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Fourth-order Convergence of a Compact Scheme for the One-dimensional Biharmonic Equation

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Abstract. The convergence of a fourth-order compact scheme to the one-dimensional biharmonic problem is established in the case of general Dirichlet boundary conditions. The compact scheme invokes value of the unknown function as well as Pade approximations of its first-order derivative. Using the Pade approximation allows us to approximate the first-order derivative within fourth-order accuracy. However, although the truncation error of the discrete biharmonic scheme is of fourth-order at interior point, the truncation error drops to first-order at near-boundary points. Nonetheless, we prove that the scheme retains its fourth-order (optimal) accuracy. This is done by a careful inspection of the matrix elements of the discrete biharmonic operator. A number of numerical examples corroborate this effect.

We also present a study of the eigenvalue problem $u_{xxxx} = \nu u$. We compute and display the eigenvalues and the eigenfunctions related to the continuous and the discrete problems. By the positivity of the eigenvalues, one can deduce the stability of of the related time-dependent problem $u_t = -u_{xxxx}$. In addition, we study the eigenvalue problem $u_{xxxx} = \nu u_{xx}$. This is related to the stability of the linear time-dependent equation $u_{xxt} = \nu u_{xxxx}$. Its continuous and discrete eigenvalues and eigenfunction (or eigenvectors) are computed and displayed graphically.

DERIVATION OF THREE-POINT COMPACT OPERATORS

We consider here the one-dimensional biharmonic equation on the interval $[a,b]$. For the simplicity of the presentation, we choose homogeneous boundary conditions. The one-dimensional biharmonic equation is

$$\begin{cases} u^{(4)}(x) = f(x), \;\; a < x < b, \\ u(a) = 0, \; u(b) = 0, \; u'(a) = 0, \; u'(b) = 0. \end{cases} \tag{1}$$

We look for a high-order compact approximation to (1). We lay out a uniform grid $a = x_0 < x_1 < \ldots < x_{N-1} < x_N = b$. Here $x_i = ih$ for $0 \le i \le N$ and $h = (b-a)/N$.

In what follows, we shall use the notion of grid functions. A *grid function* is a function defined on the discrete grid $\{x_i\}_{i=0}^{N}$. We denote grid functions with bold letters such as We have $\mathbf{u} = (\mathbf{u}(x_0), \mathbf{u}(x_1), \cdots, \mathbf{u}(x_{N-1}), \mathbf{u}(x_N))$. In addition, we denote by $u^* = (u(x_0), u(x_1), \cdots, u(x_{N-1}), u(x_N))$ the grid function, which consists of the values of $u(x)$ at grid points.

We denote by l_h^2 the functional space of grid functions. This space is equipped with a scalar product and an associated norm $(\mathbf{u},\mathbf{v})_h = h\sum_{i=0}^{N} \mathbf{u}(x_i)\mathbf{v}(x_i)$, $|\mathbf{u}|_h = (\mathbf{u},\mathbf{u})_h^{1/2}$.

We define the difference operators δ_x, δ_x^2 on grid functions by

$$\delta_x \mathbf{u}_i = \frac{\mathbf{u}_{i+1} - \mathbf{u}_{i-1}}{2h}, \qquad \delta_x^2 \mathbf{u}_i = \frac{\mathbf{u}_{i+1} - 2\mathbf{u}_i + \mathbf{u}_{i-1}}{h^2}, \qquad 1 \le i \le N-1. \tag{2}$$

In these definitions the boundary values $\mathbf{u}_0, \mathbf{u}_N$ are assumed to be known.

Suppose that we are given data u_{i-1}^*, u_i^*, u_{i+1}^* at the grid points x_{i-1}, x_i, x_{i+1}. In addition, we are given some approximations $u_{x,i-1}^*, u_{x,i+1}^*$ for $u'(x_{i-1}), u'(x_{i+1})$. We seek a polynomial of degree 4

$$p(x) = u_i^* + a_1(x - x_i) + a_2(x - x_i)^2 + a_3(x - x_i)^3 + a_4(x - x_i)^4, \tag{3}$$

