Semi-discrete Time-Dependent Fourth-Order Problems on an Interval: Error Estimate

Dalia Fishelov

Abstract We present high-order compact schemes for fourth-order time-dependent problems, which are related to the "buckling plate" or the "clamping plate" problems. Given a mesh size h, we show that the truncation error is $O(h^4)$ at interior points and $O(h)$ at near-boundary points. In addition, the convergence of these schemes is analyzed. Although the truncation error is only of first-order at near-boundary points, we have proved that the error of these schemes converges to zero as h tends to zero at least as $O(h^{3.5})$. Numerical results are performed and they calibrate the high-order accuracy of the schemes. It is shown that the numerical rate of convergence is actually four, thus the error tends to zero as $O(h^4)$.

1 Introduction

Time-dependent fourth-order differential problems play an important role in various areas of physics. In mechanics they are involved in plate problems, such as the "buckling plate" or the "clamping plate" problem. In fluid dynamics they are used in the Navier-Stokes equations. In this paper we are interested in two time dependent problems, which are related to fourth-order problems.

The first one is

$$u_{xxt} = u_{xxxx} + b\, u_{xx} + c\, u_x + d\, u + f(x,t), \quad 0 < x < 1, \quad t > 0 \tag{1}$$

and the second is

$$u_t = -u_{xxxx} + b\, u_{xx} + c\, u_x + d\, u + f(x,t), \quad 0 < x < 1, \quad t > 0. \tag{2}$$

Both problems are supplemented with boundary conditions

$$u(0,t) = \partial_x u(0,t) = 0, \quad u(1,t) = \partial_x u(1,t) = 0 \tag{3}$$

D. Fishelov (✉)
Afeka-Tel-Aviv Academic College for Engineering, 218 Bnei-Efraim St. Tel-Aviv 69107, Israel
e-mail: daliaf@post.tau.ac.il

A. Abdulle et al. (eds.), *Numerical Mathematics and Advanced Applications - ENUMATH 2013*, Lecture Notes in Computational Science and Engineering 103, DOI 10.1007/978-3-319-10705-9_13

and the initial condition

$$u(x,0) = g(x), \quad 0 \le x \le 1. \tag{4}$$

Consider now the finite interval $I = [0, 1]$ with the grid

$$x_0 = 0 < x_1 < \cdots < x_{N-2} < x_{N-1} < x_N = 1, \tag{5}$$

where $x_j = jh$, $j = 1, \cdots, N$ and $h = 1/N$. In order to approximate the solutions of Problems (1) and (2) one needs to approximate the operators ∂_x^4, ∂_x^2 and ∂_x. We approximate ∂_x^4 by δ_x^4, where δ_x^4 is the three-point compact operator defined by (see [3, 4]).

$$\delta_x^4 v_j = \frac{12}{h^2}\left(\frac{v_{x,j+1} - v_{x,j-1}}{2h} - \frac{v_{j+1} + v_{j-1} - 2v_j}{h^2}\right) = \frac{12}{h^2}(\delta_x v_{x,j} - \delta_x^2 v_j), \tag{6}$$

for $1 \le j \le N-1$. The operator ∂_x^2 is a approximated by $\tilde{\delta}_x^2$, where

$$\tilde{\delta}_x^2 v_j = 2\delta_x^2 v_j - \delta_x v_{x,j} = \delta_x^2 v_j - \frac{h^2}{12}\delta_x^4 v_j, \quad 1 \le j \le N-1. \tag{7}$$

Here $v_{x,j}$ is the Hermitian derivative of v at point x_j. It is defined by

$$\frac{1}{6}v_{x,j-1} + \frac{2}{3}v_{x,j} + \frac{1}{6}v_{x,j+1} = \frac{v_{j+1} - v_{j-1}}{2h}, \quad 1 \le j \le N-1. \tag{8}$$

