4. DETERMINATION OF A STRESS INTENSITY FACTOR USING LOCAL MESH REFINEMENT

B. Schiff and D. Fishelov
University of Tel Aviv

J.R. Whiteman*
Institute of Computational Mathematics
Brunel University

1. Introduction

A general survey of methods for adapting *standard* finite element methods to deal with singularities has been given by Whiteman and Akin in [7]; that paper will henceforth be referred to as WA and, when equations and sections are referenced, the relevant numbers will have the prefix WA. In the present paper we consider in detail the application of a *conforming local mesh refinement* finite element technique, proposed by Gregory, Fishelov, Schiff and Whiteman [2], to a model *opening mode* linear elastic fracture problem. This problem has also been treated using specially modified finite difference techniques by Bernal and Whiteman [1], and, using collocation techniques by Gross, Srawley and Brown [3] and by Whiteman [6]. In the model problem a two dimensional rectangular elastic solid containing a single edge crack is subjected to a uniform inplane load normal to the two edges parallel to the crack, the remaining edges being stress free. The problem is one of plane strain and can be formulated in three different but equivalent ways; in terms of the *Airy stress function* u(x,y), in terms of the *x- and y-displacements* U(x,y) and V(x,y), in terms of *stresses* σ_{xx}, σ_{xy}, σ_{yy}.

In WA(2.6) a singular biharmonic function was expressed in the form of the series

$$\sum_{n=1}^{\infty}\Big[(-1)^{n-1}a_{2n-1}r^{(n+\frac{1}{2})}\Big\{-\cos(n-\tfrac{3}{2})\theta+\frac{2n-3}{2n+1}\cos(n+\tfrac{1}{2})\theta\Big\} + (-1)^n a_{2n}r^{(n+1)}\Big\{-\cos(n-1)\theta+\cos(n+1)\theta\Big\}\Big], \qquad (1.1)$$

where the $\{a_i\}$ are unknown coefficients and (r,θ) are local polar coordinates about the point of singularity. For the model fracture problem the series (1.1) can be used to represent the

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Airy stress function. As was indicated in [7] the *opening mode stress intensity factor* K (see also [4]) is related to a_1 by the expression $K = -\sqrt{2\pi}\, a_1$. Thus in the computations of this paper we seek to approximate a_1 for the model problem, so producing approximations to K.

The various formulations of the model problem are given in Section 2, together with discussions of the finite element techniques used. The techniques of local mesh refinement are described in Section 3, and the results derived are given in Section 4. Although these techniques for each formulation produce approximate solutions to the field equations, they do not give approximations to the coefficients $\{a_i\}$. We therefore derive such approximations by least squares fitting to the appropriate calculated nodal values, either of a truncated form of the series (1.1) for the Airy stress function $u(r,\theta)$, or of the truncated forms of the corresponding series which can be derived from (1.1) for the displacements U and V. Results derived in this way are also presented in Section 4.

2. Two Dimensional Fracture Problem

We consider the model problem of a two dimensional solid in a plane strain situation. The solid is the cracked rectangle $\overline{OABCDEFGO}$ of Fig. 1 with width 2a and crack of length a. This is subjected to an inplane load λ normal to BC and FE, so that the normal and shearing stresses on OA and OG are zero. There is of course symmetry about AOD.

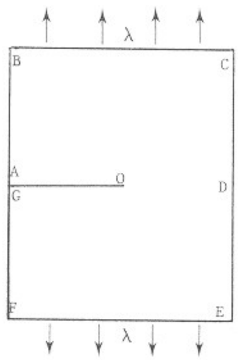

Fig. 1.

2.1. Airy Stress Function Formulation

In terms of the Airy stress function $u(x,y)$ the model problem, which is of type WA(2.2), is defined in $\Omega \equiv \overline{OABCDEFGO}$ of Fig. 1 so that

$$\Delta^2[u(x,y)] = 0, \quad (x,y) \in \Omega,$$
$$u = \frac{\partial u}{\partial n} = 0, \quad \text{on OA and AB,}$$
$$u = \lambda\left(\frac{x^2}{2} + ax + \frac{a^2}{2}\right), \quad \frac{\partial u}{\partial n} = 0 \text{ on BC,} \tag{2.1}$$
$$u = 2\lambda a^2, \; \frac{\partial u}{\partial n} = 2\lambda a \quad \text{on CD,}$$

with the boundary conditions on the remainder of the boundary being defined in the obvious way using the symmetry. In (2.1) $\partial/\partial n$ is the derivative in the direction of the outward normal to the boundary. It is easily seen from (2.1) that the series (1.1) satisfies the biharmonic equation and the boundary conditions on the crack.

