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A new fourth-order compact scheme for the Navier–Stokes equations in irregular domains

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ABSTRACT

We present a high-order finite difference scheme for Navier–Stokes equations in irregular domains. The scheme is an extension of a fourth-order scheme for Navier–Stokes equations in streamfunction formulation on a rectangular domain (Ben-Artzi et al., 2010). The discretization offered here contains two types of interior points. The first is regular interior points, where all eight neighboring points of a grid point are inside the domain and not too close to the boundary. The second is interior points where at least one of the closest eight neighbors is outside the computational domain or too close to the boundary. In the second case we design discrete operators which approximate spatial derivatives of the streamfunction on irregular meshes, using discretizations of pure derivatives in the x, y and along the diagonals of the element.

1. Introduction

In this paper we are interested in high-order discretizations of the Navier–Stokes equations. The Navier–Stokes equations play a central role in modeling fluid flows. Here we focus on incompressible flows. It is well-known that this system may be represented in pure streamfunction formulation as follows (see [1,2]).

$$\begin{cases} \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \nu\Delta^2\psi = f(x, y, t), \\ \psi(x, y, t) = \psi_0(x, y). \end{cases} \tag{1.1}$$

Recall that $\nabla^{\perp}\psi = (-\partial_y\psi, \partial_x\psi)$ is the velocity vector. The no-slip boundary condition associated with this formulation is

$$\psi = \frac{\partial\psi}{\partial n} = 0, \quad (x, y) \in \partial\Omega,\ t > 0 \tag{1.2}$$

and the initial condition is

$$\psi(x, y, 0) = \psi_0(x, y), \quad (x, y) \in \Omega. \tag{1.3}$$

In this paper we extend the fourth-order scheme [3] to irregular domains. The strategy used here is to present the biharmonic operator $\partial_x^4 + 2\partial_x^2\partial_y^2 + \partial_y^4$ as a combination of pure fourth-order derivatives in the x, y and the diagonal directions $\eta = (x + y)/\sqrt{2},\ \ \xi = (y - x)\sqrt{2}$. Then, the pure fourth-order derivatives may be approximated via a compact scheme

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using the values of the function and its directional derivatives (see also [4,5]). An alternative approach is to construct a two-dimensional polynomial which collocates the values of the function and its directional derivatives at the corners of the irregular element and then approximate the biharmonic of the function by the biharmonic of this polynomial (see [6]).

The numerical resolution of the Navier–Stokes system, governing viscous, incompressible, time-dependent flow, has been an important challenge of computational fluid dynamics. References belonging to the class of finite difference methods for the approximation of Navier–Stokes equations include projection methods [7–11]. The pure-streamfunction formulation for the time-dependent Navier–Stokes system in planar domains has been used in [12–14] some twenty years ago. It has been designed primarily for the numerical investigation of the Hopf bifurcation occurring in the driven cavity problem. Their approach was based on a finite-difference method. The application of various compact schemes to the pure streamfunction formulation is fairly recent [15–19].

We review some numerical methods for irregular domains. There are many references for finite elements methods for irregular domain (see for the example [20,21]). Several references for finite difference methods include [22–24]. In [24] a six-point scheme (star) was suggested. The disadvantage of the latter is its singularity and ill-conditioning. Several references such as [25] use coordinates transformation, however this approach is not suited to multiple irregular boundaries and may also impose singularities due to the coordinate transformation.

Liszka and Orkisz stated in [26] (1980) that "The fascination for FEM, however, caused by enormous successes or simply by fashion, has resulted in a relative stagnation in some other methods, especially in finite difference methods". In [26] a new mesh generation was constructed.

In [27,5] parabolic equations (in particular the heat equation) were solved in irregular domain, where a cartesian grid was used to approximate the solution of a time-dependent diffusion problem. At near boundary points the derivatives ∂_x^2 and ∂_y^2 were approximated using a non-uniform mesh, where one of the neighbors of the computational point was taken as a boundary point. In [28] Colella et al. suggested an embedded boundary/volume method for Navier–Stokes equations in irregular domains. It is a combination of finite differences, embedded boundary algorithm and finite volume methods. Calhoun [29] approximates the vorticity–streamfunction formulation by adding a correction term to the Poisson equation (which relates the streamfunction to the vorticity) using the immersed interface method. The purpose of this correction is to impose both boundary conditions on the streamfunction and the singular sources for the vorticity equation. The numerical results show second-order convergence rates for the solution of the Navier–Stokes equations. In [30] a fast finite difference method is proposed to solve the incompressible Navier–Stokes equations on a general domain. The method is based on the vorticity stream-function formulation and a fast Poisson solver defined on a general domain using the immersed interface method.

In [31] the discretization of the Poisson equation on irregular domains at near boundary points was carried out via quadratic polynomials, which yields second-order accuracy of the scheme. In [32] second and fourth order Cartesian grid finite difference methods were developed for second order elliptic and parabolic partial differential equations on irregular domains. The information around an irregular point was completed via a two-dimensional Taylor expansion around a boundary point using a local coordinate system. In [33] the immersed interface method is invoked for the application of the boundary conditions to the velocity–pressure formulation of Navier–Stokes equations. The approximated rates of convergence are between 2 and 3. In [34] the Poisson equation which relates the streamfunction to the vorticity was solved in two steps in order to enhance the efficiency of the scheme.

In [6,2] a two-dimensional interpolating polynomial of degree 5 and a half was constructed to approximate the solution of the biharmonic problem. This polynomial collocates the values of the function and its directional derivatives at the corners of an irregular element near the boundary (as well as of regular inner elements) and then approximates the biharmonic of the solution by the biharmonic of this polynomial. Fourth-order accuracy was achieved for the biharmonic problem in a circle and an ellipse.

In this paper we approximate spatial derivatives of the Navier–Stokes equations in streamfunction formulation. Interior points are treated via fourth-order discretizations as in [3]. Irregular elements are formed near the boundary, as in [6]. For irregular elements we write the biharmonic operator, as well as the convective term, using pure derivatives only in the directions of the axis and the diagonals of the element. Then, one-dimensional interpolating operators are used for these elements. Note that in [35] we have proved that the solution of the one-dimensional biharmonic equation by our compact high-order scheme is fourth-order accurate. Thus, it may be proved that reduction to our scheme to one dimension is fourth-order accurate (see also [36]).