This operator was extensively studied in previous works [2, 4]. Problem (1) is approximated by the semi-discrete finite-difference scheme

$$\frac{d}{dt}\tilde{\delta}_x^2 v_j = \delta_x^4 v_j + b\,\tilde{\delta}_x^2 v_j + c\,v_{x,j} + d\,v_j + f(x_j,t) \tag{9}$$

and Problem (2) by

$$\frac{d}{dt}v_j = -\delta_x^4 v_j + b\,\tilde{\delta}_x^2 v_j + c\,v_{x,j} + d\,v_j + f(x_j,t). \tag{10}$$

Let v, w be two discrete functions, defined on the grid (5) and vanishing at the two endpoints x_0, x_N. We define the discrete inner product $(v,w)_h$ and the discrete norm $|v|_h$ as

$$(v,w)_h = \sum_{j=1}^{N-1} v_j w_j h, \quad |v|_h = \sqrt{(v,v)_h}. \tag{11}$$

2 Consistency for Compact Operators on an Interval

2.1 The Truncation Error

Here we consider the truncation errors related to the operators δ_x^4, $\tilde{\delta}_x^2$ and the Hermitian derivative v_x. Let σ_x be the (Simpson) operator [4]

$$\sigma_x v_j = \frac{1}{6}v_{j-1} + \frac{2}{3}v_j + \frac{1}{6}v_{j+1}. \tag{12}$$

We consider first the truncation error related to δ_x^4. We have the following inequalities (see [4]):

$$|\sigma_x \delta_x^4 u_j^* - \sigma_x (u^{(4)})^*(x_j)| \leq Ch^4 \|u^{(8)}\|_{L^\infty}, \quad 2 \leq j \leq N-2. \tag{13}$$

$$|\sigma_x \delta_x^4 u_j^* - \sigma_x (u^{(4)})^*(x_j)| \leq Ch \|u^{(5)}\|_{L^\infty} \quad j = 1, N-1. \tag{14}$$

Consider now the operator $\tilde{\delta}_x^2$

$$-\tilde{\delta}_x^2 u_j = -2\delta_x^2 u_j + \delta_x u_{x,j} = -\delta_x^2 u_j + \frac{h^2}{12}\delta_x^4 u_j. \tag{15}$$

Operating with σ_x on the last equality, we have

$$-\sigma_x \tilde{\delta}_x^2 u_j = -\sigma_x \delta_x^2 u_j + \frac{h^2}{12}\sigma_x \delta_x^4 u_j. \tag{16}$$

Using the truncation error for $-\delta_x^2$, we have $-\delta_x^2 u_j = -\partial_x^2 u(x_j, t) - \frac{h^2}{12}\partial_x^4 u(x_j, t) + O(h^4)$. Thus,

$$-\sigma_x \delta_x^2 u_j = -\sigma_x \partial_x^2 u(x_j, t) - \frac{h^2}{12}\sigma_x \partial_x^4 u(x_j, t) + O(h^4). \tag{17}$$

Inserting the last equality in (16), we have

$$-\sigma_x \tilde{\delta}_x^2 u_j = -\sigma_x \partial_x^2 u(x_j, t) + \frac{h^2}{12}\sigma_x (\delta_x^4 u_j - \partial_x^4 u(x_j, t)) + O(h^4). \tag{18}$$

Combing the above with (13) and (14), we find that

$$-\sigma_x \tilde{\delta}_x^2 u_j = -\sigma_x \partial_x^2 u(x_j, t) + O(h^4), \quad 2 \leq j \leq N-2. \tag{19}$$

$$-\sigma_x \tilde{\delta}_x^2 u_j = -\sigma_x \partial_x^2 u(x_j, t) + O(h^3), \quad j = 1, N-1. \tag{20}$$

In addition (see [3, 4]), we have

$$u_{x,j} = \partial_x u(x_j,t) + O(h^4), \quad 1 \le j \le N-1. \tag{21}$$

3 The Time-Dependent Case $u_{xxt} = u_{xxxx} + b\, u_{xx} + c\, u_x + d\, u + f(x,t)$

Consider the time-dependent fourth-order problem

$$\begin{cases} u_{xxt} = u_{xxxx} + b\, u_{xx} + c\, u_x + d\, u + f(x,t), & 0 < x < 1,\ t > 0 \\ u(0,t) = 0,\ u(1,t) = 0,\ u_x(0,t) = 0,\ u_x(1,t) = 0, & t > 0 \\ u(x,0) = g(x), \quad 0 \le x \le 1. \end{cases} \tag{22}$$