The finite element method will be applied to a variational form of (2.1) defined on the Sobolev space $\overset{*}{H}{}^2(\Omega)$, where

$$\overset{*}{H}{}^2(\Omega) \equiv \{v : v \in H^2(\Omega) \text{ and } v \text{ satisfies the essential boundary conditions of problem (2.1) on } \partial\Omega\}.$$

We thus seek to solve the variational problem

$$\min_{\overset{*}{H}{}^2} I_1[v],$$

where

$$I_1(v) \equiv \iint_\Omega (\Delta v)^2 dxdy. \tag{2.2}$$

For the finite element method we partition Ω into rectangular elements and, in the usual way, form over this partition a finite dimensional space contained in $\overset{*}{H}{}^2(\Omega)$ which consists of piecewise bicubic functions with $C^1(\Omega)$ continuity and satisfying the essential boundary conditions of (2.1). In any element these Hermite-bicubic functions have as degrees of freedom at any node the values of the function and its x-, y- and xy-derivatives, giving 16 degrees of freedom per element.

2.2. *Displacement Formulation*

Problem (2.1) can be reformulated in terms of the x- and y-displacements U and V of the body. These minimise the *potential energy functional*

$$I_2[U,V] = \iint_\Omega (\alpha[U_x^2 + V_y^2] + \beta U_x V_y + \gamma[U_y^2 + 2U_y V_x + V_x^2])dxdy - \iint_{\partial\Omega_T} (U\,\overline{T}_x + V\,\overline{T}_y)ds, \tag{2.3}$$

over $H^1_U(\Omega) \times H^1_V(\Omega)$, where the subscripts x and y in the integrand denote derivatives.

$$H^1_*(\Omega) \equiv \{v : v \in H^1(\Omega) \text{ and } v \text{ satisfies essential boundary conditions appropriate to } *\}, \tag{2.4}$$

$\partial\Omega_T$ is the part of the boundary over which the surface tractions $\overline{T}_x$ and $\overline{T}_y$ are applied and the constants α and β are given by

$$\alpha \equiv \frac{E(1-\nu)}{2(1+\nu)(1-2\nu)}, \quad \beta = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \gamma = \frac{E}{4(1+\nu)},$$

with E the Young's modulus and ν the Poisson's ratio. The symmetry of the problem about AD, Fig. 1, is exploited, so that in (2.3) and (2.4) Ω is taken as $\overline{\text{OABCDO}}$, and the essential boundary condition $V = 0$ is imposed on OD. For this problem in the upper rectangle all surface tractions used are zero with the exception of $\overline{T}_y = \lambda$ on BC.

In applying the finite element method to (2.3) we again take a partition of rectangular elements and approximate U and V with trial functions, respectively from H^1_U and H^1_V, of the usual piecewise-bilinear/$C^0(\Omega)$ type which satisfy the required essential boundary conditions.

2.3. *Assumed Stress Hybrid Formulation*

This formulation was originally developed by Pian et al. [5]. The starting point is the principle of *minimum complementary energy,* in which the functional to be minimized is

$$I_3(\sigma) = \frac{1}{2}\iint_\Omega C_{ijkl}\sigma_{ij}\sigma_{kl}\,dxdy - \int_{\partial\Omega_u} (\overline{U}\,T_x + \overline{V}\,T_y)\,ds \tag{2.5}$$

where the σ_{ij} are the components of the stress tensor, T_x and T_y are the corresponding tractions, and $\partial\Omega_u$ is the portion of $\partial\Omega$ over which the displacements $\overline{U}$ and $\overline{V}$ are described. In the stress hybrid scheme the domain Ω is divided into elements, and the conforming condition that the tractions $T = \sigma.n$ be continuous over the interelement boundaries is replaced by the introduction of Lagrange multiplier terms. The multipliers turn out to be the corresponding displacement components. Thus independent trial functions may be assumed in each element for the stresses σ_{ij} and for the displacements $\tilde{U}$ and $\tilde{V}$ on the element boundaries. The energy functional may be written as a sum of integrals over each element $\sum_e I^e_3(\sigma,\tilde{U},\tilde{V})$, where

$$I^e_3(\sigma,\tilde{U},\tilde{V}) = \frac{1}{E}\iint_{\Omega_e}[\frac{1-\nu^2}{2}(\sigma^2_{xx}+\sigma^2_{yy}) + (1+\nu)(\sigma^2_{xy} - \nu\sigma_{xx}\sigma_{yy})]dxdy$$
$$- \int_{\partial\Omega_e}(\tilde{U}T_x+\tilde{V}T_y)\,ds + \int_{(\partial\Omega_T)_e}(\tilde{U}\overline{T}_x+\tilde{V}\overline{T}_y)\,ds$$

in which T_x and T_y are the tractions corresponding to the assumed stresses σ, and $\overline{T}_x$, $\overline{T}_y$ are the prescribed tractions over