The outline of the paper is as follows. In Section 2 we describe fourth order approximations of the Navier–Stokes equations in regular domains. All spatial operators appearing in the evolution equation, i.e., the Laplacian, the biharmonic operator and the nonlinear convective term, are approximated via fourth-order schemes. We also describe a time-marching scheme for the temporal evolution.

In Section 3 we suggest a new scheme for the Navier–Stokes system in streamfunction formulation for irregular domains. Here we assign different flags to the cartesian grid points in the rectangle, in which the irregular domain is embedded. At near-boundary points we approximate the spatial operators via combinations of pure spatial derivatives in the directions of the axis x, y and the diagonals.

In Section 4 we detail the approximations of $\partial_x\psi$, $\partial_x^4\psi$ and $\partial_x^2\psi$ for an irregular element. Similar representations are valid to $\partial_y\psi$, $\partial_y^4\psi$ and $\partial_y^2\psi$. The fourth-order derivatives along the diagonals, $\partial_\eta^4\psi$ and $\partial_\xi^4\psi$, are approximated in the same fashion. In Section 5 we describe the approximation of the convective term at near boundary points. This involves discretizations of pure third-order derivatives in x, y and along the diagonals. We show in detail how to approximate $\partial_x^3\psi$ by a fifth-order polynomial, which interpolates values as well as first-order derivatives of the function. Other pure third-order derivatives are similarly discretized.

Finally, in Section 6 we present several numerical results, including results for the full Navier–Stokes system, which demonstrate the high-order accuracy of the new scheme for irregular domains. Finally, in Section 7 conclusions are drawn.

2. Approximation of the Navier–Stokes equations on regular grids

Spatial derivatives in Eq. (1.1) are discretized as we describe next. The fourth order biharmonic $\tilde{\Delta}_h^2\psi$ operator introduced in [3,2] is a perturbation of the second order operator $\Delta_h^2\psi = (\delta_x^4 + \delta_y^4 + 2\delta_x^2\delta_y^2)\psi$. It is designed as follows.

$$\tilde{\Delta}_h^2\psi_{i,j} = \delta_x^4\psi_{i,j} + \delta_y^4\psi_{i,j} + 2\delta_x^2\delta_y^2\psi_{i,j} - \frac{h^2}{6}(\delta_x^4\delta_y^2\psi_{i,j} + \delta_y^4\delta_x^2\psi_{i,j}) = \Delta^2\psi_{i,j} + O(h^4), \tag{2.1}$$

where δ_x^4 and δ_y^4 are the compact approximations of ∂_x^4 and ∂_y^4, respectively.

$$\delta_x^4\psi_{i,j} = \frac{12}{h^2}\left((\delta_x\psi_x)_{i,j} - \delta_x^2\psi_{i,j}\right), \qquad \delta_x^4\psi = \partial_x^4\psi - \frac{1}{720}h^4\partial_x^8\psi + O(h^6), \tag{2.2}$$

$$\delta_y^4\psi_{i,j} = \frac{12}{h^2}\left((\delta_y\psi_y)_{i,j} - \delta_y^2\psi_{i,j}\right), \qquad \delta_y^4\psi = \partial_y^4\psi - \frac{1}{720}h^4\partial_y^8\psi + O(h^6). \tag{2.3}$$

Here, ψ_x, ψ_y are the fourth-order Hermitian approximations to $\partial_x\psi$, $\partial_y\psi$ described as

$$\begin{cases} (\sigma_x\psi_x)_{i,j} = \frac{1}{6}(\psi_x)_{i-1,j} + \frac{2}{3}(\psi_x)_{i,j} + \frac{1}{6}(\psi_x)_{i+1,j} = \delta_x\psi_{i,j}, & 1 \le i,j \le N-1 \\ (\sigma_y\psi_y)_{i,j} = \frac{1}{6}(\psi_y)_{i,j-1} + \frac{2}{3}(\psi_y)_{i,j} + \frac{1}{6}(\psi_y)_{i,j+1} = \delta_y\psi_{i,j}, & 1 \le i,j \le N-1. \end{cases} \tag{2.4}$$

We use the standard central difference operators δ_x, δ_y, δ_x^2, δ_y^2.

The fourth order Laplacian $\tilde{\Delta}_h\psi$ operator introduced in [3,2] is a perturbation of the second order operator $\Delta_h\psi = (\delta_x^2 + \delta_y^2)\psi$. It is designed as follows.

$$\begin{aligned} \tilde{\Delta}_h\psi_{i,j} &= 2\delta_x^2\psi_{i,j} - \delta_x(\psi_x)_{i,j} + 2\delta_y^2\psi_{i,j} - \delta_y(\psi_y)_{i,j} \\ &= 2\Delta_h\psi_{i,j} - (\delta_x(\psi_x)_{i,j} + \delta_y(\psi_y)_{i,j}) = (\Delta\psi)_{i,j} + O(h^4). \end{aligned} \tag{2.5}$$

Equivalently, the approximation of the Laplacian $\Delta\psi$ is

$$\tilde{\Delta}_h\psi_{i,j} = \tilde{\delta}_x^2\psi_{i,j} + \tilde{\delta}_y^2\psi_{i,j}, \tag{2.6}$$

where

$$\begin{aligned} \tilde{\delta}_x^2\psi_{i,j} &= 2\delta_x^2\psi_{i,j} - \delta_x\psi_{x,i,j} = \partial_x^2\psi_{i,j} + O(h^4), \\ \tilde{\delta}_y^2\psi_{i,j} &= 2\delta_y^2\psi_{i,j} - \delta_y\psi_{y,i,j} = \partial_y^2\psi_{i,j} + O(h^4). \end{aligned} \tag{2.7}$$

The convective term in (1.1) is $C(\psi) = -\partial_y\psi\,\Delta(\partial_x\psi) + \partial_x\psi\,\Delta(\partial_y\psi)$. Its fourth-order approximation needs special care. The mixed derivative $\partial_x\partial_y^2\psi$ may be approximated to fourth-order accuracy by $\tilde{\psi}_{yyx}$ using a suitable combination of lower order approximations.

$$(\tilde{\psi}_{yyx})_{i,j} = (\delta_y^2\psi_x + \delta_x\delta_y^2\psi - \delta_x\delta_y\psi_y)_{i,j} = (\partial_x\partial_y^2\psi)_{i,j} + O(h^4). \tag{2.8}$$