The canonical semi-discrete approximation of (22) on the grid (5) is

$$\begin{aligned} &\tfrac{\partial}{\partial t}\tilde{\delta}_x^2 v_j(t) = \delta_x^4 v_j(t) + b\, \tilde{\delta}_x^2 v_j(t) + c\, v_{x,j} + d\, v_j(t) + f_j(t), \quad j = 1,\ldots,N-1, \\ &v_0(t) = 0,\ v_N(t) = 0,\ v_{x,0}(t) = 0,\ v_{x,N}(t) = 0, \quad t > 0 \\ &v_j(0) = g_j := g(x_j), \quad j = 0,\ldots,N. \end{aligned} \tag{23}$$

Although the truncation error deteriorate at near-boundary points, we prove in the following Proposition that the convergence of the approximate solution to the exact one is of high accuracy. A similar result was shown in [1, 6, 7] in cases where the accuracy of the scheme deteriorates near the boundary. In [6] and [7] a hyperbolic system of first order and a parabolic problem were analyzed. In [1] it was proved for a parabolic equation that if the scheme is of order $O(h^\alpha)$ at inner points and of order $O(h^{\alpha-s})$ near the boundary, then if $s = 0, 1$ the accuracy of the scheme is $O(h^\alpha)$. However, if $s \ge 2$ then the overall accuracy the scheme is $O(h^{\alpha-s+3/2})$. In our case $\alpha = 4$ and $s = 3$ so this result yields a convergence rate of 2.5, but we actually prove that the convergence rate is at least 3.5.

Theorem 1 *Let $u(x,t)$ be the exact solution of (1) satisfying the boundary conditions (3) and the initial condition (4). Assume that u has continuous derivatives with respect to x up to order eight on $[0,1]$ and up to order 1 with respect to t. Let $v(t)$ be the approximation to u, given by the (23). Then, the error $e_j(t) = v_j(t) - u(x_j,t)$ satisfies*

$$\max_{0 \le t \le T} |e(t)|_h \le \max_{0 \le t \le T} |\delta_x^+ e(t)|_h \le C(T) h^{3.5}, \tag{24}$$

where $C(T)$ depends only on f, g and T.

Proof The error $e_j(t)$ satisfies

$$\begin{aligned}
&-\tfrac{\partial}{\partial t}\tilde{\delta}_x^2 e_j(t) = -\delta_x^4 e_j(t) - b\, \tilde{\delta}_x^2 e_j(t) - c\, e_{x,j}(t) - d\, e_j(t) + r_j(t), \quad j = 1, \ldots, N-1, \\
&e_0(t) = 0,\; e_N(t) = 0,\; e_{x,0}(t) = 0,\; e_{x,N}(t) = 0, \quad t > 0, \\
&e_j(0) = 0, \quad j = 0, \ldots, N,
\end{aligned} \tag{25}$$

where $r_j(t)$ is the truncation error at point x_j at time t. Taking the inner product of (25) with $e(t)$ and using

$$\frac{1}{2}\frac{\partial}{\partial t}(-\tilde{\delta}_x^2 e(t), e(t))_h = (-\frac{\partial}{\partial t}\tilde{\delta}_x^2 e(t), e(t))_h, \tag{26}$$

we find that

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\partial}{\partial t}(-\tilde{\delta}_x^2 e(t), e(t))_h &= -(\delta_x^4 e(t), e(t))_h - b\, (\tilde{\delta}_x^2 e(t), e(t))_h \\
&\quad - c\, (e_x(t), e(t))_h - d\, (e(t), e(t))_h + (r(t), e(t))_h.
\end{aligned} \tag{27}$$

First consider the term $(e_x(t), e(t))_h$. Using the Cauchy-Schwartz inequality, we have

$$|(e_x(t), e(t))_h| \le |e(t)|_h\, |e_x(t)|_h \le \frac{1}{2}|e(t)|_h^2 + \frac{1}{2}|e_x(t)|_h^2. \tag{28}$$