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which interpolates the data $u^*_{i-1}, u^*_i, u^*_{i+1}, u^*_{x,i-1}, u^*_{x,i+1}$. The coefficients a_1, a_2, a_3, a_4 of the polynomial are

$$\begin{cases} a_1 = \frac{3}{4h}(u^*_{i+1} - u^*_{i-1}) - (\frac{1}{4}u^*_{x,i+1} + \frac{1}{4}u^*_{x,i-1}), \\ a_2 = \frac{1}{h^2}(u^*_{i+1} + u^*_{i-1} - 2u^*_i) - \frac{1}{4h}(u^*_{x,i+1} - u^*_{x,i-1}) = \delta_x^2 u^*_i - \frac{1}{2}(\delta_x u^*_x)_i, \\ a_3 = -\frac{1}{4h^3}(u^*_{i+1} - u^*_{i-1}) + \frac{1}{4h^2}(u^*_{x,i+1} + u^*_{x,i-1}), \\ a_4 = -\frac{1}{2h^4}(u^*_{i+1} + u^*_{i-1} - 2u^*_i) + \frac{1}{4h^3}(u^*_{x,i+1} - u^*_{x,i-1}) = \frac{1}{2h^2}\left((\delta_x u^*_x)_i - \delta_x^2 u^*_i\right). \end{cases} \tag{4}$$

The coefficients above require the data u^*_i and $u^*_{x,i}$. In the case where only the values of u^*_i are given, then $\left\{u^*_{x,i}\right\}_{i=1}^{N-1}$ have to be evaluated in terms of $\{u^*_i\}_{i=0}^N$. Looking at the first equation in (4), we see that a natural candidate for $u^*_{x,i}$ is $u^*_{x,i} = a_1$. This yields

$$\frac{1}{6}u^*_{x,i} + \frac{2}{3}u^*_{x,i} + \frac{1}{6}u^*_{x,i+1} = \delta_x u^*_i. \tag{5}$$

This is by definition the Hermitian derivative. If we introduce the three-point operator σ_x on grid functions by $\sigma_x \mathbf{v}_i = \frac{1}{6}\mathbf{v}_{i-1} + \frac{2}{3}\mathbf{v}_i + \frac{1}{6}\mathbf{v}_{i+1}, \quad 1 \le i \le N-1$, can rewrite (5) as

$$\sigma_x u^*_{x,i} = \delta_x u^*_i, \ \ 1 \le i \le N-1. \tag{6}$$

This suggests that $\delta_x^4 u^*_i$ is an approximation to the fourth-order derivative of u at x_i, namely,

$$\delta_x^4 u^*_i = \frac{12}{h^2}\left((\delta_x u^*_x)_i - \delta_x^2 u^*_i\right). \tag{7}$$

This approximation, called the discrete biharmonic approximation (see also [1, 2]). Note that, in the non-periodic setting, boundary values of u_x should be given in order to compute δ_x^4 at near boundary points x_1, x_{N-1}.

We refer to δ_x^4 as the *discrete biharmonic operator (DBO)*. We define

Definition 1 (Discrete biharmonic operator (DBO)) *Let* $\mathbf{u} \in l_h^2$ *be a given grid function. The* **discrete biharmonic operator** *is defined by*

$$\delta_x^4 \mathbf{u}_i = \frac{12}{h^2}(\delta_x \mathbf{u}_{x,i} - \delta_x^2 \mathbf{u}_i), \ \ 1 \le i \le N-1. \tag{8}$$

Here $\mathbf{u}_x$ *is the Hermitian derivative of* $\mathbf{u}$ *satisfying (6) with given boundary values* $\mathbf{u}_{x,0}$ *and* $\mathbf{u}_{x,N}$.

Using (7) and (5), the solution of (1) may be approximated by the scheme

$$\begin{cases} (a) \qquad \delta_x^4 \mathbf{u}_i = f(x_i) \quad 1 \le i \le N-1, \\ (b) \qquad \frac{1}{6}\mathbf{u}_{x,i-1} + \frac{2}{3}\mathbf{u}_{x,i} + \frac{1}{6}\mathbf{u}_{x,i+1} = \delta_x \mathbf{u}_i, \quad 1 \le i \le N-1, \\ (c) \qquad \mathbf{u}_0 = 0,\ \mathbf{u}_N = 0,\ \mathbf{u}_{x,0} = 0,\ \mathbf{u}_{x,N} = 0. \end{cases} \tag{9}$$