$(\partial\Omega_T)_e$, the part of $\partial\Omega_T$ lying on $\partial\Omega_e$. The finite element solution is obtained by assuming in each element polynomial approximations for the stresses σ_{ij} and for the displacements $\tilde{U}$, $\tilde{V}$. Specifically, we take $(\sigma_{xx},\sigma_{xy},\sigma_{yy})^T = \mathbf{S}(\beta_1,\beta_2,\ldots,\beta_{n_\beta})^T$, where the elements of the matrix $\mathbf{S}$ are polynomials, and $(\beta_1,\ldots,\beta_{n_\beta})^T$ is a vector of n_β coefficients. Similarly, we take $(U,V)^T = \mathbf{L}(q_1,q_2,\ldots,q_{n_Q})^T$, where $\mathbf{L}$ is a matrix of interpolation polynomials, and $(q_1,\ldots,q_{n_Q})^T$ is the vector of the n_Q nodal values specifying the displacements along $\partial\Omega_e$. Thus we obtain for each element that

$$I_3^e(\sigma,\tilde{U},\tilde{V}) = \tfrac{1}{2}\beta^T\mathbf{H}\beta - \beta^T\mathbf{G}\mathbf{q} + \mathbf{q}^T\mathbf{Q}^e,$$

where

$$\tfrac{1}{2}H_{ij} = \frac{1}{E}\iint_{\Omega_e}\left[\frac{1-\nu^2}{2}(S_{1i}S_{1j}+S_{2i}S_{2j}) + (1+\nu)(S_{3i}S_{3j}-\nu S_{1i}S_{2j}\right)\,dxdy$$

with $\mathbf{G} = \int_{\partial\Omega_e} \mathbf{R}^T\mathbf{L}dl$ and $\mathbf{Q}^e = \int_{(\partial\Omega_T)_e} \mathbf{L}^T\bar{\mathbf{T}}dl$.

As the conforming condition has been replaced by Lagrange multiplier terms, the assumed stresses in each element are completely independent of one another. Hence we may make each $I_3^e(\sigma,\tilde{U},\tilde{V})$ stationary with respect to its coefficients β_i, $i = 1,\ldots,n_\beta$. The resulting equations will be $\mathbf{H}\beta = \mathbf{G}\mathbf{q}$ so that we may substitute for $\mathbf{H}$ in the expression for $-I_3^e$, obtaining $-I^e(\sigma,\tilde{U},\tilde{V}) = \frac{1}{2}\mathbf{q}^T\mathbf{k}^e\mathbf{q}-\mathbf{q}^T\mathbf{Q}^e$, where $\mathbf{k}^e = \mathbf{G}^T\mathbf{H}^{-1}\mathbf{G}$ is the element stiffness matrix. We note that I_3^e is now expressed entirely in terms of the nodal values for the displacements, in analogy to the displacement formulation. We can now assemble, and make $I(\sigma,\tilde{U},\tilde{V})$ stationary with respect to the $\{q_i\}$, resulting in the usual set of equations $\mathbf{K}\mathbf{q} = \mathbf{Q}$.

In choosing the particular approximations to be used for σ, $\tilde{U}$ and $\tilde{V}$, there is a restriction that $n_\beta \geq n_Q-\ell$, where ℓ is the number of rigid-body degrees of freedom ($\ell = 3$ in two dimensions). Computations have been carried out for the following two cases:

(i) σ_{ij} bilinear ($n_\beta = 7$) in Ω_e, $\tilde{U}$ and $\tilde{V}$ linear over the $\partial\Omega_e$ ($n_Q = 7$),

(ii) σ_{ij} biquartic ($n_\beta = 25$) and $\tilde{U}$, $\tilde{V}$ Lagrange cubic ($n_Q = 24$).

3. Local Mesh Refinement

In order to deal effectively with the singularity at the crack tip 0 of the problem of Section 2 we have applied the technique of local mesh refinement with rectangular elements, proposed by Gregory, Fishelov, Schiff and Whiteman [2] and described in WA Section 4.2.This produces in the mesh *transitional* *elements*, each containing five nodal points, see WA Fig. 2. For the Airy stress function formulation of Section 2.1 the C^1 form of the local trial function [2,eq.6] is used in these elements, thus giving 20 degrees of freedom in the element. The corresponding C^0 form of [2,eq.6] has been used in these elements for the displacement formulation of Section 2.2. For the stress hybrid formulation of Section 2.3 the stresses in a transitional element are treated as for a standard four node element, whilst the displacements over the edge containing the extra nodal point are taken to be linear between each pair of nodal points.