For the pure third order derivative $\partial_x^3\psi$ we note that if ψ is smooth then

$$(\psi_{xxx})_{i,j} = \frac{3}{2h^2}\left(10\delta_x\psi - h^2\delta_x^2\partial_x\psi - 10\partial_x\psi\right)_{i,j} = (\partial_x^3\psi)_{i,j} + O(h^4). \tag{2.9}$$

One needs to approximate $\partial_x\psi$ to sixth-order accuracy in order to obtain from (2.9) a fourth-order approximation for $\partial_x^3\psi$. Denoting this approximation by $\tilde{\psi}_x$, we invoke the Pade formulation [37], having the following form.

$$\frac{1}{3}(\tilde{\psi}_x)_{i+1,j} + (\tilde{\psi}_x)_{i,j} + \frac{1}{3}(\tilde{\psi}_x)_{i-1,j} = \frac{14}{9}\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2h} + \frac{1}{9}\frac{\psi_{i+2,j} - \psi_{i-2,j}}{4h}. \tag{2.10}$$

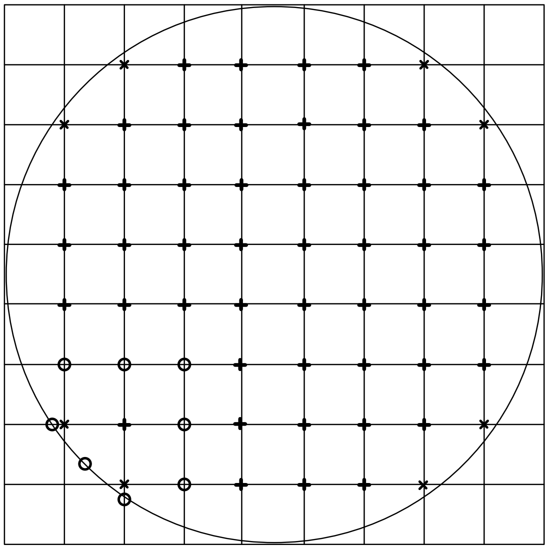


Fig. 1. Grid: '+' computational point, 'o' eight neighbors of a computational point, 'x' point too close to the boundary.

At near-boundary points we apply a special treatment as in [37]. Carrying out the same procedure for $\partial_y \psi$, which yields the approximate value $\tilde{\psi}_y$, and combining with all other mixed derivatives, a fourth order approximation of the convective term is

$$\begin{aligned}\tilde{C}_h(\psi) &= -\psi_y \cdot \left(\Delta_h \tilde{\psi}_x + \frac{5}{2}\left(6\frac{\delta_x \psi - \tilde{\psi}_x}{h^2} - \delta_x^2 \tilde{\psi}_x\right) + \delta_x \delta_y^2 \psi - \delta_x \delta_y \tilde{\psi}_y\right) \\ &\quad + \psi_x \cdot \left(\Delta_h \tilde{\psi}_y + \frac{5}{2}\left(6\frac{\delta_y \psi - \tilde{\psi}_y}{h^2} - \delta_y^2 \tilde{\psi}_y\right) + \delta_y \delta_x^2 \psi - \delta_y \delta_x \tilde{\psi}_x\right) \\ &= C(\psi) + O(h^4). \end{aligned} \tag{2.11}$$

Our implicit time-stepping scheme is of the Crank–Nicholson type as follows.

$$\frac{(\tilde{\Delta}_h \psi_{i,j})^{n+1/2} - (\tilde{\Delta}_h \psi_{i,j})^n}{\Delta t/2} = -\tilde{C}_h \psi^{(n)} + \frac{\nu}{2}[\tilde{\Delta}_h^2 \psi_{i,j}^{n+1/2} + \tilde{\Delta}_h^2 \psi_{i,j}^n] \tag{2.12}$$

$$\frac{(\tilde{\Delta}_h \psi_{i,j})^{n+1} - (\tilde{\Delta}_h \psi_{i,j})^n}{\Delta t} = -\tilde{C}_h \psi^{(n+1/2)} + \frac{\nu}{2}[\tilde{\Delta}_h^2 \psi_{i,j}^{n+1} + \tilde{\Delta}_h^2 \psi_{i,j}^n]. \tag{2.13}$$

3. Approximation of the Navier–Stokes equations on irregular domains

In the previous section we described the approximation of the Navier–Stokes equations in streamfunction formulation in rectangular domains. If the domain is not a rectangular, one can either map the domain onto a rectangle or design an approximation of the equations on a cartesian grid embedded inside the domain Ω. In case we chose to map the domain onto a rectangle, then the differential equations take a new form, as the derivatives of the new coordinate system are involved in the equations, which may complicate the equations. In addition, the transformation (such as a polar coordinate system) is sometimes singular at certain points and special treatment is needed near singular points.

In this paper we embed the domain Ω in a rectangle. Then, a uniform mesh is laid out inside the rectangle. Some of the mesh points are outside Ω, some are inside Ω and some may be on the boundary $\partial\Omega$.

If a mesh point is outside the computational domain Ω (flag $= -1$), then an arbitrary value, such as zero, is given to this point. Points which are outside the computational domain do not affect the values of the function at interior or at boundary points.

If a mesh point is on the boundary of the domain $\partial\Omega$ (flag $= 0$), then the boundary values ψ of the function and its first-order normal derivative $\partial_n \psi$ are assigned to this point.

If a mesh point is inside the domain it may be labeled as follows.

Case 1: the point is in the center of a rectangle for which all the vertices are inside the domain (flag $= 1$). This point is labeled by '+' in Fig. 1 if in addition all its eight nearest neighbors are inside Ω or on the boundary and they are all on the Cartesian grid. In this case the differential operators for this point are approximated as in Section 2.

Case 2: the point is too close to the boundary (flag $= 2$) then this point is not included in the set of computational points. It is labeled as 'x' in Fig. 1. In this case, neither the differential equations nor the boundary conditions are imposed at this point. In our computations we have labeled a point with flag $= 2$ if its distance to the boundary was less then βh, where h is the mesh size at the interior of the domain and $0 < \beta < 1$. In practice we have picked $\beta = 0.2$.

Case 3: the point is not too close to the boundary, but at least one of its eight nearest neighbors is outside the computational domain or at least one of its eight nearest neighbors is too close to the boundary (this point is labeled with flag $= 3$). This point is labeled by '+' in Fig. 1 if in addition all of its eight nearest neighbors are inside Ω or on the boundary and at least one of them is not on the Cartesian grid. Its eight nearest neighbors are labeled as 'o' in Figs. 1 and 2. In this case the computational point (denoted by '+') is the center of an irregular element. Thus, special discretization of the differential operator is needed.