Since $e_x = \sigma_x^{-1}\delta_x e$ and σ_x^{-1} is bounded (see [2] Equation (51)), we have that $|e_x(t)|_h^2 \le C|\delta_x^+ e(t)|_h^2$. By discrete integration by parts $|\delta_x^+ e(t)|_h^2 = -(\delta_x^2 e(t), e(t))_h$. Using the definition (7) of $-\tilde{\delta}_x^2$ and the coercivity of δ_x^4, we have

$$-(\tilde{\delta}_x^2 e(t), e(t))_h \ge -(\delta_x^2 e(t), e(t))_h. \tag{29}$$

Thus, $|e_x(t)|_h^2 \le -C(\tilde{\delta}_x^2 e(t), e(t))_h$. Therefore,

$$|(e_x(t), e(t))_h| \le -C(\tilde{\delta}_x^2 e(t), e(t))_h + \frac{1}{2}(e(t), e(t))_h. \tag{30}$$

Combining (30) with (27) we obtain

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\partial}{\partial t}(-\tilde{\delta}_x^2 e(t), e(t))_h &\le -(\delta_x^4 e(t), e(t))_h - C_1(\tilde{\delta}_x^2 e(t), e(t))_h \\
&\quad + \tilde{C}(e(t), e(t))_h + (r(t), e(t))_h.
\end{aligned} \tag{31}$$

Let us now consider the term $\tilde{C}(e(t), e(t))_h$. Using the definition (7) of $-\tilde{\delta}_x^2$, and the coercivity of δ_x^4, we have

$$-(\tilde{\delta}_x^2 e(t), e(t))_h \geq -(\delta_x^2 e(t), e(t))_h = (\delta_x^+ e(t), \delta_x^+ e(t))_h. \tag{32}$$

By the discrete Poincaré inequality, we find

$$(e(t), e(t))_h \leq C\ (\delta_x^+ e(t), \delta_x^+ e(t))_h. \tag{33}$$

Therefore, $(e(t), e(t))_h \leq -C\ (\tilde{\delta}_x^2 e(t), e(t))_h$. Combining (31) with the last inequality, we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}(-\tilde{\delta}_x^2 e(t), e(t))_h \leq -(\delta_x^4 e(t), e(t))_h - C_1\ (\tilde{\delta}_x^2 e(t), e(t))_h + (e(t), r(t))_h. \tag{34}$$

Consider now the term $(r(t), e(t))_h$. Using the Cauchy-Schwartz inequality, we have

$$\begin{aligned} &|(r(t), e(t))_h| = ((\delta_x^{-4})^{1/2} r(t), (\delta_x^4)^{1/2} e(t))_h \\ &\leq |(\delta_x^{-4})^{1/2} r(t)|_h\ |(\delta_x^4)^{1/2} e(t)|_h \leq \tfrac{1}{2}(r(t), \delta_x^{-4} r(t))_h + \tfrac{1}{2}(\delta_x^4 e(t), e(t))_h \\ &= \tfrac{1}{2}(\sigma_x r(t), \sigma_x^{-1}\delta_x^{-4}\sigma_x^{-1}\sigma_x r(t)) + \tfrac{1}{2}(\delta_x^4 e(t), e(t))_h. \end{aligned} \tag{35}$$

Combining the truncation errors (13) and (14) for $\sigma_x \delta_x^4$, (19) and (20) for $\sigma_x \tilde{\delta}_x^2$ and (21) for the Hermitian derivative e_x, we have

$$(PR(t))^T = [O(h), O(h^4), \ldots, O(h^4), O(h)]. \tag{36}$$

Here, P is the matrix representing the operator $6\sigma_x$ and $R(t)$ is the vector corresponding to $r(t)$. As a result of [5] Equations (111) and (116), we have

$$\begin{aligned} &(a) \qquad |(P^{-1}S^{-1}P^{-1}PR)_i| \leq Ch^4, \ \ 2 \leq i \leq N-2, \\ &(b) \qquad |(P^{-1}S^{-1}P^{-1}PR)_i| \leq Ch^5, \ \ i = 1, N-1, \end{aligned} \tag{37}$$