OPTIMAL RATE OF CONVERGENCE OF THE ONE-DIMENSIONAL STEPHENSON SCHEME

In order to prove the fourth-order convergence of the scheme, we invoke the matrix representation fo the discrete biharmonic operator. In [3] we carried out an error analysis based on the coercivity of δ_x^4. The analysis presented there was based on an energy (l^2) method and led to a "sub-optimal" convergence rate of $h^{\frac{3}{2}}$. In [4] we have improved this result by showing that the convergence rate is almost three (the error is bounded by $Ch^3\log(|h|)$. Here we prove the optimal (fourth-order) convergence of the scheme. Let u be the exact solution of (1) and let $\mathbf{u}$ be its approximation by the Stephenson scheme (9). Let u^* be the grid function corresponding to u. We consider the error between the approximated solution $\mathbf{v}$ and the collocated exact solution u^*, $\mathbf{e} = \mathbf{v} - u^*$. We prove the following error estimate.

Theorem 2 *Let u be the exact solution of (1) and assume that u has continuous derivatives up to order eight on* $[a,b]$*. Let* $\mathbf{v}$ *be the approximation to u, given by the Stephenson scheme (9). Let* u^* *be the grid function corresponding to u. The, the error* $\mathbf{e} = \mathbf{v} - u^*$ *satisfies*

$$|\mathbf{e}|_h \leq Ch^4, \tag{10}$$

where C depends only on f*.*

EIGENFUNCTIONS AND EIGENVALUES OF FOURTH-ORDER OPERATORS

Eigenfunction Approach for $u_{xxxx} = \nu u$

Consider the eigenvalue problem for the one-dimensional biharmonic problem.

$$\phi^{(4)} = \lambda\phi, \quad \phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0. \tag{11}$$

It may be easily checked that the eigenfunctions for this problem are

$$\begin{aligned}\phi_k(x) = &(\sin\omega_k - \sinh\omega_k)\big(\cos(\omega_k x) - \cosh(\omega_k x)\big)\\ &-(\cos\omega_k - \cosh\omega_k)\big(\sin(\omega_k x) - \sinh(\omega_k x)\big) \quad x \in [0,1], \quad k = 1,2,...,\end{aligned} \tag{12}$$

and $\lambda_k = \omega_k^4, \quad k = 1,2,...$, where ω_k satisfy

$$\cos(\omega_k)\cosh(\omega_k) = 1, \quad k = 1,2,.., \quad \omega_k > 0. \tag{13}$$

We consider now the discrete problem. Let $\mathbf{v}$ be the solution of the discrete biharmonic problem.

$$\delta_x^4\mathbf{v} = \lambda\mathbf{v}, \quad \mathbf{v}(0) = \mathbf{v}_x(0) = \mathbf{v}(1) = \mathbf{v}_x(1) = 0, \tag{14}$$

where $\mathbf{v}_x$ is the Hermitian derivative of $\mathbf{v}$.

Claim 3 *A complete set of* $(N-1)$ *linearly independent eigenfunctions (in* $l^2_{h,0}$*) for problem (14) is given by*

$$\begin{aligned}\mathbf{v}_i^{(k)} = &\left(\sin(\omega_1^{(k)}) - T_k\sinh(\omega_2^{(k)})\right)\cdot\left(\cos(\omega_1^{(k)}x_i) - \cosh(\omega_2^{(k)}x_i)\right)\\ &-\left(\cos(\omega_1^{(k)}) - \cosh(\omega_2^{(k)})\right)\left(\sin(\omega_1^{(k)}x_i) - T_k\sinh(\omega_2^{(k)}x_i)\right), \quad k = 1,..,N-1,\end{aligned} \tag{15}$$

where $T_k = \frac{\sinh^3(\omega_2^{(k)}h)}{\sin^3(\omega_1^{(k)}h)}$ *and* $\omega_1^{(k)}, \omega_2^{(k)}$ *satisfy the following set of equations.*

$$\frac{\sin^4(\frac{\omega_1^{(k)}h}{2})}{3 - 2\sin^2(\frac{\omega_1^{(k)}h}{2})} = \frac{\sinh^4(\frac{\omega_2^{(k)}h}{2})}{3 + 2\sinh^2(\frac{\omega_2^{(k)}h}{2})}, \tag{16}$$

$$2\cos(\omega_1^{(k)})\cosh(\omega_2^{(k)}) + R_k \quad \sin(\omega_1^{(k)})\sinh(\omega_2^{(k)}) = 2, \quad R_k = T_k - \tfrac{1}{T_k}. \tag{17}$$