We first have to describe how one constructs the element around such a computational point. Suppose the point under consideration is (x_i, y_j). If, for example (x_{i+1}, y_j) is outside the domain, then we define by (x_{east}, y_j) the point which is the closest on the right to (x_i, y_j) lying on the line $y = y_j$ and intersects with the boundary. We note by h_1 the distance from x_{east} to x_i, i.e., $h_1 = x_{east} - x_i$. Similarly in the case where (x_{i-1}, y_j) is outside Ω, for which we define $h_2 = x_i - x_{west}$. In the same fashion we treat the cases where (x_i, y_{j+1}) and (x_i, y_{j-1}) are outside the domain and define $h_3 = y_{north} - y_j$ and $h_4 = y_j - y_{south}$, respectively.

We also look at points along the line $x - x_i = y - y_j$. If (x_{i+1}, y_{j+1}) is outside the domain Ω, then we denote by $(x_{north-east}, y_{north-east})$ the intersection of the line $x - x_i = y - y_j$ going north-east of (x_i, y_j) with the boundary. We denote by h_5 the distance of $(x_{north-east}, y_{north-east})$ to (x_i, y_j), thus $(x_{north-east}, y_{north-east}) = (x_i + h_5/\sqrt{2}, y_j + h_5/\sqrt{2})$. Similarly $(x_{south-west}, y_{south-west}) = (x_i - h_6/\sqrt{2}, y_j - h_6/\sqrt{2})$. We also treat the points along the line $x - x_i = y_j - y$, thus defining h_7 as the distance of the point $(x_{north-west}, y_{north-west})$ to (x_i, y_j) and h_8 is the distance from $(x_{south-east}, y_{south-east})$ to (x_i, y_j).

Now we have to approximate $\Delta^2\psi$ at (x_i, y_j) in case where (x_i, y_j) is a computational point which is in the center of an irregular element.

Define a new coordinate system

$$\eta = (x + y)/\sqrt{2}, \qquad \xi = (y - x)/\sqrt{2}. \tag{3.1}$$

This yields

$$y = (\eta + \xi)/\sqrt{2}, \qquad x = (\eta - \xi)/\sqrt{2}. \tag{3.2}$$

Expressing $\psi_{\eta\eta\eta\eta}$ and $\psi_{\xi\xi\xi\xi}$ in terms of ψ_{xxxx}, ψ_{xxyy} and ψ_{yyyy}, we have

$$\begin{cases} \psi_\eta = \dfrac{1}{\sqrt{2}}(\psi_x + \psi_y), \\ \psi_{\eta\eta} = \dfrac{1}{2}(\psi_{xx} + 2\psi_{xy} + \psi_{yy}) \\ \psi_{\eta\eta\eta} = \dfrac{1}{2\sqrt{2}}(\psi_{xxx} + 3\psi_{xxy} + 3\psi_{xyy} + \psi_{yyy}), \\ \psi_{\eta\eta\eta\eta} = \dfrac{1}{4}(\psi_{xxxx} + 4\psi_{xxxy} + 6\psi_{xxyy} + 4\psi_{xyyy} + \psi_{yyyy}). \end{cases} \tag{3.3}$$

$$\begin{cases} \psi_\xi = \dfrac{1}{\sqrt{2}}(\psi_y - \psi_x), \\ \psi_{\xi\xi} = \dfrac{1}{2}(\psi_{xx} - 2\psi_{xy} + \psi_{yy}), \\ \psi_{\xi\xi\xi} = \dfrac{1}{2\sqrt{2}}(-\psi_{xxx} + 3\psi_{xxy} - 3\psi_{xyy} + \psi_{yyy}), \\ \psi_{\xi\xi\xi\xi} = \dfrac{1}{4}(\psi_{xxxx} - 4\psi_{xxxy} + 6\psi_{xxyy} - 4\psi_{xyyy} + \psi_{yyyy}). \end{cases} \tag{3.4}$$

Therefore,

$$2(\psi_{\eta\eta\eta\eta} + \psi_{\xi\xi\xi\xi}) = \psi_{xxxx} + 6\psi_{xxyy} + \psi_{yyyy}. \tag{3.5}$$

Thus,

$$2\psi_{xxyy} = \frac{2}{3}(\psi_{\eta\eta\eta\eta} + \psi_{\xi\xi\xi\xi}) - \frac{1}{3}(\psi_{xxxx} + \psi_{yyyy}). \tag{3.6}$$

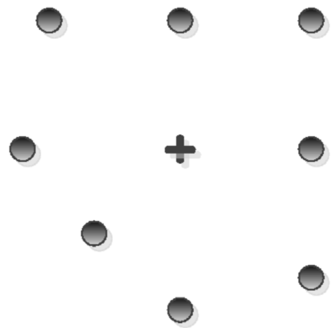


Fig. 2. Single computational element: '+' computational point, 'o' eight neighbors of a computational point.

This yields

$$\Delta^2\psi = \psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy} = \frac{2}{3}(\psi_{\eta\eta\eta\eta} + \psi_{\xi\xi\xi\xi} + \psi_{xxxx} + \psi_{yyyy}). \quad (3.7)$$

Thus, the operator Δ^2 can be expressed via pure fourth-order derivatives in the directions of x, y and η, ξ. We can therefore approximate $\Delta^2\psi$ by $\tilde{\Delta}_h^2\psi$, where

$$\tilde{\Delta}_h^2\psi = \frac{2}{3}(\delta_\eta^4\psi + \delta_\xi^4\psi + \delta_x^4\psi + \delta_y^4\psi). \quad (3.8)$$

The discretizations of ψ_{xxxx}, ψ_{yyyy} by $\delta_x^4\psi, \delta_y^4\psi$ and those of $\psi_{\eta\eta\eta\eta}, \psi_{\xi\xi\xi\xi}$ by $\delta_\eta^4\psi, \delta_\xi^4\psi$, respectively, are carried out via one-dimensional approximations of pure fourth-order derivatives.

