by

$$\text{(176)} \qquad \tau_0 = \sup\{t > 0 \text{ such that } |\delta_x^+ e|_h + |\delta_y^+ e|_h \le Kh\}.$$

We have $\tau_0 \ge \tau$ and, as in (175), we obtain

$$\text{(177)} \qquad |\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2 \le C_1 e^{C(1+K^2)t} h^3, \quad t \le \tau_0.$$

We can now select $h_0$ so small that

$$\text{(178)} \qquad C_1 e^{C(1+K^2)T} h_0 < K^2.$$

Now the definition of $\tau_0$ and (177)–(178) imply that, for any $0 < h \le h_0$, we have $\tau_0(h) \ge T$ and, in particular, for such $h$, the estimate (175) holds true for all $t \le T$. This concludes the proof of the theorem. $\square$

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