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A FIXED GRID FOR VORTEX METHODS[1]

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A FIXED GRID FOR VORTEX METHODS

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Abstract. The purpose of this paper is to suggest a fixed grid for vortex methods. Originally, vortex methods are grid-free methods, for which the initial uniform grid is moving in time with the particles. It was observed in numerical experiments that there was a deterioration in the accuracy of the vortex method as time progressed. This was understood as lose of accuracy due to the distortion of the initial grid. We suggest a fixed grid calculation for vortex methods to overcome this difficulty. We give error estimates and prove stability for the linearized Euler and Navier-Stokes equations.

Key words: Vortex Methods, Moving and fixed grids, Spatial differentiation.

AMS(MOS) Subject Classifications: 65D25, 76C05, 76D05.

1. Introduction

Vortex methods are numerical methods for the simulation of the incompressible Euler and Navier-Stokes equations. In vortex methods, one uses the vorticity formulation of the Euler and the Navier-Stokes equations. This representation has the advantage that some physical phenomena, such as turbulence [8], [9], are better understood by the realization of the evolution of the vorticity. Another feature of vortex methods, as suggested by Rosenhead [21], is that for vortex methods we follow particle trajectories, along which the evolution of the vorticity is tracked.

Chorin [7] introduced the blob-vortex method, for which the kernel, which connects velocity and vorticity for incompressible flows, was smoothed. This was done by convolving this singular kernel against a cutoff function, which approximated a delta function in the sense of moments. Stability and convergence of vortex methods was first proved by Hald [17], and further improved by Beale and Majda [2], [3], and Hald [16], which introduced high order cutoff functions for vortex methods.

It was observed numerically [4], that the formal accuracy of vortex methods is lost as time progresses. Beale and Majda suggested, therefore, the rezoning for vortex methods. Every several time steps the vorticity is interpolated into a uniform grid. For this purpose, they used the natural continuous representation of the vorticity, which is given by the discrete convolution of the vorticity against the cutoff function. As was suggested in [4], the rezoning should not be done too often in time, as it introduces numerical viscosity to the vortex scheme.

To avoid the deterioration in the accuracy of vortex schemes, we suggest to approximate the vorticity on a non-moving, for example uniform, grid. Since the grid is fixed, there is no need to interpolate the vorticity into a uniform grid as time progresses. For the fixed grid one has to to approximate the convective term. The idea is to convolve the vorticity with a cutoff function, and then explicitly differentiate the cutoff function to approximate the first order derivatives in the convective term. In fact, other numerical methods, such as spectral methods, can be represented in a similar way [15]. We prove stability for the linearized Euler equations and give error estimates. It is sufficient to require that the Fourier transform of the cutoff function is real to ensure the stability of the scheme. We also prove the consistency of this scheme and give error estimates. The discretization error is determined by the order of the cutoff function. One may choose the cutoff function, such that arbitrary order of convergence is obtained.

For the Navier-Stokes equations, we have to approximate the viscous term as well. The latter is approximated by an explicit differentiation of the cutoff function, as suggested in [14], this time on a fixed grid. In more detail, we approximate the vorticity by convolving it with a cutoff function, and then approximate the Laplacian of the vorticity by explicit calculation of the Laplacian of the cutoff function. Therefore, all the spatial derivatives involved in the Navier-Stokes equation are approximated in the same manner. We prove the stability for the linearized Navier-Stokes equations in case the Fourier transform of the cutoff function is non-negative (see [14]). We prove the consistency of this scheme for the linearized Navier-Stokes equations and give error estimates.

We applied the scheme to the Euler equations with radially symmetric initial conditions, and for the Navier-Stokes equations with periodic initial conditions. For both problems the analytical solution is known. The numerical results demonstrate the accuracy of the scheme, and in most cases the error from the fixed grid is smaller compared to the one from the moving grid.

The paper is organized as follows. In section 2 the new scheme for the Euler equations is represented and in section 3 and 4 we prove the stability the consistency of the scheme for the linearized equations. In section 5 we represent the new scheme for the Navier-Stokes equation and in section 6 and 7 we prove stability and consistency for the linearized Navier-Stokes equations. Numerical results are represented in section 8.

2. A Fixed Grid for Euler Equations

The object of this paper is to construct a high-order numerical approximation for the Euler and the Navier-Stokes equations, using a vortex method. The Euler equations, formulated for the vorticity ξ are given below.

$$\begin{aligned} &\partial_t \xi + (\mathbf{u} \cdot \nabla)\xi - (\xi \cdot \nabla)\mathbf{u} = 0, \\ &\operatorname{div} \mathbf{u} = 0, \end{aligned} \tag{2.1}$$

where $\xi = \operatorname{curl} \mathbf{u}$, $\mathbf{u} = (u, v, w)$ is the velocity vector.