We describe now the approximation of the convective term $C(\psi) = \nabla^\perp\psi \cdot \nabla\Delta\psi$. This may be written as

$$C(\psi) = \nabla^\perp\psi \cdot \nabla\Delta\psi = -(\partial_y\psi)\cdot\partial_x(\Delta\psi) + (\partial_x\psi)\cdot\partial_y(\Delta\psi).$$

Thus,

$$C(\psi) = -(\partial_y\psi)\cdot(\partial_{xxx}\psi + \partial_{xyy}\psi) + (\partial_x\psi)\cdot(\partial_{xxy}\psi + \partial_{yyy}\psi). \quad (3.9)$$

The mixed third-order derivatives ψ_{xxy} and ψ_{xyy} may be written using (3.3)–(3.4) by

$$\psi_{xxy} = \frac{\sqrt{2}}{3}(\psi_{\eta\eta\eta} + \psi_{\xi\xi\xi}) - \frac{1}{3}\psi_{yyy} \quad (3.10)$$

and

$$\psi_{xyy} = \frac{\sqrt{2}}{3}(\psi_{\eta\eta\eta} - \psi_{\xi\xi\xi}) - \frac{1}{3}\psi_{xxx}. \quad (3.11)$$

Inserting Eqs. (3.10)–(3.11) in Eq. (3.9), we obtain

$$C(\psi) = -\psi_y \cdot \Big(\frac{2}{3}\psi_{xxx} + \frac{\sqrt{2}}{3}(\psi_{\eta\eta\eta} - \psi_{\xi\xi\xi})\Big) + \psi_x \cdot \Big(\frac{2}{3}\psi_{yyy} + \frac{\sqrt{2}}{3}(\psi_{\eta\eta\eta} + \psi_{\xi\xi\xi})\Big). \quad (3.12)$$

In Section 4 we concentrate on the approximation of the biharmonic and the Laplacian operators. We discuss the truncation error due to the various discretizations. In Section 5 we describe the approximation of the convective term at near-boundary points.

4. Approximation of $\partial_x\psi$, $\partial_x^4\psi$ and $\partial_x^2\psi$ on an irregular mesh

We describe how to approximate $\partial_x\psi$, $\partial_x^4\psi$ and $\partial_x^2\psi$ in case the mesh is irregular. Let (x_i, y_j) be a grid point where at least one of its neighbors to the right or to the left is inside the domain or on the boundary but its distance to (x_i, y_j) is not h. Define the neighbor of (x_i, y_j) to the right by (x_{east}, y_j) and its neighbor to the left by (x_{west}, y_j). Let $h_1 = x_{east} - x_i$ and $h_2 = x_i - x_{west}$.

By the requirements we set in Section 2 on a computational point, we find that there exist positive constants, which does not depend on the mesh size, such that

$$\beta \le h_1/h \le 1 + \beta, \qquad \beta < h_2/h \le 1 + \beta, \quad 0 < \beta < 1. \quad (4.1)$$

Therefore,

$$\gamma \le \frac{h_2}{h_1} \le \frac{1}{\gamma}, \qquad \gamma \le \frac{h_1}{h_2} \le \frac{1}{\gamma}, \qquad \gamma = \frac{\beta}{1+\beta}. \tag{4.2}$$

Let $Q(x)$ be a polynomial of degree less or equal to 4.

$$Q(x) = a_0 + a_1(x-x_i) + a_2(x-x_i)^2 + a_3(x-x_i)^3 + a_4(x-x_i)^4. \tag{4.3}$$

The interpolating data is

$$\psi(x_{west}, y_j),\ \psi(x_i, y_j),\ \psi(x_{east}, y_j),\ \psi_x(x_{west}, y_j),\ \psi_x(x_{east}, y_j). \tag{4.4}$$

Then, a_1 is set as an approximation to $\partial_x\psi$ at (x_i, y_j) and is denoted by $\psi_{x,i,j}$. We also set $24a_4$ as an approximation to $\partial_x^4\psi$ at (x_i, y_j) and denote it by $\delta_x^4\psi_{i,j}$. In a similar manner one approximates the first order derivative with respect to y and the pure fourth-order derivatives with respect to y, ξ and η.

4.1. Approximation of the first-order derivatives

We now describe in detail the approximation of $\partial_x\psi$ at (x_i, y_j). Define

$$\begin{cases} c = \dfrac{4h_1h_2^3 - 4h_1^3h_2 + 2h_2^4 - 2h_1^4}{h_1h_2(h_1+h_2)^3}, \\ c_p = \dfrac{2h_2^4 + 4h_1h_2^3}{h_1h_2(h_1+h_1)^3}, \\ c_m = \dfrac{2h_1^4 + 4h_2h_1^3}{h_1h_2(h_1+h_2)^3}, \\ c_{x,p} = \dfrac{h_2^3h_1^2 + h_2^4h_1}{h_1h_2(h_1+h_2)^3}, \\ c_{x,m} = \dfrac{h_2^2h_1^3 + h_1^4h_2}{h_1h_2(h_1+h_2)^3}. \end{cases} \tag{4.5}$$

Then, the approximation $\psi_{x,i,j}$ to $\partial_x\psi_{i,j}$ is given by

$$\psi_{x,i,j} + c_{x,p}\cdot\psi_x(x_{east}, y_j) + c_{x,m}\cdot\psi_x(x_{west}, y_j) = c_p\cdot\psi(x_{east}, y_j) - c_m\cdot\psi(x_{west}, y_j) - c\cdot\psi_{i,j}. \tag{4.6}$$

Note that in case $h_1 = h_2 = h$ then (4.6) is equivalent to (2.4).

Suppose the values of ψ are known exactly on the points (x_{west}, y_j), (x_i, y_j) and (x_{east}, y_j). We analyze the error due to the approximation (4.6), which supplies an approximation $\partial_x\psi$ to the exact first-order derivative $\partial_x\psi$.

