Highly Accurate Discretization of the Navier–Stokes Equations in Streamfunction Formulation

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Dedicated to the memory of Professor David Gottlieb for his Wisdom and Generosity

Abstract A discrete version of the pure streamfunction formulation of the Navier–Stokes equation is presented. The proposed scheme is fourth order in both two and three spatial dimensions.

1 Fourth Order Scheme for the Navier–Stokes Equations in Two Dimensions

We consider the Navier–Stokes equations in pure streamfunction form, which in the two-dimensional case leads to the scalar equation

$$\begin{cases} \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \nu\Delta^2\psi = f(x,y,t), \\ \psi(x,y,t) = \psi_0(x,y). \end{cases} \tag{1}$$

Recall that $\nabla^{\perp}\psi = (-\partial_y\psi, \partial_x\psi)$ is the velocity vector. The no-slip boundary condition associated with this formulation is

$$\psi = \frac{\partial\psi}{\partial n} = 0, \quad (x,y) \in \partial\Omega, \quad t > 0 \tag{2}$$

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and the initial condition is

$$\psi(x, y, 0) = \psi_0(x, y), \quad (x, y) \in \Omega. \tag{3}$$

The spatial derivatives in Equation (1) are discretized as we describe next. The fourth order discrete Laplacian $\tilde{\Delta}_h\psi$ and biharmonic $\tilde{\Delta}_h^2\psi$ operators introduced in [4] are perturbations of the second order operators $\Delta_h\psi = (\delta_x^2 + \delta_y^2)\psi$ and $\Delta_h^2\psi = (\delta_x^4 + \delta_y^4 + 2\delta_x^2\delta_y^2)\psi$. They are designed as follows.

$$\tilde{\Delta}_h\psi_{i,j} = 2\Delta_h\psi_{i,j} - (\delta_x(\psi_x)_{i,j} + \delta_y(\psi_y)_{i,j}) = (\Delta\psi)_{i,j} + O(h^4). \tag{4}$$

Here, ψ_x, ψ_y are the fourth-order Hermitian approximations to $\partial_x\psi$, $\partial_y\psi$ described as

$$\begin{cases} \sigma_x\psi_x = \frac{1}{6}(\psi_x)_{i-1,j} + \frac{2}{3}(\psi_x)_{i,j} + \frac{1}{6}(\psi_x)_{i+1,j} = \delta_x\psi_{i,j}, & 1 \le i, j \le N-1 \\ \sigma_y\psi_y = \frac{1}{6}(\psi_y)_{i,j-1} + \frac{2}{3}(\psi_y)_{i,j} + \frac{1}{6}(\psi_y)_{i,j+1} = \delta_y\psi_{i,j}, & 1 \le i, j \le N-1. \end{cases} \tag{5}$$

We use the standard central difference operators $\delta_x, \delta_y, \delta_x^2, \delta_y^2$.

The fourth-order approximation to the biharmonic operator $\Delta^2\psi$ is

$$\tilde{\Delta}_h^2\psi = \delta_x^4\psi + \delta_y^4\psi + 2\delta_x^2\delta_y^2\psi - \frac{h^2}{6}(\delta_x^4\delta_y^2\psi + \delta_y^4\delta_x^2\psi) = \Delta^2\psi + O(h^4), \tag{6}$$

where δ_x^4 and δ_y^4 are the compact approximations of ∂_x^4 and ∂_y^4, respectively.

$$\delta_x^4\psi_{i,j} = \frac{12}{h^2}\left((\delta_x\psi_x)_{i,j} - \delta_x^2\psi_{i,j}\right), \quad \delta_x^4\psi = \partial_x^4\psi - \frac{1}{720}h^4\partial_x^8\psi + O(h^6), \tag{7}$$

$$\delta_y^4\psi_{i,j} = \frac{12}{h^2}\left((\delta_y\psi_y)_{i,j} - \delta_y^2\psi_{i,j}\right), \quad \delta_y^4\psi = \partial_y^4\psi - \frac{1}{720}h^4\partial_y^8\psi + O(h^6). \tag{8}$$