We first describe the convensional formulation of vortex methods, in which the grid is moving with the particles. We follow the characteristic lines $\frac{d\mathbf{x}}{dt} = \mathbf{u}$, along which the vorticity evolution is given by $\frac{d\xi}{dt} = (\xi \cdot \nabla)\mathbf{u}$ (see (2.1)). Note that in the right hand side of the last equation we have the stretching term, which vanishes in a two-dimensional problem. In addition, the following relation between velocity and vorticity holds for incompressible flows [10].

$$\mathbf{u}(\mathbf{x}, t) = \int K(\mathbf{x} - \mathbf{x}')\xi(\mathbf{x}', t)d\mathbf{x}'. \tag{2.2}$$

Here, $K = (-y, x)/2\pi r^2$ in two dimensions, and in three dimensions K is a matrix, which is singular at the origin. See, for example, [1],[13] for the definition of K in three-dimensions. Upon replacing $\mathbf{u}$ by the convolution of K with ξ, one finds

$$\frac{d\mathbf{x}}{dt} = \int K(\mathbf{x} - \mathbf{x}')\xi(\mathbf{x}', t)d\mathbf{x}', \tag{2.3}$$

$$\frac{d\xi}{dt} = \xi \cdot \nabla \int K(\mathbf{x} - \mathbf{x}')\xi(\mathbf{x}', t)d\mathbf{x}'. \tag{2.4}$$

Equations (2.3)-(2.4) are a set of ordinary differential equations for the location of the particles $\mathbf{x}$ and the vorticity ξ. We set an initial uniform grid $\mathbf{x}_j(0), j = 1, ..., n$ with spacing h_1, h_2, h_3 for a three-dimensional problem and h_1, h_2 for a two-dimensional one. For simlicity, we assume $h_1 = h_2 = h_3 = h$. Let $\mathbf{x}_j^h(t), \xi_j^h(t)$ be the approximate particle locations and the approximate vorticity respectively at time t. Equation (2.3) is discretized by (see [6],[7])

$$\frac{d\mathbf{x}_i^h(t)}{dt} = \sum_{j=1}^{n} K_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\xi_j^h(t)h^N , \tag{2.5}$$

where $N = 2, 3$ is the dimension of the problem. Here we approximate the singular kernel $K(\mathbf{x})$ by a smoothed one $K_\delta(\mathbf{x})$, where $K_\delta = \phi_\delta * K$ and $\phi_\delta(\mathbf{x}) = \frac{1}{\delta^N}\phi(\mathbf{x}/\delta)$. The function $\phi(\mathbf{x})$ is called a cutoff function. We also have to discretize spatial derivatives which appear in $\xi \cdot \nabla \mathbf{u}$, which is the stretching term. For a three-dimensional problem we approximate the stretching term by an explicit differentiation of the smoothed kernel, as was suggested in [1]. More explicitly, we approximate this term by

$$\xi_i^h(t) \cdot \sum_{j=1}^{n} \nabla_{\mathbf{x}} K_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\xi_j^h(t)h^N , \tag{2.6}$$

where $\nabla_{\mathbf{x}} K_\delta$ is an explicit differentiation of the smoothed kernel in Euleran coordinates. To conclude, the three-dimensional vortex scheme on a moving grid is given by

$$\frac{d\mathbf{x}_i^h(t)}{dt} = \sum_{j=1}^{n} K_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\xi_j^h(t)h^N , \tag{2.7}$$

$$\frac{d\xi_i^h(t)}{dt} = \xi_i^h(t) \cdot \sum_{j=1}^{n} \nabla_{\mathbf{x}} K_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\xi_j^h(t)h^N . \tag{2.8}$$

We now represent the approximation for the non-moving grid. Let $\mathbf{x}_j, j = 1, ..., n$ be unifomly distributed grid points in R^N. The stretching term is approximated as in (2.6), this time on a fixed grid. We describe now our approximation for the convective term $\mathbf{u} \cdot \nabla \xi$. The idea is to approximate the vorticity by convolving it with a cutoff function, i.e., ξ is approximated by $\phi_\delta * \xi$. We then derive an approximation to the gradient of the vorticity by differentiating this convolution, i.e., by $\nabla(\phi_\delta * \xi) = \nabla \phi_\delta * \xi$. Finally, we approximate the integrals involved in the convolution by the trapezoid rule, and obtain

$$\begin{aligned} \frac{\partial \xi_i^h}{\partial t} = &- \sum_{j=1}^{n} K_\delta(\mathbf{x}_i - \mathbf{x}_j)\xi_j^h h^N \cdot \sum_{j=1}^{n} \nabla \phi_\delta(\mathbf{x}_i - \mathbf{x}_j)\xi_j^h h^N \\ &+ \xi_i^h \cdot \sum_{j=1}^{n} \nabla K_\delta(\mathbf{x}_i - \mathbf{x}_j)\xi_j^h h^N . \end{aligned} \tag{2.9}$$