where S the matrix representing δ_x^4. Using (37) (a),(b) and (36), we find that

$$|(PR(t))^T P^{-1} S^{-1} R(t)| \leq Ch^6. \tag{38}$$

Therefore, $|\sigma_x r(t), \sigma_x^{-1}\delta_x^{-4}\sigma_x^{-1}\sigma_x r(t)| \leq Ch^7$. Combining the last inequality with (35), we obtain

$$|(r(t), e(t))_h| \leq \tfrac{1}{2}(\delta_x^4 e(t), e(t))_h + Ch^7. \tag{39}$$

Inserting (39) in (34), we have

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}(-\tilde{\delta}_x^2 e, e)_h &\le -\frac{1}{2}(\delta_x^4 e(t), e(t))_h - C_1\,(\tilde{\delta}_x^2 e(t), e(t))_h + Ch^7\\ &\le C_1(-\tilde{\delta}_x^2 e, e)_h + Ch^7.\end{aligned} \tag{40}$$

By Gronwall's inequality $-(\tilde{\delta}_x^2 e(t), e(t))_h \le C(t)h^7$. Using the coercivity property

$$-(e(t), \tilde{\delta}_x^2 e(t))_h \ge (\delta_x^+ e(t), \delta_x^+ e(t))_h, \tag{41}$$

and the discrete Poincaré inequality, we obtain the estimate

$$|e(t)|_h \le C|\delta_x^+ e(t)|_h \le C(T)h^{3.5}, \quad 0 \le t \le T. \tag{42}$$

□

4 The Time-Dependent Case $u_t = -u_{xxxx} + b\, u_{xx} + c\, u_x + d\, u + f(x,t)$

Consider the time-dependent biharmonic problem

$$\begin{cases} u_t = -u_{xxxx} + b\, u_{xx} + c\, u_x + d\, u + f(x,t), & 0 < x < 1, \\ u(0,t) = 0,\ u(1,t) = 0,\ u_x(0,t) = 0,\ u_x(1,t) = 0, & t > 0 \\ u(x,0) = g(x), & 0 \le x \le 1. \end{cases} \tag{43}$$

The canonical semi-discrete approximation of (43) on the grid (5) is

$$\begin{aligned} &\frac{\partial v_j(t)}{\partial t} = -\delta_x^4 v_j(t) + b\,\tilde{\delta}_x^2 v_j + c\, v_{x,j}(t) + d\, v_j(t) + f_j(t), \quad j = 1, \dots, N-1, \\ &v_0(t) = 0,\ v_N(t) = 0,\ v_{x,0}(t) = 0,\ v_{x,N}(t) = 0, \quad t > 0 \\ &v_j(0) = g_j := g(x_j), \quad j = 0, \dots, N. \end{aligned} \tag{44}$$

Define the error $e_j(t)$ by $e_j(t) = v_j(t) - u(x_j, t)$.

Theorem 2 *Let $u(x,t)$ be the exact solution of (2) satisfying the boundary conditions (3) and the initial condition (4). Assume that u has continuous derivatives with respect to x up to order eight on $[0,1]$ and up to order 1 with respect to t. Let $v(t)$ be* *the approximation to u, given by the (44). Then, the error* $e_j(t) = v_j(t) - u(x_j, t)$ *satisfies*

$$\max_{0\le t\le T} |e(t)|_h \le C(T)h^{3.5}, \tag{45}$$

where $C(T)$ *depends only on* f, g *and* T.