4. Numerical Results

All the computations have been carried out with $BC = 2a = 0.8$ and $AB = 0.7$. The load λ is taken to be unity and the partition consists of elements of size 0.1×0.1 (x- and y-directions) for the Airy stress function calculations and 0.1×0.0875 for all others.

Table 1 lists results obtained using the Airy stress function formulation (2.1) and (2.2). The approximations to the stress function u, obtained using various levels of local mesh refinement, are given at representative points in the domain. The last column of the Table contains results obtained by Whiteman, [6], using a collocation technique involving the fitting of a truncated form of the series (1.1) to the boundary data at a large number of points on AB, BC and CD and solving the resulting overdetermined linear system for the values $\{a_j\}$ with linear

TABLE 1

Values of Airy Stress Function (Multiplied by 10^4*)*

Position		From Airy Stress Function Formulation Using Hermite Bicubic Elements — Levels of Refinement				From Collocation Using L.P,[6]
x	y	0	4	7	14	
-0.1	0.1	19.0	20.1	20.2	20.2	20.2
0	0.1	133.1	145.7	146.4	146.5	146.6
0.1	0.1	586.8	615.5	617.1	617.3	617.3
0.1	0	470.4	505.7	507.8	508.1	508.1
-0.2	0.2	23.0	23.8	23.8	23.8	23.9
0	0.3	495.8	503.1	503.5	503.5	503.7
0.2	0.2	1501.2	1514.6	1515.4	1515.5	1515.5
0.3	0.1	2313.7	2321.3	2321.7	2321.8	2321.6
-0.2	0.4	122.4	123.4	123.4	123.4	123.4
0	0.5	720.5	722.4	722.5	722.5	722.8
0.2	0.4	1699.5	1703.0	1703.1	1703.2	1703.2

programming methods. These collocation values are treated as *benchmark results* for the model problem. The agreement between the results of the last two columns of Table 1 is particularly satisfactory in view of the fact that the two methods used for their derivation are completely unrelated.

TABLE 2

Values of $\tilde{U}$ and $\tilde{V}$ obtained with stress hybrid formulation (Bilinear stresses and linear displacements)

Position			Levels of Refinement				
x	y		2	6	10	15	20
-0.05	0	$\tilde{U}$	-0.0321	0.0046	0.0142	0.0169	0.0174
		$\tilde{V}$	0.8488	0.8651	0.8661	0.8662	0.8662
0	0.04375	$\tilde{U}$	0.3391	0.3793	0.3892	0.3919	0.3924
		$\tilde{V}$	0.3643	0.3780	0.3788	0.3788	0.3788
0.05	0	$\tilde{U}$	0.2088	0.2457	0.2553	0.2580	0.2584
-0.2	0.175	$\tilde{U}$	0.9189	0.9633	0.9734	0.9761	0.9766
		$\tilde{V}$	1.8717	1.8912	1.8925	1.8926	1.8926
0	0.35	$\tilde{U}$	1.5529	1.6048	1.6154	1.6181	1.6186
		$\tilde{V}$	1.0264	1.0366	1.0373	1.0373	1.0373
0.2	0.175	$\tilde{U}$	0.7238	0.7678	0.7779	0.7806	0.7811
		$\tilde{V}$	0.1241	0.1250	0.1251	0.1251	0.1251

The values of $\tilde{U}$ and $\tilde{V}$ obtained with the stress hybrid method and local mesh refinement using bilinear and linear trial functions respectively for the stresses and displacements are given in Table 2, where we have used values $E = 5/4$ and $\nu = 1/4$. The convergence of the results with successive levels of refinement, although satisfactory, is not as rapid as in the Airy stress function formulation. However, less computation is required because there are now only two degrees of freedom per node $(\tilde{U},\tilde{V})$ instead of four (u, u_x, u_y, u_{xy}).

The use of local mesh refinement in the finite element solution of (2.2), (2.3) and (2.4) does not produce approximations to any of the coefficients $\{a_i\}$ in the series of (1.1). In order to calculate such approximations we perform least squares fittings of a truncated form of (1.1) to the calculated values of $r^{-\frac{3}{2}}u$ and of modified truncated forms of (1.1) to $\tilde{U}$ and $\tilde{V}$. In the fitting to the values of the Airy stress function the factor $r^{-\frac{3}{2}}$ is introduced in order to compensate for the smallness of u near 0. Table 3 lists values so obtained together with values from [6]. Convergence is again more rapid with the Airy stress function approach.