3. Stability for the Linearized Euler Equations

We prove stability for the linearized Euler equations in two dimensions

$$\frac{\partial \xi}{\partial t} = -\mathbf{a} \cdot \nabla \xi, \tag{3.1}$$

where a is a constant vector. In our proof, we consider the continuous representation of the scheme, rather than the discrete one.

$$\frac{\partial \xi^h(\mathbf{x},t)}{\partial t} = -\mathbf{a} \cdot (\nabla \phi_\delta(\mathbf{x}) * \xi^h(\mathbf{x},t)) \tag{3.2}$$

$$\xi^h(\mathbf{x},0) = \xi_0(\mathbf{x}).$$

Let us define the Sobolev spaces

$$W^{m,p} = \{ f, \partial^\alpha f \in L^p(R^n), |\alpha| \le m \}$$

and the norm $\| \cdot \|_{m,p}$

$$\|f\|_{m,p} = \left(\sum_{0 \le |\alpha| \le m} \|\partial^\alpha f\|^p_{0,p} \right)^{1/p}.$$

Stability Theorem 1. Let $\phi \in W^{1,1}(R^2)$ and let the Fourier transform of the cutoff function be real, i.e.,

$$\hat{\phi}(\mathbf{s}) \quad \text{is real} \tag{3.3}$$

then

$$\int (\xi^h(\mathbf{x},t))^2 d\mathbf{x} = \int (\xi^h(\mathbf{x},0))^2 d\mathbf{x}, \tag{3.4}$$

i.e., (3.2) is stable.

Proof. Taking the Fourier transform of (3.2) yields

$$\frac{\partial \hat{\xi}^h}{\partial t}(\mathbf{s},t) = i(\mathbf{a} \cdot \mathbf{s}) \hat{\phi}_\delta(\mathbf{s}) \hat{\xi}^h(\mathbf{s},t).$$

Multiplying the last equality by the complex conjugate of $\hat{\xi^h}(\mathbf{s},t)$, we have

$$\frac{\partial \xi^h(\hat{\mathbf{s}},t)}{\partial t}\bar{\hat{\xi}}^h(\mathbf{s},t) = i(\mathbf{a}\cdot\mathbf{s})\hat{\phi_\delta}(\mathbf{s})|\hat{\xi^h}(\mathbf{s},t)|^2. \tag{3.5}$$

Adding to (3.5) its complex conjugate, we find by (3.3) that

$$\frac{\partial|\hat{\xi^h}(\mathbf{s},t)|^2}{\partial t} = 0.$$

If we integrate the last equality over $\mathbf{s}$, we find by the parseval equality that

$$\frac{\partial}{\partial t}\int (\xi^h(\mathbf{x},t))^2 d\mathbf{x} = 0.$$

Hence (3.4) results.

Remark: All the examples listed in [14], as well as other radially symmetric cutoff, satisfy condition (3.3).

4. Consistency for the Linearized Euler Equations

We prove consistency for the linearized equation (3.1), and give error estimates. We consider the following approximation to (3.1).

$$\frac{\partial \xi^h(\mathbf{x},t)}{\partial t} = -\mathbf{a}\cdot\sum_{j=1}^{n}\nabla\phi_\delta(\mathbf{x}-\mathbf{x}_j)\xi_j^h h^2. \tag{4.1}$$

Consistency Theorem 1. Let ξ be of compact support and belongs to $W^{d+1,2}(R^2)\cap W^{m,2}(R^2)$ for $m>1$, and let the cutoff function ϕ satisfy the following conditions.

$$\phi \in W^{m+1,1}(R^2), m > 1 \tag{4.2}$$

$$\int_{R^2}\phi(\mathbf{x})d\mathbf{x} = 1, \int_{R^2}\mathbf{x}^\alpha\phi(\mathbf{x})d\mathbf{x} = 0, |\alpha| \le d-1, \int_{R^2}|\mathbf{x}|^d\phi(\mathbf{x})d\mathbf{x} < \infty. \tag{4.3}$$

Let $\mathbf{x}_j, j = 1,\ldots,n$ be unifomly distributed grid points in R^2. Then, there exist a constant C such that

$$\|e_t\|_{0,2} = \|\mathbf{a}\cdot\nabla\xi - \mathbf{a}\cdot\sum_{j=1}^{n}\nabla\phi_\delta(\mathbf{x}-\mathbf{x}_j)\xi_j h^2\|_{0,2} \le C(\delta^d + \frac{h^m}{\delta^{m+1}}). \tag{4.4}$$

Proof. We shall write the truncation error in the linearized version (4.1) as the sum of the regularization and the discretization error.