The convective term in (1) is $C(\psi) = -\partial_y\psi\Delta(\partial_x\psi) + \partial_x\psi\Delta(\partial_y\psi)$. Its fourth-order approximation needs special care. The mixed derivative $\partial_x\partial_y^2\psi$ may be approximated to fourth-order accuracy by $\tilde{\psi}_{yyx}$ using a suitable combination of lower order approximations.

$$\tilde{\psi}_{yyx} = \delta_y^2\psi_x + \delta_x\delta_y^2\psi - \delta_x\delta_y\psi_y = \partial_x\partial_y^2\psi + O(h^4). \tag{9}$$

For the pure third order derivative $\partial_x^3\psi$ we note that if ψ is smooth then

$$\psi_{xxx} = \frac{3}{2h^2}\left(10\delta_x\psi - h^2\delta_x^2\partial_x\psi - 10\partial_x\psi\right)_{i,j} + O(h^4). \tag{10}$$

One needs to approximate $\partial_x \psi$ to sixth-order accuracy in order to obtain from (10) a fourth-order approximation for $\partial_x^3 \psi$. Denoting this approximation by $\tilde{\psi}_x$, we invoke the Pade formulation [5], having the following form.

$$\frac{1}{3}(\tilde{\psi}_x)_{i+1,j} + (\tilde{\psi}_x)_{i,j} + \frac{1}{3}(\tilde{\psi}_x)_{i-1,j} = \frac{14}{9}\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2h} + \frac{1}{9}\frac{\psi_{i+2,j} - \psi_{i-2,j}}{4h}. \tag{11}$$

At near-boundary points we apply a special treatment as in [5]. Carrying out the same procedure for $\partial_y \psi$, which yields the approximate value $\tilde{\psi}_y$, and combining with all other mixed derivatives, a fourth order approximation of the convective term is

$$\begin{aligned} \tilde{C}_h(\psi) &= -\psi_y\big(\Delta_h \tilde{\psi}_x + \frac{5}{2}\big(6\frac{\delta_x \psi - \tilde{\psi}_x}{h^2} - \delta_x^2 \tilde{\psi}_x\big) + \delta_x \delta_y^2 \psi - \delta_x \delta_y \tilde{\psi}_y\big) \quad (12) \\ &+ \psi_x\big(\Delta_h \tilde{\psi}_y + \frac{5}{2}\big(6\frac{\delta_y \psi - \tilde{\psi}_y}{h^2} - \delta_y^2 \tilde{\psi}_y\big) + \delta_y \delta_x^2 \psi - \delta_y \delta_x \tilde{\psi}_x\big) \\ &= C(\psi) + O(h^4). \end{aligned}$$

Our implicit–explicit time-stepping scheme is of the Crank–Nicholson type as follows.

$$\frac{(\tilde{\Delta}_h \psi_{i,j})^{n+1/2} - (\tilde{\Delta}_h \psi_{i,j})^n}{\Delta t/2} = -\tilde{C}_h \psi^{(n)} + \frac{\nu}{2}[\tilde{\Delta}_h^2 \psi_{i,j}^{n+1/2} + \tilde{\Delta}_h^2 \psi_{i,j}^n] \tag{13}$$

$$\frac{(\tilde{\Delta}_h \psi_{i,j})^{n+1} - (\tilde{\Delta}_h \psi_{i,j})^n}{\Delta t} = -\tilde{C}_h \psi^{(n+1/2)} + \frac{\nu}{2}[\tilde{\Delta}_h^2 \psi_{i,j}^{n+1} + \tilde{\Delta}_h^2 \psi_{i,j}^n]. \tag{14}$$

Due to stability reasons we have chosen an Explicit–Implicit time stepping scheme. It is possible however to use an explicit time-stepping scheme if one can afford a small time step in order to advance the solution in time. The set of linear equations is solved via a FFT solver using the Sherman–Morrison formula (see [2]). This solver is of $O(N^2 logN)$ operations, where N is the number of grid points in each spatial direction. For the application of the pure streamfunction formulation on an irregular domain see [3].