The eigenvalues λ_k *are* $\lambda_k = \frac{48}{h^4}\frac{\sin^4(\frac{\omega_1^{(k)}h}{2})}{3-2\sin^2\frac{\omega_1^{(k)}h}{2}}, \quad k = 1,...,N-1.$

Eigenfunction Approach for $u_{xxt} = \nu u_{xxxx}$

We consider a simple linear one-dimensional model for the Navier-Stokes equations in streamfunction formulation

$$u_{xxt} = \nu u_{xxxx}, \tag{18}$$

where $u(x,t)$ is a function defined for $(x,t) \in [0,1] \times [0,\infty)$. We define an eigenfunction $\phi(x)$ to (18) as a function satisfying, for some real number μ (which is the associated eigenvalue),

$$\mu \phi''(x) = \phi^{(4)}(x), \quad x \in [0,1], \tag{19}$$

along with the boundary conditions

$$\phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0. \tag{20}$$

Claim 4 *A complete set of orthogonal eigenfunctions and their associate eigenvalues is given by the union of the following two families*

$$\begin{aligned} &\phi_k^{(1)}(x) = 1 - \cos(2\pi k x), \quad \mu_k^{(1)} = -(2\pi k)^2 \quad k = 1,2,..., \\ &\phi_k^{(2)}(x) = \tfrac{1}{\pi q_k} \sin(2q_k \pi x) - \cos(2q_k \pi x) - 2x + 1, \quad \mu_k^{(2)} = -(2\pi q_k)^2, \\ &\qquad\qquad k = 2,3,..., \end{aligned} \tag{21}$$

where (for the second family), q_k is the (unique) solution of

$$\tan(q_k \pi) = q_k \pi, \quad q_k \in (k-1, k-\frac{1}{2}). \tag{22}$$

In the discrete problem the second-order and fourth-order spatial derivatives are replaced by their three-point counterparts, δ_x^2 and δ_x^4, respectively.We look for the discrete eigenvalues and eigenfunctions, namely, grid functions satisfying

$$\mu \delta_x^2 \mathbf{v} = \delta_x^4 \mathbf{v}, \tag{23}$$

subject to the homogeneous boundary conditions

$$\mathbf{v}(x_0) = \mathbf{v}_x(x_0) = \mathbf{v}(x_N) = \mathbf{v}_x(x_N) = 0. \tag{24}$$

Claim 5 *Suppose that N is even. Then a complete set of $(N-1)$ linearly independent eigenfunctions (in $l_{h,0}^2$) is given by the union of the following two families*

$$\begin{aligned} &\mathbf{v}_k^{(1)}(x_i) = 1 - \cos(2\pi k x_i), \quad 0 \le i \le N, \quad k = 1,2,...,\tfrac{N}{2}, \\ &\mathbf{v}_k^{(2)}(x_i) = A_k \sin(2r_k \pi x_i) - \cos(2r_k \pi x_i) - 2x_i + 1, \quad 0 \le i \le N, \quad k = 2,...,\tfrac{N}{2}, \end{aligned} \tag{25}$$

where (for the second family), A_k, r_k are uniquely determined by the pair of equations

$$\frac{1}{A_k} = \tan(r_k \pi), \quad A_k = \frac{2h}{3} \frac{2 + \cos(2r_k \pi h)}{\sin(2r_k \pi h)}, \quad r_k \in (k-1, k-\frac{1}{2}). \tag{26}$$

Figure 1 displays the eigenvalues of Problem (19) versus the discrete spectrum $N = 64$, using a Log-Log scale.

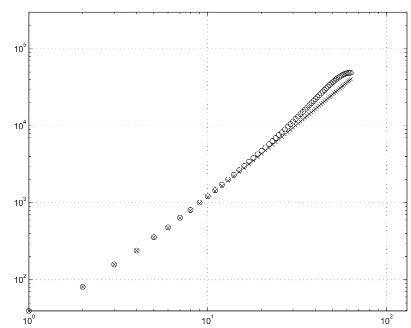

FIGURE 1. Spectral problem (19). Continuous ('x') versus discrete ('o') eigenvalues with $N = 64$ in Log-Log scale.

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