Proof The error $e_j(t)$ satisfies

$$\begin{aligned}
&\tfrac{\partial e_j(t)}{\partial t} = -\delta_x^4 e_j(t) + b\,\tilde{\delta}_x^2 e_j(t) + c\,e_{x,j}(t) + d\,e_j(t) - r_j(t), \quad j = 1, \ldots, N-1,\\
&e_0(t) = 0,\ e_N(t) = 0,\ e_{x,0}(t) = 0,\ e_{x,N}(t) = 0, \quad t > 0\\
&e_j(0) = 0, \quad j = 0, \ldots, N,
\end{aligned} \tag{46}$$

where $r_j(t)$ is the truncation error at point x_j at time t. Taking the inner product of (46) with $e(t)$, and using

$$\frac{1}{2}\frac{\partial}{\partial t}(e(t), e(t))_h = (\frac{\partial}{\partial t}e(t), e(t))_h, \tag{47}$$

we find that

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\partial}{\partial t}(e(t), e(t))_h &= -(\delta_x^4 e(t), e(t))_h + b\,(\tilde{\delta}_x^2 e(t), e(t))_h\\
&\quad + c\,(e_x(t), e(t))_h + d\,(e(t), e(t))_h - (r(t), e(t))_h.
\end{aligned} \tag{48}$$

Considering first the term $(e_x(t), e(t))_h$. Combining (30) with (48) we obtain

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\partial}{\partial t}(e(t), e(t))_h &= -(\delta_x^4 e(t), e(t))_h + \tilde{b}\,(\tilde{\delta}_x^2 e(t), e(t))_h\\
&\quad + \tilde{d}\,(e(t), e(t))_h - (r(t), e(t))_h.
\end{aligned} \tag{49}$$

Let us now consider the term $(\tilde{\delta}_x^2 e(t), e(t))_h$. Using the definition of $-\tilde{\delta}_x^2$, we have

$$-(\tilde{\delta}_x^2 e(t), e(t))_h = -(\delta_x^2 e(t), e(t))_h + \frac{h^2}{12}(\delta_x^4 e(t), e(t))_h. \tag{50}$$

Therefore,

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\partial}{\partial t}(e(t), e(t))_h &= -(1 + \tilde{b}\tfrac{h^2}{12})(\delta_x^4 e(t), e(t))_h + \tilde{b}\,(\delta_x^2 e(t), e(t))_h\\
&\quad + \tilde{d}\,(e(t), e(t))_h - (r(t), e(t))_h.
\end{aligned} \tag{51}$$

Consider now the term $(\delta_x^2 e(t), e(t))_h$. Using Cauchy-Schwartz inequality, we have

$$-(\delta_x^2 e(t), e(t))_h \leq |e(t)|_h \, |\delta_x^2 e(t)|_h \leq \frac{1}{2\epsilon}|e(t)|_h^2 + \frac{\epsilon}{2}|\delta_x^2 e(t)|_h^2. \tag{52}$$

Using the coercivity property $|\delta_x^2 e(t)|_h^2 \leq C(e(t), \delta_x^4 e(t))_h$, we obtain

$$-(\delta_x^2 e(t), e(t))_h \leq \frac{1}{2\epsilon}|e(t)|_h^2 + \frac{C\epsilon}{2}(\delta_x^4 e(t), e(t))_h. \tag{53}$$

Inserting (53) in (51), we obtain that for $h \leq h_0$ and $\epsilon \leq \epsilon_0$,

$$\tfrac{1}{2}\tfrac{\partial}{\partial t}(e(t), e(t))_h \leq -\tfrac{1}{2}(\delta_x^4 e(t), e(t))_h + C_2\,(e(t), e(t))_h - (r(t), e(t))_h. \tag{54}$$

Inserting (39) in (54), we have

$$\frac{1}{2}\frac{\partial}{\partial t}(e(t), e(t))_h \leq C_2\;\;(e(t), e(t))_h + Ch^7. \tag{55}$$

By Gronwall's inequality $|e|_h^2 \leq Ch^7$. Thus, $|e(t)|_h \leq C(T)h^{3.5}, \;\; 0 \leq t \leq T.$ □

5 Numerical Results

Consider the exact solution $u(x,t) = e^{-t}e^x$ of the problem

$$\begin{cases} u_{xxt} = u_{xxxx} + u_{xx} + f(x,t), \quad 0 < x < 1, \quad t > 0, \\ u(0,t) = e^{-t}, \quad u_x(0,t) = e^{-t}, \quad t > 0, \\ u(1,t) = e^{1-t}, \quad u_x(1,t) = e^{1-t}, \quad t > 0, \\ u(x,0) = e^x, \quad 0 \leq x \leq 1. \end{cases} \tag{56}$$

Here u and u_x are given on the boundary points and $f(x,t)$ is chosen as $u(x,t) = e^{-t}e^x$ is the solution of the differential equation above. The results are given in Table 1. They demonstrate the fourth-order accuracy of the scheme.