2 The Pure Streamfunction Formulation in Three Dimensions

Let Ω be a bounded domain in R^3. The three-dimensional Navier–Stokes equations in vorticity-velocity formulation is

$$\begin{aligned}
&\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) - \nu \Delta \boldsymbol{\omega} = \nabla \times \mathbf{f}, \quad \text{in} \quad \Omega \\
&\boldsymbol{\omega} = \nabla \times \mathbf{u}, \;\; \nabla \cdot \mathbf{u} = 0, \quad \text{in} \quad \Omega \\
&\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \\
&\boldsymbol{\omega}(\mathbf{x}, 0) = \boldsymbol{\omega}_0(\mathbf{x}) := \nabla \times \mathbf{u}_0, \quad \text{in} \quad \Omega.
\end{aligned} \tag{15}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ and the no-slip boundary condition has been imposed. The pure streamfunction formulation for this system is obtained by introducing a streamfunction $\psi(\mathbf{x}, t) \in R^3$, such that

$$\mathbf{u} = -\nabla \times \psi. \tag{16}$$

This is always possible since $\nabla \cdot \mathbf{u} = \mathbf{0}$. Thus,

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \Delta\psi - \nabla(\nabla \cdot \psi). \tag{17}$$

Imposing a gauge condition

$$\nabla \cdot \psi = 0, \tag{18}$$

yields

$$\omega = \Delta\psi. \tag{19}$$

The system (15) can now be rewritten as

$$\frac{\partial \Delta\psi}{\partial t} - \nabla \times (\Delta\psi \times (\nabla \times \psi)) = \nu \Delta^2 \psi + \nabla \times \mathbf{f}, \quad \text{in} \quad \Omega. \tag{20}$$

The boundary conditions $\mathbf{u} = 0$ translates to $\nabla \times \psi = 0$ on $\partial\Omega$. We require that

$$\mathbf{n} \times \psi = \mathbf{0}, \quad \mathbf{n} \times (\nabla \times \psi) = \mathbf{0}, \quad \text{on} \quad \partial\Omega. \tag{21}$$

The condition $\mathbf{n} \times \psi = \mathbf{0}$ means that ψ is parallel to $\mathbf{n}$, hence the normal component of the velocity vector is zero on the boundary. Adding the condition $\mathbf{n} \times (\nabla \times \psi) = \mathbf{0}$ ensures that the full velocity vector vanishes on the boundary. The requirements in (21) are equivalent to four scalar conditions, namely the vanishing of the two tangential components of ψ and $\nabla \times \psi$.

Turning now to the gauge condition $\nabla \cdot \psi = 0$, we add the condition

$$\frac{\partial(\psi \cdot \mathbf{n})}{\partial n} = 0, \quad \text{on} \quad \partial\Omega. \tag{22}$$

Together with the vanishing of the tangential components of ψ, it implies that $\nabla \cdot \psi = 0$ on $\partial\Omega$.

Equations (21) and (22) consist of five scalar conditions for ψ on the boundary. We can still add one more scalar boundary condition, as the equations for the 3-component streamfunction ψ contain the fourth order biharmonic operator. The sixth scalar boundary condition that we choose to add is

$$\Delta(\nabla \cdot \psi) = 0, \quad \text{on} \quad \partial\Omega. \tag{23}$$

We thus obtain

$$\nabla \cdot \psi = 0, \quad \Delta(\nabla \cdot \psi) = 0, \quad \text{on} \quad \partial\Omega. \tag{24}$$

We assume that the initial value $\psi(\mathbf{x}, 0)$ satisfies $(\nabla \cdot \psi)(\mathbf{x}, 0) = 0$. Taking the divergence of (20) we obtain an evolution equation for $\nabla \cdot \psi$.

$$\frac{\partial \Delta(\nabla \cdot \psi)}{\partial t} = \nu \Delta^2(\nabla \cdot \psi), \quad \text{in} \quad \Omega. \tag{25}$$

Equations (24) and (25) together with the assumption that $\nabla \cdot \psi = 0$ initially ensure that $\nabla \cdot \psi = 0$ for all $t > 0$. See also [1, 6, 7]. Finally, we have the following three-dimensional pure streamfunction formulation

$$\begin{cases} \frac{\partial \Delta\psi}{\partial t} - \nabla \times (\Delta\psi \times (\nabla \times \psi)) = \nu\Delta^2\psi + \nabla \times \mathbf{f}, & \text{in} \quad \Omega \\ \mathbf{n} \times \psi = \mathbf{0}, \ \frac{\partial(\psi \cdot \mathbf{n})}{\partial n} = 0, & \text{on} \quad \partial\Omega \\ \mathbf{n} \times (\nabla \times \psi) = \mathbf{0}, \quad \Delta(\nabla \cdot \psi) = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{26}$$