$$e_t = e_r + e_d,$$

where

$$e_r = \mathbf{a} \cdot \nabla \xi - \mathbf{a} \cdot (\nabla \phi_\delta * \xi),$$

$$e_d = \mathbf{a} \cdot (\nabla \phi_\delta * \xi) - \mathbf{a} \cdot \sum_{j=1}^{n} \nabla \phi_\delta (\mathbf{x}_i - \mathbf{x}_j) \xi_j^h(t) h^2.$$

We approximate the regularization error by expanding its Fourier transform in Taylor series ([1], [20, pp. 267]). This yields

$$\|e_r\|_{0,2} = \|\mathbf{a} \cdot \nabla \xi - \mathbf{a} \cdot (\nabla \phi_\delta * \xi)\|_{0,2} = \|\mathbf{a} \cdot (\nabla \xi - \phi_\delta * \nabla \xi)\|_{0,2}.$$

Therefore, we find that

$$\|e_r\|_{0,2} \leq C \delta^d \|\xi\|_{d+1,2}. \tag{4.5}$$

The discretization error originates from the replacement of the integral involved in the convolution by the trapezoidal rule. It was proven in [20, pp. 262] that if $g \in W^{m,p}(R^2) \cap L^1(R^2)$ for $m \leq 2$ or if $g \in W^{m,2}(R^2) \cap W^{m-1,1}(R^2)$ for $m \geq 3$, then

$$\left| \int g d\mathbf{x} - \sum_{j=1}^{n} g(\mathbf{x}_j) h^2 \right| \leq h^m \|g\|_{m,2}. \tag{4.6}$$

Therefore, since ξ is of compact support and belongs to $W^{m,2}(R^2)$ and $\nabla \phi_\delta \in W^{m,1}(R^2)$,

$$|e_d| \leq h^m \|\mathbf{a} \cdot (\nabla \phi_\delta * \xi)\|_{m,2}.$$

We also apply the inequality

$$\|f * g\|_2 \leq \|f\|_1 \|g\|_2, \tag{4.7}$$

which was proved in [20, pp. 267], and find

$$\|\mathbf{a} \cdot (\nabla \phi_\delta * \xi)\|_{m,2} \leq \|\mathbf{a} \cdot \nabla \phi_\delta\|_{m,1} \|\xi\|_{m,2} \leq \|\phi_\delta\|_{m+1,1} \|\xi\|_{m,2}.$$

Since $\|\phi_\delta\|_{m+1,1} \le C\delta^{-(m+1)}$ (see [20, pp. 275]), we find

$$\|e_d\|_{0,2} \le C\frac{h^m}{\delta^{m+1}}. \tag{4.8}$$

Combining (4.5) and (4.8) yields the desired result.

5. A Fixed Grid for Navier-Stokes Equations

The Navier-Stokes equations, formulated for the vorticity ξ are given below.

$$\partial_t\xi + (\mathbf{u}\cdot\nabla)\xi - (\xi\cdot\nabla)\mathbf{u} = R^{-1}\Delta\xi,$$

$$\operatorname{div}\mathbf{u} = 0,$$

and $\Delta = \nabla^2$ is the Laplace operator. $R = UL/\nu$ is the Reynolds number, where U and L are typical velocity and length, respectively, and ν is the viscosity. We first describe the formulation of vortex methods [14], in which the grid is moving with the particles. We follow the characteristic lines

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}, \tag{5.1}$$

along which the vorticity evolution is given by

$$\frac{d\xi}{dt} = (\xi\cdot\nabla)\mathbf{u} + R^{-1}\Delta\xi. \tag{5.2}$$

In addition, we use the relation (2.2) between velocity and vorticity holds for incompressible flow. We get the following system of ordinary differential equations.

$$\frac{d\mathbf{x}}{dt} = \int K(\mathbf{x}-\mathbf{x}')\xi(\mathbf{x}',t)d\mathbf{x}', \tag{5.3}$$

$$\frac{d\xi}{dt} = \xi\cdot\nabla\int K(\mathbf{x}-\mathbf{x}')\xi(\mathbf{x}',t)d\mathbf{x}' + R^{-1}\Delta\xi. \tag{5.4}$$

We set an initial uniform grid $\mathbf{x}_j(0), j = 1, ..., n$ with spacing h_1, h_2, h_3 for a three-dimensional problem and h_1, h_2 for a two-dimensional one. For simlicity, we assume $h_1 = h_2 = h_3 = h$. Let $\mathbf{x}_j^h(t), \xi_j^h(t)$ be the approximate particle locations and approximate vorticity respectively at time t. Hence, the formulation of the vortex method for the Navier-Stokes equations on a moving grid is (see [14])

$$\frac{d\mathbf{x}_i^h(t)}{dt} = \sum_{j=1}^{n} K_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\xi_j^h(t)h^N, \tag{5.5}$$

$$\begin{aligned}\frac{d\xi_i^h(t)}{dt} &= \xi_i^h(t) \cdot \sum_{j=1}^{n} \nabla\mathbf{x}K_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\xi_j^h(t)h^N \\ &+ R^{-1}\sum_{j=1}^{n} \Delta\phi_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\xi_j^h(t)h^N.\end{aligned} \tag{5.6}$$