Table 1 Compact scheme for $u_{xxt} = u_{xxxx} + u_{xx} + f$ with exact solution: $u = e^{-t}e^x$ on $[0,1], t > 0$. We present $|e_h|_h$ the error in u, and $|e_x|_h$ the error in u_x in the l_2 norm at $t = 0.5$

Mesh	$N=8$	Rate	$N=16$	Rate	$N=32$	Rate	$N=64$
$\lvert e\rvert_h$	1.5742(−7)	4.07	9.3544(−9)	4.02	5.7490(−10)	4.00	3.5844(−11)
$\lvert e_x\rvert_h$	1.5500(−6)	4.00	9.7033(−8)	4.00	6.0672(−9)	4.00	3.7893(−10)

Table 2 Compact scheme for $u_{xxt} = u_{xxxx} + u_{xx} + f$ with exact solution: $u = e^{-t}\sin(\pi x)/\pi^2$ on $[0, 1]$, $t > 0$. We present $|e|_h$ the error in u, and $|e_x|_h$ the error in u_x in the l_2 norm at $t = 0.5$

Mesh	$N=8$	Rate	$N=16$	Rate	$N=32$	Rate	$N=64$
$\|e\|_h$	5.2800(−6)	4.01	3.2679(−7)	4.00	2.0381(−8)	4.00	1.2732(−9)
$\|e_x\|_h$	2.0038(−6)	4.07	1.1926(−7)	4.02	7.3451(−9)	4.01	4.5737(−10)

Next we consider the solution $u(x,t) = e^{-t} sin(\pi x)/\pi^2$ of the problem

$$\begin{cases} u_{xxt} = u_{xxxx} + u_{xx} + f(x,t), & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, \quad u_x(0,t) = e^{-t}/\pi, & t > 0, \\ u(1,t) = 0, \quad u_x(1,t) = -e^{-t}/\pi, & t > 0, \\ u(x,0) = sin(\pi x)/\pi^2, & 0 \le x \le 1. \end{cases} \tag{57}$$

Here u and u_x are given at the two boundary points and $f(x,t)$ is chosen such that $u(x,t) = e^{-t}\sin(\pi x)/\pi^2$ if the exact solution of the problem above. The numerical results are shown in Table 2. The calibrate the fourth-order accuracy of the scheme.

References

1. S. Abarbanel, A. Ditkowski, B. Gustafsson, On error bound of finite difference approximations for partial differential equations. J. Sci. Comput. **15**, 79–116 (2000)
2. M. Ben-Artzi, J-P. Croisille, D. Fishelov, Convergence of a compact scheme for the pure streamfunction formulation of the unsteady Navier–Stokes system. SIAM J. Numer. Anal. **44**, 1997–2024 (2006)
3. M. Ben-Artzi, J.-P. Croisille, D. Fishelov, A fast direct solver for the biharmonic problem in a rectangular grid. SIAM J. Sci. Comput. **31**(1), 303–333 (2008)
4. M. Ben-Artzi, J.-P. Croisille, D. Fishelov, *Navier-Stokes Equations in Planar Domains* (Imperial College Press, London, 2013)
5. D. Fishelov, M. Ben-Artzi, J-P. Croisille, Recent advances in the study of a fourth-order compact scheme for the one-dimensional biharmonic equation. J. Sci. Comput. **53**, 55–70 (2012)
6. B. Gustafsson, The convergence rate for difference approximations to mixed initial boundary value problems. Math. Comput. **29**, 386–406 (1975)
7. B. Gustafsson, The convergence rate for difference approximations to general mixed initial boundary value problems. SIAM J. Numer. Anal. **18**, 179–190 (1981)