3 The Numerical Scheme

Our numerical scheme is based on the approximation of the following equation

$$\frac{\partial \Delta\psi}{\partial t} - ((\nabla \times \psi) \cdot \nabla)\Delta\psi + (\Delta\psi \cdot \nabla)(\nabla \times \psi) - \nu\Delta^2\psi = \nabla \times \mathbf{f}, \quad \text{in} \quad \Omega, \tag{27}$$

assuming that $\psi \in H_0^2(\Omega)$. For the vector function ψ we construct a fourth-order approximation to the biharmonic operator as follows. The pure fourth-order derivatives are approximated by $\delta_x^4, \delta_y^4, \delta_z^4$ as in (7) and (8).

The mixed terms ψ_{xxyy}, ψ_{yyzz} and ψ_{zzxx} are approximated by

$$\begin{cases} \tilde{\delta}_{xy}^2 \psi_{i,j,k} = 3\delta_x^2\delta_y^2\psi_{i,j,k} - \delta_x^2\delta_y\psi_{y,i,j,k} - \delta_y^2\delta_x\psi_{x,i,j,k} = \partial_x^2\partial_y^2\psi_{i,j,k} + O(h^4) \\ \tilde{\delta}_{yz}^2 \psi_{i,j,k} = 3\delta_y^2\delta_z^2\psi_{i,j,k} - \delta_y^2\delta_z\psi_{z,i,j,k} - \delta_z^2\delta_y\psi_{y,i,j,k} = \partial_y^2\partial_z^2\psi_{i,j,k} + O(h^4) \\ \tilde{\delta}_{zx}^2 \psi_{i,j,k} = 3\delta_z^2\delta_x^2\psi_{i,j,k} - \delta_z^2\delta_x\psi_{z,i,j,k} - \delta_x^2\delta_z\psi_{x,i,j,k} = \partial_z^2\partial_x^2\psi_{i,j,k} + O(h^4). \end{cases} \tag{28}$$

A fourth order approximation of the biharmonic operator is then obtained as

$$\tilde{\Delta}_h^2\psi = \delta_x^4\psi + \delta_y^4\psi + \delta_z^4\psi + 2\tilde{\delta}_{xy}^2\psi + 2\tilde{\delta}_{yz}^2\psi + 2\tilde{\delta}_{zx}^2\psi. \tag{29}$$

The approximate derivatives ψ_x, ψ_y and ψ_z are related to ψ via the Hermitian derivatives as in (5).

Equation (29) provides a fourth order compact operator for $\Delta^2\psi$, which involves values of ψ, ψ_x, ψ_y and ψ_z at (i, j, k) and at its 26 nearest neighbors. The Laplacian operator is approximated by a fourth order operator via

$$\tilde{\Delta}_h \psi = 2\Delta_h \psi - (\delta_x \psi_x + \delta_y \psi_y + \delta_z \psi_z). \tag{30}$$

The nonlinear part in (27) consists of two terms, the convective term and the stretching term. We design a fourth-order scheme which approximates the convective term. The convective term in the three-dimensional case is

$$C(\psi) = -((\nabla \times \psi) \cdot \nabla)\Delta \psi = u\Delta \partial_x \psi + v\Delta \partial_z \psi + w\Delta \partial_z \psi. \tag{31}$$

Here $(u, v, w) = \mathbf{u} = -\nabla \times \psi$ is the velocity vector, whose components contain first order derivatives of the streamfunction, and thus may be approximated to fourth-order accuracy. The terms $\Delta \partial_x \psi$, $\Delta \partial_z \psi$, $\Delta \partial_z \psi$ may be approximated as in the two-dimensional case. The term $\Delta \partial_x \psi$, for example, may be written as

$$\Delta \partial_x \psi = \partial_x^3 \psi + \partial_x \partial_y^2 \psi + \partial_x \partial_z^2 \psi. \tag{32}$$

Here, the pure and mixed type derivatives may be approximated as in the two-dimensional Navier–Stokes equations (see (9) and (10)). We denote the approximation to the convective term by $\tilde{C}_h(\psi)$.