We now represent the approximation for the non-moving grid. Let $\mathbf{x}_j, j = 1, ..., n$ be unifomly distributed grid points in R^N. The stretching and the conventive terms are approximated as for the Euler equations. We describe the discretization for the viscous term. The idea is similar to that in [14], but this time the convolution involved in the discretization is taken on a fixed grid, rather on a moving one. We approximate the vorticity by convolving it with a cutoff function, and then derive an approximation to the Laplacian of the vorticity by differentiating this convolution, i.e., by $\Delta(\phi_\delta * \xi) = \Delta\phi_\delta * \xi$. Finally, we approximate the integrals involved in the convolution by the trapezoid rule, and obtain

$$\begin{aligned}\frac{\partial \xi_i^h}{\partial t} &= -\sum_{j=1}^{n} K_\delta(\mathbf{x}_i - \mathbf{x}_j)\xi_j^h h^N \cdot \sum_{j=1}^{n} \nabla\phi_\delta(\mathbf{x}_i - \mathbf{x}_j)\xi_j^h h^N \\ &+ \xi_i^h \cdot \sum_{j=1}^{n} \nabla K_\delta(\mathbf{x}_i - \mathbf{x}_j)\xi_j^h h^N + R^{-1}\sum_{j=1}^{n} \Delta\phi_\delta(\mathbf{x}_i - \mathbf{x}_j)\xi_j^h h^N.\end{aligned} \tag{5.7}$$

6. Stability for the linearized Navier-Stokes Equations

We prove stability for the linearized Navier-Stokes equations in two dimensions.

$$\frac{\partial \xi}{\partial t} = -\mathbf{a} \cdot \nabla\xi + R^{-1}\Delta\xi,$$

where a is a constant vector. We consider the continuous representation.

$$\frac{\partial \xi^h(\mathbf{x},t)}{\partial t} = -\mathbf{a} \cdot (\nabla\phi_\delta(\mathbf{x}) * \xi^h(\mathbf{x},t)) + R^{-1}\Delta\phi_\delta(\mathbf{x}) * \xi^h(\mathbf{x},t) \tag{6.1}$$

$$\xi^h(\mathbf{x},0) = \xi_0(\mathbf{x}).$$

Stability Theorem 2. Let $\phi \in W^{2,1}(R^2)$ and let the Fourier transform of the cutoff function be non-negative, i.e.,

$$\hat{\phi}(\mathbf{s}) \geq 0, \tag{6.2}$$

then (6.1) is stable, i.e.,

$$\int (\xi^h(\mathbf{x},t))^2 d\mathbf{x} \leq \int (\xi^h(\mathbf{x},0))^2 d\mathbf{x}. \tag{6.3}$$

Proof. Taking the Fourier transform of (6.1) yields

$$\frac{\partial \hat{\xi^h}}{\partial t}(\mathbf{s},t) = i(\mathbf{a}\cdot\mathbf{s})\hat{\phi_\delta}(\mathbf{s})\hat{\xi^h}(\mathbf{s},t) - R^{-1}(\mathbf{s}\cdot\mathbf{s})\hat{\phi_\delta}(\mathbf{s})\hat{\xi^h}(\mathbf{s},t).$$

Multiplying the last equality by the complex conjugate of $\hat{\xi^h}(\mathbf{s},t)$, we have

$$\frac{\partial \hat{\xi^h}(\mathbf{s},t)}{\partial t}\bar{\hat{\xi^h}}(\mathbf{s},t) = i(\mathbf{a}\cdot\mathbf{s})\hat{\phi_\delta}(\mathbf{s})|\hat{\xi^h}(\mathbf{s},t)|^2 - R^{-1}(\mathbf{s}\cdot\mathbf{s})\hat{\phi}(\delta\mathbf{s})|\hat{\xi^h}(\mathbf{s},t)|^2 \tag{6.4}$$

Adding to (6.4) its complex conjugate, we find that

$$\frac{\partial |\hat{\xi^h}(\mathbf{s},t)|^2}{\partial t} = -2R^{-1}(\mathbf{s}\cdot\mathbf{s})\hat{\phi}(\delta\mathbf{s})|\hat{\xi^h}(\mathbf{s},t)|^2$$

If we integrate the last equality over $\mathbf{s}$ and use condition (6.2) and the parseval equality, we find

$$\frac{\partial}{\partial t}\int |\xi^h(\mathbf{x},t)|^2 d\mathbf{x} \leq 0.$$

Hence, (6.3) results.