Lemma 4.1. *Suppose the values of ψ are known exactly on the points (x_{west}, y_j), (x_i, y_j) and (x_{east}, y_j). Let $\psi_{x,i,j}$ be defined by (4.6). Then the truncation error, T, of this approximation is given by*

$$|T| \le Ch^4\|\psi^{(5)}\|_{L^\infty}. \tag{4.7}$$

Proof. Suppose the values of ψ are known exactly on the points (x_{west}, y_j), $(x_i, y_j)(x_{east}, y_j)$. Inserting the exact derivatives of ψ in (4.6) then the truncation error, T, satisfy

$$\partial_x\psi_{i,j} + c_{x,p}\cdot\partial_x\psi(x_{east}, y_j) + c_{x,m}\cdot\partial_x\psi(x_{west}, y_j) - c_p\cdot\psi(x_{east}, y_j) + c_m\cdot\psi(x_{west}, y_j) + c\cdot\psi_{i,j} + T, \tag{4.8}$$

where T is the truncation error. Taylor expansion yields

$$\begin{aligned} \psi(x_i+h_1) &= \psi_i + h_1\partial_x\psi_i + \frac{h_1^2}{2}\partial_x^2\psi_i + \frac{h_1^3}{3!}\partial_x^3\psi_i + \frac{h_1^4}{4!}\partial_x^4\psi_i + \frac{h_1^5}{5!}\partial_x^5\psi(x_i^{(1)}), \\ \psi(x_i-h_2) &= \psi_i - h_2\partial_x\psi_i + \frac{h_2^2}{2}\partial_x^2\psi_i - \frac{h_2^3}{3!}\partial_x^3\psi_i + \frac{h_2^4}{4!}\partial_x^4\psi_i - \frac{h_2^5}{5!}\partial_x^5\psi(x_i^{(2)}), \\ \partial_x\psi(x_i+h_1) &= \partial_x\psi_i + h_1\partial_x^2\psi_i + \frac{h_1^2}{2}\partial_x^3\psi_i + \frac{h_1^3}{3!}\partial_x^4\psi_i + \frac{h_1^4}{4!}\partial_x^5\psi(x_i^{(3)}), \\ \partial_x\psi(x_i-h_2) &= \partial_x\psi_i - h_2\partial_x^2\psi_i + \frac{h_2^2}{2}\partial_x^3\psi_i - \frac{h_2^3}{3!}\partial_x^4\psi_i + \frac{h_2^4}{4!}\partial_x^5\psi(x_i^{(4)}). \end{aligned} \tag{4.9}$$

Inserting (4.9) in (4.8) and collecting the terms that multiply ψ_i, $\partial_x\psi_i$ $\partial_x^2\psi_i$, $\partial_x^3\psi_i$ and $\partial_x^4\psi_i$, we have

$$\begin{aligned}
&\psi_i:\ c - c_p + c_m = 0,\\
&\partial_x\psi_i:\ 1 + c_{x,p} + c_{x,m} - h_1 c_p - h_2 c_m = 0,\\
&\partial_x^2\psi_i:\ h_1 c_{x,p} - h_2 c_{x,m} - \frac{h_1^2}{2}c_p + \frac{h_2^2}{2}c_m = 0,\\
&\partial_x^3\psi_i:\ \frac{h_1^2}{2}c_{x,p} + \frac{h_2^2}{2}c_{x,m} - \frac{h_1^3}{3!}c_p - \frac{h_2^3}{3!}c_m = 0,\\
&\partial_x^4\psi_i:\ \frac{h_1^3}{3!}c_{x,p} - \frac{h_2^3}{3!}c_{x,m} - \frac{h_1^4}{4!}c_p + \frac{h_2^4}{4!}c_m = 0.
\end{aligned} \tag{4.10}$$

Using (4.1) the truncation error T satisfies

$$\partial_x\psi_{i,j} + c_{x,p}\cdot\partial_x\psi(x_{east}, y_j) + c_{x,m}\cdot\partial_x\psi(x_{west}, y_j) = c_p\cdot\psi(x_{east}, y_j) - c_m\cdot\psi(x_{west}, y_j) - c\cdot\psi_{i,j} + T, \tag{4.11}$$

where $|T| \le Ch^4\|\psi^{(5)}\|_{L^\infty}$. ■

Suppose that for a given j there are $M+1$ grid points $x_0, x_1, \ldots, x_M$ inside Ω. Define the error in the first-order derivative as

$$e_{x,i,j} = \psi_{x,i,j} - \partial_x\psi(x_i, y_j), \quad (x_i, y_j) \in \Omega,\ i = 1, 2, \ldots, M-1, . \tag{4.12}$$

The error at the two end-points $i = 0, M$ is zero.

In the next lemma we bound the error in the approximation of the first-order derivative.

Lemma 4.2. *Suppose the values of ψ are known exactly on the points (x_{west}, y_j), (x_i, y_j) and (x_{east}, y_j). Let $\psi_{x,i,j}$ be defined by (4.6). Then the error in $e_x = \psi_x - \partial_x\psi$ is bounded as follows.*

$$|e_x|_{l^\infty} \le Ch^4\|\psi^{(5)}\|_{L^\infty}. \tag{4.13}$$

Proof. Extracting (4.8) from (4.6) and noting that the values of ψ are given exactly in both equations, we find that the error e_x satisfies

$$e_{x,i,j} + c_{x,p}^{(i)}\cdot e_x(x_{east}, y_j) + c_{x,m}^{(i)}\cdot e_x(x_{west}, y_j) = -T, \quad i = 1, 2, \ldots, M-1 e_{x,0,j} = 0,\ e_{x,M,j} = 0. \tag{4.14}$$

This tridiagonal system (4.14) is diagonally dominant, since

$$|c_{x,p}^{(i)}| + |c_{x,m}^{(i)}| = \frac{h_2^2 h_1 + h_2^3 + h_2 h_1^2 + h_1^3}{(h_1 + h_1)^3} < 1. \tag{4.15}$$

Letting $h_2/h_1 = \alpha$, we have

$$|c_{x,p}^{(i)}| + |c_{x,m}^{(i)}| = 1 - 2\frac{\alpha+\alpha^2}{(1+\alpha)^3} = 1 - 2\alpha\frac{1}{(1+\alpha)^2} = 1 - 2\frac{1}{(1+\alpha)}(1 - \frac{1}{(1+\alpha)}). \tag{4.16}$$

Define

$$q = 1 - 2\frac{1}{(1+\alpha)}(1 - \frac{1}{(1+\alpha)}. \tag{4.17}$$

Using (4.2) we have

$$\frac{\gamma}{1+\gamma} \le \frac{1}{1+\alpha} \le \frac{1}{1+\gamma}. \tag{4.18}$$

Therefore,

$$0 < q < 1 - 2\big(\frac{\gamma}{\gamma+1}\big)^2. \tag{4.19}$$

Since $\gamma = \beta/(1+\beta)$, it follows from the last inequality that

$$0 < q < 1 - 2\big(\frac{\gamma}{\gamma+1}\big)^2 = 1 - 2\big(\frac{\beta}{1+2\beta}\big)^2 < 1. \tag{4.20}$$