Now, we construct a fourth-order approximation to the stretching term $S = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = -(\Delta \psi \cdot \nabla)(\nabla \times \psi)$. Note that the stretching term contains $\Delta \psi$ and mixed second order derivatives of the streamfunction. The Laplacian of ψ may be approximated to fourth-order accuracy, as in (30). The second order mixed terms, such as $\partial_x \partial_y \psi$, may be approximated using a Hermitian approximation of the type

$$(\sigma_x \sigma_y)(\psi_{xy})_{i,j,k} = \delta_x \delta_y \psi_{i,j,k}. \tag{33}$$

Hence,

$$(I + \frac{h^2}{6}\delta_x^2)(I + \frac{h^2}{6}\delta_y^2)(\psi_{xy})_{i,j,k} = \delta_x \delta_y \psi_{i,j,k} \quad , 1 \le i, j, k \le N-1 \tag{34}$$

is an implicit equation for ψ_{xy}. We denote the approximation of the stretching term by $\tilde{S}_h(\psi)$. For the approximation in time, we apply a Crank–Nicholson scheme (see the comment after (13) and (14)).

We obtain the following scheme

$$\frac{(\tilde{\Delta}_h \psi_{i,j,k})^{n+1/2} - (\tilde{\Delta}_h \psi_{i,j,k})^n}{\Delta t/2} = -\tilde{C}_h \psi_{i,j,k}^{(n)} + \tilde{S}_h \psi_{i,j,k}^{(n)} + \frac{\nu}{2}[\tilde{\Delta}_h^2 \psi_{i,j,k}^{n+1/2} + \tilde{\Delta}_h^2 \psi_{i,j,k}^n] \tag{35}$$

$$\frac{(\tilde{\Delta}_h \psi_{i,j,k})^{n+1} - (\tilde{\Delta}_h \psi_{i,j,k})^n}{\Delta t} = -\tilde{C}_h \psi_{i,j,k}^{(n+1/2)} + \tilde{S}_h \psi_{i,j,k}^{(n+1/2)} + \frac{\nu}{2}[\tilde{\Delta}_h^2 \psi_{i,j}^{n+1} + \tilde{\Delta}_h^2 \psi_{i,j,k}^n]. \tag{36}$$

At present, a direct solver is invoked to solve the linear set of equations (35) and (36).

Some preliminary MATLAB computations with coarse grids confirm the fourth order accuracy of the scheme. We first show numerical results for the time-dependent

Stokes equations

$$\frac{\partial \Delta \psi}{\partial t} = \nu \Delta^2 \psi + \mathbf{f}, \quad \text{in} \quad \Omega. \tag{37}$$

We have picked the exact solution ψ

$$\psi^T(\mathbf{x}, t) = -\frac{1}{4} e^{-t} \left(z^4, x^4, y^4\right) \tag{38}$$

in the cube $\Omega = (0,1)^3$. Here, $\mathbf{f}$ is chosen such that ψ in (38) satisfied (37) exactly. Infg the numerical results shown here we have chosen the time step Δt of order h^2 in order to retain the overall fourth-order accuracy of the scheme. In practice, if we are interested mainly in the steady state solution, a larger time step, which is independent of h, may be used. In Table 1 we show results for the Stokes problem with $\Delta t = 0.1h^2$ and $t = 0.00625$. Here e is the error in the l_h^2 norm, i.e.,

$$e^2 = \sum_i \sum_j \sum_k (\psi_3(x_i, y_j, z_k) - \tilde{\psi}_3(x_i, y_j, z_k))^2 h^3,$$

where ψ_3 is the z component of the exact solution and $\tilde{\psi}_3$ is the z component of the approximate solution. e_y is the l_h^2 in the y derivative of ψ_3. In Table 2 we display the results for $t = 0.0625$ using $\Delta t = h^2$.