Remark: All the examples listed in [14] satisfy condition (6.2).

7. Consistency for Navier-Stokes Equations

We prove consistency for the linearized scheme.

$$\frac{\partial \xi^h(\mathbf{x},t)}{\partial t} = -\mathbf{a}\cdot\sum_{j=1}^{n}\nabla\phi_\delta(\mathbf{x}-\mathbf{x}_j)\xi_j^h h^2 + R^{-1}\sum_{j=1}^{n}\Delta\phi_\delta(\mathbf{x}-\mathbf{x}_j)\xi_j^h h^2.$$

Consistency Theorem 2. Let ξ be of compact support and belongs to $W^{d+2,2}(R^2) \cap W^{m,2}(R^2)$ for $m > 1$, and let the cutoff function ϕ be in $W^{m+2,1}(R^2), m > 1$, and satisfy condition (4.3). Let $\mathbf{x}_j, j = 1, ..., n$ be unifomly distributed grid points in R^2. Then, there exist a constant C such that

$$\|e_t\|_{0,2} = \| -\mathbf{a} \cdot \nabla \xi + \mathbf{a} \cdot \sum_{j=1}^{n} \nabla \phi_\delta(\mathbf{x} - \mathbf{x}_j)\xi_j h^2 + R^{-1}[\Delta \xi - \sum_{j=1}^{n} \Delta \phi_\delta(\mathbf{x} - \mathbf{x}_j)\xi_j h^2]\|_{0,2}$$

$$\leq C[\delta^d + \frac{h^m}{\delta^{m+1}} + R^{-1}(\delta^d + \frac{h^m}{\delta^{m+2}})].$$

Proof. We shall write the truncation error in the two-dimensional linearized version of (5.7) as the sum of the convective error and the viscous error.

$$e_t = e_c + e_v,$$

where

$$e_c = \mathbf{a} \cdot \nabla \xi - \mathbf{a} \cdot \sum_{j=1}^{n} \nabla \phi_\delta(\mathbf{x} - \mathbf{x}_j)\xi_j h^2,$$

$$e_v = R^{-1}[\Delta \xi - \sum_{j=1}^{n} \Delta \phi_\delta(\mathbf{x} - \mathbf{x}_j)\xi_j h^2)].$$

The error from the convective term e_c is bounded by (4.4). We shall bound now the error e_v from the viscous term. By composing the viscous error to the regularization error and discretization error, we find

$$e_v = e_r + e_d,$$

where

$$e_r = R^{-1}[\Delta \xi - \Delta \phi_\delta * \xi],$$

$$e_d = R^{-1}[\Delta \phi_\delta * \xi - \sum_{j=1}^{n} \Delta \phi_\delta(\mathbf{x} - \mathbf{x}_j)\xi_j h^N].$$

We rewrite the regularization error in the form

$$R\|e_r\|_{0,2} = \|\Delta \xi - \Delta \phi_\delta * \xi\|_{0,2} = \|\Delta \xi - \phi_\delta * \Delta \xi\|_{0,2},$$

and expand it in Taylor series ([1], [20, pp. 267]). This yields

$$R\|e_r\|_{0,2} \leq C\delta^d \|\Delta\xi\|_{d,2}.$$

Hence,

$$R\|e_r\|_{0,2} \leq C\delta^d \|\xi\|_{d+2,2}. \tag{7.3}$$

The discretization error originates from the replacement of the integral involved in the covolution by the trapezoidal rule. By (4.6), since ξ is of compact support and belongs to $W^{m,2}(R^2)$ and $\Delta\phi_\delta \in W^{m,1}(R^2)$,

$$R|e_d| \leq h^m \|\Delta\phi_\delta * \xi\|_{m,2}.$$

We also apply the inequality (4.7), and find

$$\|\Delta\phi_\delta * \xi\|_{m,2} \leq \|\Delta\phi_\delta\|_{m,1}\|\xi\|_{m,2} \leq \|\phi_\delta\|_{m+2,1}\|\xi\|_{m,2}.$$

Since $\|\phi_\delta\|_{m+2,1} \leq C\delta^{-(m+2)}$ (see [20, pp. 275]), we find

$$R\|e_d\|_{0,2} \leq C\frac{h^m}{\delta^{m+2}}. \tag{7.4}$$

Combining (7.3) and (7.4) yields

$$\|e_v\|_{0,2} \leq CR^{-1}(\delta^d + \frac{h^m}{\delta^{m+2}}). \tag{7.5}$$

Finally, by (4.4) and (7.5) we get the desired result.