Using (4.16), (4.17), (4.20) and $0 < \beta < 1$, it results that

$$|c_{x,p}^{(i)}| + |c_{x,m}^{(i)}| = q < 1. \tag{4.21}$$

Therefore, the matrix above is invertible and its inverse is bounded in the maximum norm. Thus,

$$|e_{x,i,j}| \le Ch^4 \|\psi^{(5)}\|_{L^\infty}. \quad \blacksquare \tag{4.22}$$

Similar representations are valid for $\partial_y \psi$. The derivatives along the diagonals $\partial_\eta \psi$ and $\partial_\xi \psi$ are approximated using the chain rule

$$\begin{aligned} \psi_\eta &= \frac{1}{\sqrt{2}}(\psi_x + \psi_y), \\ \psi_\xi &= \frac{1}{\sqrt{2}}(\psi_y - \psi_x). \end{aligned} \tag{4.23}$$

4.2. Approximation of the fourth-order derivatives

For the approximation of $\partial_x^4 \psi$ at an irregular point we define

$$\begin{cases} b = 24\dfrac{(h_1+h_2)^3}{h_1^2 h_2^2 (h_1+h_2)^3}, \\ b_p = 24\dfrac{h_2+3h_1}{h_1^2(h_1+h_2)^3}, \\ b_m = 24\dfrac{h_1+3h_2}{h_2^2(h_1+h_2)^3}, \\ b_{x,p} = 24\dfrac{h_1+h_2}{h_1(h_1+h_2)^3}, \\ b_{x,m} = 24\dfrac{h_1+h_2}{h_2(h_1+h_2)^3}. \end{cases} \tag{4.24}$$

Then, the approximation $\bar{\delta}_x^4 \psi_{i,j}$ to $\partial_x^4 \psi$ is given by

$$\bar{\delta}_x^4 \psi_{i,j} = b_{x,p} \cdot \psi_x(x_{east}, y_j) - b_{x,m} \cdot \psi_x(x_{west}, y_j) - (b_p \cdot \psi(x_{east}, y_j) + b_m \cdot \psi(x_{west}, y_j) - b \cdot \psi_{i,j}). \tag{4.25}$$

Note that in case $h_1 = h_2 = h$ then (4.25) is equivalent to (2.2). Similar representations are valid to $\partial_y^4 \psi$, and to derivatives along the diagonals $\partial_\eta^4 \psi$ and $\partial_\xi^4 \psi$, given that $\partial_\eta \psi$ and $\partial_\xi \psi$ are approximated by (4.23).

Lemma 4.3. *Suppose the values of ψ are known exactly on the points (x_{west}, y_j), (x_i, y_j) and (x_{east}, y_j). In addition, suppose that the values of $\psi_x(x_{east}, y_j)$ and $\psi_x(x_{west}, y_j)$ are taken as the exact values of $\partial_x \psi$ at $x_{east}, y_j)$ and $\psi_x(x_{west}, y_j)$, respectively. Let $\delta_x^4 \psi_{i,j}$ be defined by (4.25). Then, the truncation error $\tilde{T}_4$ of this approximation is given by*

$$\begin{aligned} \partial_x^4 \psi_{i,j} = {} & b_{x,p} \cdot \partial_x \psi(x_{east}, y_j) - b_{x,m} \cdot \partial_x \psi(x_{west}, y_j) \\ & - (b_p \cdot \psi(x_{east}, y_j) + b_m \cdot \psi(x_{west}, y_j) - b \cdot \psi_{i,j}) + \tilde{T}_4 \end{aligned} \tag{4.26}$$

and satisfies

$$|\tilde{T}_4| \le Ch\|\psi^{(5)}\|_{L^\infty}. \tag{4.27}$$

Proof. Using the Taylor expansion (4.9) we find that

$$\begin{aligned} &\psi_i: \; b - b_p + b_m = 0, \\ &\partial_x \psi_i: \; b_{x,p} - b_{x,m} - h_1 b_p + h_2 b_m = 0, \\ &\partial_x^2 \psi_i: \; h_1 b_{x,p} - h_2 b_{x,m} - \frac{h_1^2}{2} b_p - \frac{h_2^2}{2} b_m = 0, \\ &\partial_x^3 \psi_i: \; \frac{h_1^2}{2} b_{x,p} - \frac{h_2^2}{2} b_{x,m} - \frac{h_1^3}{3!} b_p + \frac{h_2^3}{3!} b_m = 0, \\ &\partial_x^4 \psi_i: \; \frac{h_1^3}{3!} b_{x,p} + \frac{h_2^3}{3!} b_{x,m} - \frac{h_1^4}{4!} b_p - \frac{h_2^4}{4!} b_m = 1. \end{aligned} \tag{4.28}$$

Using (4.1), we find that the truncation error $\tilde{T}_4$ satisfies

$$|\tilde{T}_4| \leq Ch\|\psi^{(5)}\|_{L^\infty}. \quad \blacksquare \tag{4.29}$$

Lemma 4.4. *Suppose the values of ψ are known exactly on the points (x_{west}, y_j), (x_i, y_j) and (x_{east}, y_j). In addition, suppose that the values of $\psi_x(x_{east}, y_j)$ and $\psi_x(x_{west}, y_j)$ are taken as the discrete first-order derivative given in (4.6), then the truncation error T_4 for the approximation of the $\partial_x^4\psi$ for an irregular element is bounded as follows.*

$$|\bar{\delta}_x^4\psi_{i,j} - \partial_x^4\psi| \leq Ch\|\psi^{(5)}\|_{L^\infty}. \tag{4.30}$$

Proof. Relating T_4 with $\tilde{T}_4$ and using (4.1), and the bounds $|b_{x,p}| \leq C/h^3$ and $|b_{x,m}| \leq C/h^3$, we obtain that

$$\begin{aligned} |T_4| &\leq |\tilde{T}_4| + |b_{x,p}||e_x(x_{east}, y_j)| + |b_{x,m}||e_x(x_{west}, y_j)| \\ &\leq (Ch + C_1\frac{h_1+h_2}{h_1(h_1+h_2)^3}h^4 + C_2\frac{h_1+h_2}{h_2(h_1+h_2)^3}h^4)\|\psi^{(5)}\|_{L^\infty} \leq Ch\|\psi^{(5)}\|_{L^\infty}. \end{aligned} \tag{4.31}$$