Table 4 outlines the situation for the first seven coefficients, $a_1, \ldots, a_7$, which includes results from the stress

TABLE 3

Values of a_j obtained by stress hybrid and Airy stress function formulations

	Stress Hybrid Values obtained from Displacement $\tilde{U}$			Stress Hybrid Values obtained from Displacement $\tilde{V}$			Airy Stress Function Formulation			Collocation, [6].
	Levels of Refinement			Levels of Refinement			Levels of Refinement			
	2	10	20	2	10	20	0	7	14	
a_1	-0.990	-1.234	-1.250	-1.210	-1.254	-1.255	-1.1707	-1.2642	-1.2649	-1.2651
a_2	0.056	-0.114	-0.125	-0.079	0.028	0.028	-0.1093	-0.0946	-0.0945	-0.0944
a_3	-1.495	-0.848	-0.802	-1.076	-0.803	-0.801	-0.6004	-0.9329	-0.9354	-0.9361
a_4	-0.102	-0.015	-0.010	-0.065	-0.418	-0.420	-0.0456	-0.1004	-0.1007	-0.1004
a_5	-0.031	0.734	0.788	0.950	0.532	0.530	0.3954	0.7983	0.8013	0.7985
a_6	-0.440	0.583	0.657	0.591	0.587	0.587	0.2513	0.4668	0.4684	0.4603
a_7	-1.884	-1.210	-1.164	-0.196	-0.917	-0.920	-0.0107	-0.9962	-1.0040	-0.9961

TABLE 4

Coefficients a_j *obtained by the various methods*

	Displacement Formulation (Bilinear) with Mesh Refinement		Stress Hybrid Formulation (Linear) with Mesh Refinement		Stress Hybrid Formulation (Cubic Displacement Biquartic Stresses)		Airy Stress Formulation with Mesh Refinement
	U	V	$\tilde{U}$	$\tilde{V}$	$\tilde{U}$	$\tilde{V}$	
a_1	-1.170	-1.212	-1.250	-1.255	-1.308	-1.309	-1.265
a_2	-0.089	-0.051	-0.125	0.028	-0.103	-0.089	-0.095
a_3	-0.885	-0.886	-0.802	-0.801	-0.933	-0.950	-0.935
a_4	0.029	-0.175	-0.010	-0.420	-0.096	-0.141	-0.101
a_5	0.590	0.640	0.788	0.530	0.809	0.907	0.801
a_6	0.652	0.568	0.657	0.587	0.429	0.418	0.468
a_7	-2.118	-1.040	-1.164	-0.920	-0.893	-0.894	-1.004

TABLE 5

Final values for the coefficient a_1

Method		a_1
Displacement: Bilinear Displacements with Refinement	U	-1.1699
	V	-1.2122
Stress Hybrid: Linear Displacements and Bilinear Stresses	$\tilde{U}$	-0.9897
	$\tilde{V}$	-1.2095
Linear Displacements and Bilinear Stresses with Refinement	$\tilde{U}$	-1.2502
	$\tilde{V}$	-1.2545
Cubic Displacements, Biquartic Stresses	$\tilde{U}$	-1.3080
	$\tilde{V}$	-1.3091
Airy Stress Function: Hermite Bicubic		-1.1707
Hermite Bicubic with Refinement		-1.2649
		-1.2651
Collocation, [6]		

hybrid scheme using biquartic trial functions for the stresses and Lagrange cubics for the displacements, but without mesh refinement. It will be noted that these results give better approximations to a_j, $j > 1$ than those obtained using bilinear/linear trial functions, plus mesh refinement. Thus, since we are primarly interested in the stress intensity factor $K \equiv (-\sqrt{2\pi}a_1)$

it seems that the bilinear/linear technique produces the best approximation. The results obtained from the displacement formulation converge less rapidly than the corresponding results for the stress hybrid case. The results for the coefficient a_1 are summarised in Table 5.

From the above we conclude that the results obtained for the stress intensity factor are sufficiently accurate for all practical purposes. The Airy stress function formulation leads to results of higher accuracy, but at the cost of a larger computing time on account of the C^1 conforming condition, which necessitates the use of piecewise bicubic trial functions with the resulting large matrices. The excellent agreement of the Airy stress function results with the bench mark solutions [6] suggests the definitive value of -1.265 for a_1 in the model problem.

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