8. Numerical Results

We show numerical results for two test problems. The first is the two-dimensional Euler equations for a radially symmetric flow. The second is the two-dimensional Navier-Stokes equation for a periodic flow. For both problems we used the fourth-order cutoff of Beale and Majda [4]

$$\phi(r) = \frac{1}{2\pi}[4e^{-r^2} - e^{-r^2/2}]. \tag{8.1}$$

In this case (see [14], [22, pp. 393])

$$\hat{\phi}(s) = 2e^{-s^2/4} - e^{-s^2/2} = e^{-s^2/4}[2 - e^{-s^2/4}] \geq 0.$$

Therefore, the Fourier transform of the cutoff function is real and non-negative, hence it satisfies conditions (3.3) and (6.2).

The first test problem is the Euler equations with radially symmetric initial conditions. The set of radially symmetric problems is often used in vortex methods to check the accuracy of the vortex schemes. As was pointed in [4], this set might not be general enough to represent arbitrary situations for the Euler equations. We chose the initial conditions to be

$$\xi(\mathbf{x}, 0) = \begin{cases} (1 - |\mathbf{x}|^2)^3 & 0 \leq |\mathbf{x}| \leq 1 \\ 0 & |\mathbf{x}| \geq 1. \end{cases}$$

This problem was solved numerically by Beale and Majda [4], by Nordmark [18], and Perlman [19]. In all those tests, lose of the high accuracy, that was expected from the vortex schemes, was detected.

We represent numerical results for the fixed grid scheme (2.9) and compare them with the moving-grid results (2.7)-(2.8). One has to compute the gradient of the cutoff function in (2.9). This is given by

$$\nabla \phi_\delta = \frac{1}{2\pi\delta^4}(x, y)(4e^{-r^2/\delta^2} - e^{-r^2/2\delta^2}).$$

In table 1 the discrete L_2 error $e(t)$ is shown for different time-levels, where

$$\|e\|_2^2 = \frac{1}{n}\sum_{j=1}^{n} |\xi_{exact} - \xi_{comput}|^2.$$

For both schemes we used initial spacing between the particles $h = h_1 = h_2 = 0.1$. We chose the cutoff parameter δ to be $\sqrt{h}$, since it was observed numerically [18], [14], that in this way, the accuracy of the scheme is kept for a longer time. We stepped the equation in time via the second-order Modified Euler scheme [11],[12], for which the time step was chosen as $\Delta t = 1, 0.5, 0.1$ in Tables 1, 2 and 3 respectively.

time	moving grid	fixed grid
t= 5	0.6167E-3	0.2698E-5
t=10	0.1220E-2	0.5294E-5
t=15	0.1819E-2	0.7703E-5
t=20	0.2446E-2	0.9875E-5

Table 1. $\Delta t = 1$

time	moving grid	fixed grid
t= 5	0.8166E-4	0.2696E-5
t=10	0.1631E-3	0.5289E-5
t=15	0.2682E-3	0.7928E-5
t=20	0.5774E-3	0.9871E-5

Table 2. $\Delta t = 0.5$

time	moving grid	fixed grid
t= 5	0.8822E-6	0.2690E-5
t=10	0.2145E-5	0.5278E-5
t=15	0.1008E-3	0.7685E-5
t=20	0.4694E-3	0.9848E-5

Table 3. $\Delta t = 0.1$

One can learn from Tables 1-3 that the fixed-grid results are more accurate than the moving-grid ones, for this test problem. Furthermore, one can hardly notice any change in the fixed-grid error as we decrease the time step. Therefore, the error is mainly a spatial one. However, for the moving grid, the errors get smaller when one decreases the time step, but the improved accuracy is not kept for a long time, as is evident from Table 3. For the moving grid we lost three digits in the accuracy of the numerical solution over the total time of integration, whereas for the fixed grid we lost less that one digit throughout the same period of time.

The second problem for which we checked the accuracy of our scheme is a periodic one. This problem served as a test problem for Chorin's finite-difference scheme for the Navier-Stokes equations [5]. The initial vorticity is given by $\xi(x,y,0) = 2cos(x)cos(y)$. We perfomed our computations for $0 \leq x, y \leq 2\pi$. The exact solution for this problem is $\xi(x,y,t) = 2e^{-2t/R}cos(x)cos(y)$. We ran the scheme for $R = 1000$. The Laplacian of the cutoff function is

$$\Delta\phi_\delta(r) = \frac{1}{2\pi\delta^4}[16(\frac{r^2}{\delta^2} - 1)e^{-r^2/\delta^2} + (2 - \frac{r^2}{\delta^2})e^{-r^2/2\delta^2}].$$

The periodic boundary conditions were imposed as follows. For each computational particle we added the contributions of another eight particles, located at $(x \pm 2\pi, y), (x, y \pm 2\pi), (x \pm 2\pi, y \pm 2\pi), (x \pm 2\pi, y \mp 2\pi)$. This is reasonable, since the further are the particles from the computational domain the smaller is their contribution.