Next we show results for the Navier–Stokes Equations

$$\frac{\partial \Delta \psi}{\partial t} - ((\nabla \times \psi) \cdot \nabla)\Delta \psi + (\Delta \psi \cdot \nabla)(\nabla \times \psi) - \nu \Delta^2 \psi = \nabla \times \mathbf{f}, \quad \text{in} \quad \Omega \tag{39}$$

in the cube $\Omega = (0,1)^3$. Here, the source term $\mathbf{g} = \nabla \times \mathbf{f}$ is chosen such that $\psi^T(\mathbf{x}, t) = -\frac{1}{4} e^{-t} \left(z^4, x^4, y^4\right)$ is an exact solution of (39). In Table 3 we present results for $t = 0.00625$ using $\Delta t = 0.1h^2$.

Table 1 Stokes equations for $t = 0.00625$ using $\Delta t = 0.1h^2$

	Grid $5 \times 5 \times 5$	Rate	Grid $9 \times 9 \times 9$	Rate	Grid $17 \times 17 \times 17$
e	2.5460(−9)	3.82	1.8017(−10)	3.98	1.1443(−11)
e_y	7.7417(−9)	3.73	5.8037(−10)	3.96	3.7391(−11)
div (ψ)	1.3409(−8)	3.74	1.0052(−9)	3.96	6.4621(−11)

Table 2 Stokes equations with $\Delta t = h^2$ for $t = 0.0625$

	Grid $5 \times 5 \times 5$	Rate	Grid $9 \times 9 \times 9$	Rate	Grid $17 \times 17 \times 17$
e	9.6461(−7)	4.41	4.5309(−8)	4.00	2.8291(−9)
e_y	3.0293(−6)	4.33	1.5049(−7)	3.99	9.4269(−9)
div (ψ)	5.2470(−6)	4.33	2.6066(−7)	4.00	1.6328(−8)

Table 3 Navier–Stokes equations for $t = 0.00625$ using $\Delta t = 0.1h^2$

	Grid $5 \times 5 \times 5$	Rate	Grid $9 \times 9 \times 9$	Rate	Grid $17 \times 17 \times 17$
e	2.4497(−9)	3.86	1.6924(−10)	4.01	1.0473(−11)
e_y	7.6486(−9)	3.75	5.6845(−10)	3.98	3.5917(−11)
div (ψ)	1.2294(−8)	3.71	9.3619(−10)	3.92	6.1700(−11)

Table 4 Navier–Stokes equations for $t = 0.0625$ using $\Delta t = h^2$

	Grid $5 \times 5 \times 5$	Rate	Grid $9 \times 9 \times 9$	Rate	Grid $17 \times 17 \times 17$
e	9.4418(−7)	4.46	4.2709(−8)	4.04	2.5934(−9)
e_y	2.9836(−6)	4.38	1.4334(−7)	4.03	8.7800(−9)
div (ψ)	5.0471(−6)	4.40	2.3944(−7)	4.02	1.4778(−8)

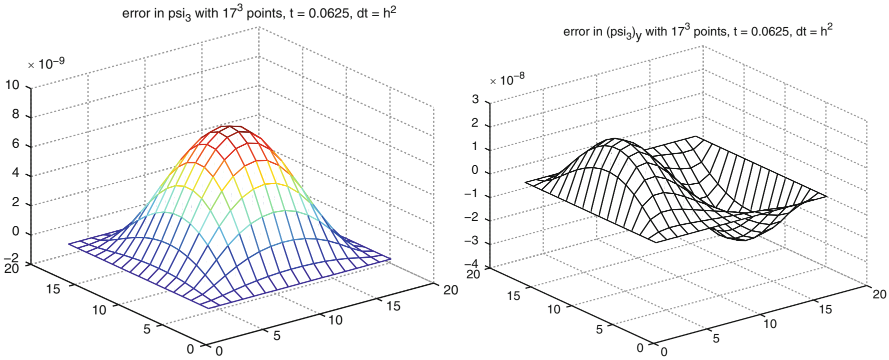

Fig. 1 Navier–Stokes : Errors in (**a**) ψ_3 and (**b**) $(\psi_3)_y$ for $N = 17, t = 0.0625,\ dt = h^2$

In Table 4 we show results for the Navier–Stokes Equations with $\Delta t = h^2$ for $t = 0.0625$. In Fig. 1a, b we display the errors for Navier–Stokes equations in ψ_3 and $(\psi_3)_y$ at $t = 0.0625$ with $dt = h^2$ and a 17^3 grid.

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