We checked the error in the discrete L_2 norm, and chose the initial spacing between the particles to be $h = h_1 = h_2 = 2\pi/16$. It is possible to pick a different cutoff-parameter (δ_1) for smoothing the singular kernel by replacing it with K_δ in (5.5)-(5.6) or (5.7), and a different one (δ_2) for the smoothing of the vorticity by the convolution with a cutoff-function ϕ_δ in (5.5)-(5.6) or (5.7). We chose $\delta_1 = 8\sqrt{h}$, and $\delta_2 = \sqrt{h}$. Choosing larger δ_1 smoothes further the singular kernel K, and therefore decreases the discretization error. On the other hand, it should not be taken too large, since it increases the regularization error. The particular constant C, where $\delta_1 = C\sqrt{h}$, might depend on the differential problem. Tables 4, 5 and 6 refer to $\Delta t = 1, 0.5, 0.1$ respectively. In all tables we give the error when we applied the scheme once for the moving grid (5.5)-(5.6) and once for the fixed grid (5.7).

time	moving grid	fixed grid
t= 5	0.1412E-1	0.6189E-4
t=10	0.2503E-1	0.1227E-3
t=15	0.3524E-1	0.1823E-3
t=20	0.4476E-1	0.2409E-3

Table 4. $\Delta t = 1, h = 2\pi/16, \delta_1 = 8\sqrt{h}$.

time	moving grid	fixed grid
t= 5	0.1240E-1	0.6189E-4
t=10	0.2326E-1	0.1227E-3
t=15	0.3362E-1	0.1823E-3
t=20	0.4349E-1	0.2409E-3

Table 5. $\Delta t = 0.5, h = 2\pi/16, \delta_1 = 8\sqrt{h}$.

time	moving grid	fixed grid
t= 5	0.1220E-1	0.6189E-4
t=10	0.2337E-1	0.1227E-3
t=15	0.3384E-1	0.1824E-3
t=20	0.4377E-1	0.2409E-3

Table 6. $\Delta t = 0.1, h = 2\pi/16, \delta_1 = 8\sqrt{h}$.

One can deduce from Tables 4-6 that the error is smaller for the fixed-grid calculations, for all the given choices of Δt. Note that results on the moving grid [14] were of order 10^{-3} or 10^{-2} for $t = 1$ to $t = 4$, but they increase as time progresses. In this case, as for the non-viscous one, the error for the fixed grid is almost indepent of the time step, and therefore the error is mostly a spatial one.

In Tables 7, 8 and 9 we show similar results as in Tables 4, 5 and 6, except $\delta_1 = 50$. We chose δ_1 to be a large constant, since it keeps the grid less distorted, which improves the accuracy of the moving-grid scheme. Note that the moving-grid errors are smaller in Tables 7-9, compared to those in Tables 4-6, but are not that different for the fixed grid.

time	moving grid	fixed grid
t= 5	0.1308E-2	0.4854E-4
t=10	0.2255E-2	0.9612E-4
t=15	0.3232E-2	0.1427E-3
t=20	0.4183E-2	0.1885E-3

Table 7. $\Delta t = 1, h = 2\pi/16, \delta_1 = 50\sqrt{h}$.

time	moving grid	fixed grid
t= 5	0.1151E-2	0.4854E-4
t=10	0.2185E-2	0.9612E-4
t=15	0.3192E-2	0.1474E-3
t=20	0.4159E-2	0.1885E-3

Table 8. $\Delta t = 0.5, h = 2\pi/16, \delta_1 = 50\sqrt{h}$.

time	moving grid	fixed grid
t= 5	0.1112E-2	0.1952E-4
t=10	0.2177E-2	0.9614E-4
t=15	0.3195E-2	0.1428E-3
t=20	0.4168E-2	0.1885E-3

Table 9. $\Delta t = 0.1, h = 2\pi/16, \delta_1 = 50\sqrt{h}$.

9. Conclusions

The numerical experiments performed here show that it is preferable to use a fixed grid rather than a moving grid for vortex methods. One should note, however, that the test problems that we used might not represent an arbitrary flow, and therefore the fixed-grid schemes should be further checked for more complicated flow problems. It is advisable to use the fixed grid in problems in which the vorticity is smooth and is not concentrated in a small region. For this set of problems spatial derivatives can be accurately resolved on a fixed grid. We found theoretical support for our fixed-grid scheme for the linearized Euler and Navier-Stokes equations. Furthermore, it is easy to construct a highly accurate stable cutoff function, and therefore an accurate stable vortex method on a fixed grid. If we choose a cutoff function whose Fourier transform is real or positive for the Euler or

Navier-Stokes equations respectively, stability is assured. The rate of convergence can be made as high as desired by choosing a high-order cutoff function.

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