Therefore, the truncation error for the approximation of the $\partial_x^4\psi$ for an irregular element is bounded as follows.

$$|\bar{\delta}_x^4\psi_{i,j} - \partial_x^4\psi| \leq Ch\|\psi^{(5)}\|_{L^\infty}. \quad \blacksquare \tag{4.32}$$

4.3. Approximation of the second-order derivatives

For the approximation of $\partial_x^2\psi$ at an irregular point we define

$$\begin{cases} d = 2\dfrac{8(h_2^2h_1^3 + h_2^3h_1^2) + h_2h_1^4 + h_2^4h_1 - h_2^5 - h_1^5}{h_2^2h_1^2(h_1+h_2)^3}, \\ d_p = 2\dfrac{h_2(-h_2^2 + 8h_1^2 + h_2h_1)}{h_1^2(h_1+h_2)^3}, \\ d_m = 2\dfrac{h_1(-h_1^2 + 8h_2^2 + h_2h_1)}{h_2^2(h_1+h_2)^3}, \\ d_{x,p} = 2\dfrac{h_2(2h_1^2 + h_2h_1 - h_2^2)}{h_1(h_1+h_2)^3}, \\ d_{x,m} = 2\dfrac{h_1(2h_2^2 + h_2h_1 - h_1^2)}{h_2(h_1+h_2)^3}. \end{cases} \tag{4.33}$$

Then, the approximation $\bar{\delta}_x^2\psi_{i,j}$ to $\partial_x^2\psi$ is given by

$$\bar{\delta}_x^2\psi_{i,j} = d_p \cdot \psi(x_{east}, y_j) + d_m \cdot \psi(x_{west}, y_j) - d \cdot \psi_{i,j} - (d_{x,p} \cdot \psi_x(x_{east}, y_j) - d_{x,m} \cdot \psi_x(x_{west}, y_j)). \tag{4.34}$$

Note that in case $h_1 = h_2 = h$ then (4.34) is equivalent to the approximation of $\delta_x^2\psi$ in (2.5). Similar representations are valid for $\partial_y^2\psi$.

One can show, using Taylor expansion and (4.1), that if the values of ψ_x are chosen as the exact values of the first-order derivative, then the truncation error for the approximation of the $\partial_x^2\psi$ for an irregular element is bounded as follows.

Lemma 4.5. *Suppose the values of ψ are known exactly on the points (x_{west}, y_j), (x_i, y_j) and (x_{east}, y_j). In addition, suppose that the values of $\psi_x(x_{east}, y_j)$ and $\psi_x(x_{west}, y_j)$ are taken as the discrete first-order derivative given in (4.6), then the truncation error T_2 for the approximation of the $\partial_x^2\psi$ for an irregular element is bounded as follows.*

$$|\bar{\delta}_x^2\psi_{i,j} - \partial_x^2\psi| \leq Ch^3\|\psi^{(5)}\|_{L^\infty}. \tag{4.35}$$

Proof. First assume that the values of the first-order derivatives are given exactly. Then using the Taylor expansion (4.9) we find that

$$
\begin{aligned}
&\psi_i: \ d_p + d_m - d = 0,\\
&\partial_x\psi_i: d_{x,p} + d_{x,m} + h_1 d_p - h_2 d_m = 0,\\
&\partial_x^2\psi_i: \ -h_1 d_{x,p} - h_2 d_{x,m} + \frac{h_1^2}{2} d_p + \frac{h_2^2}{2} d_m = 1,\\
&\partial_x^3\psi_i: \ -\frac{h_1^2}{2} d_{x,p} + \frac{h_2^2}{2} d_{x,m} + \frac{h_1^3}{3!} d_p - \frac{h_2^3}{3!} d_m = 0,\\
&\partial_x^4\psi_i: \ -\frac{h_1^3}{3!} d_{x,p} - \frac{h_2^3}{3!} d_{x,m} + \frac{h_1^4}{4!} d_p + \frac{h_2^4}{4!} d_m = 0.
\end{aligned}
\tag{4.36}
$$

Using (4.1) the truncation error $\tilde{T}_2$ satisfies

$$|\tilde{T}_2| \le Ch^3 \|\psi^{(5)}\|_{L^\infty}. \tag{4.37}$$

Relating the error in the approximation of the second-order derivative may be related with $\tilde{T}_2$. Using (4.1), we find that

$$|\bar{\delta}_x^2\psi_{i,j} - \partial_x^2\psi| \le |\tilde{T}_2| + |d_{x,p}||e_x(x_{east}, y_j)| + |d_{x,m}||e_x(x_{west}, y_j)|. \tag{4.38}$$

Since $|d_{x,p}|$ and $|d_{x,m}|$ are bounded by C/h, we obtain that

$$|\bar{\delta}_x^2\psi_{i,j} - \partial_x^2\psi| \le Ch^3 \|\psi^{(5)}\|_{L^\infty}. \quad \blacksquare \tag{4.39}$$

The proof is similar to the proof of Lemma 4.4.

5. Approximation of convective term on an irregular mesh

In order to approximate the convective term (3.9) (or its equivalent form (3.12)), we have to discretize pure third-order derivatives of ψ in x, y and in ξ, η. Note that we have already obtained fourth-order approximations to $\partial_x\psi$ and $\partial_y\psi$ (see (4.6)).

In [3,2] we have constructed a sixth-order approximation to the first-order derivative, using a sixth-order interpolating polynomial based on the interpolating values $\psi_{i-2,j}, \psi_{i-1,j}, \psi_{i,j}, \psi_{i+1,j}, \psi_{i+1,j}$ and $\psi_{x,i-1,j}, \psi_{x,i,j}, \psi_{x,i+1,j}$. Then we inserted these values into an approximation of $\partial_x^3\psi$, based on a fifth-order polynomial. The latter interpolates the values $\psi_{i-1,j}, \psi_{x,i-1,j}, \psi_{i,j}, \psi_{x,i,j}, \psi_{i+1,j}, \psi_{x,i+1,j}$ and the resulting approximation was fourth-order accurate for $\partial_x^3\psi$.

We first describe the approximation to $\partial_x^3\psi$ and then show how to obtain a higher-order approximation to the first-order derivative. Let (x_i, y_j) be a grid point where two of its neighbors to the right (x_{west}, y_j) and to the left (x_{east}, y_j) are inside the domain or on the boundary. Define $h_1 = x_{east} - x_i$ and $h_2 = x_i - x_{west}$.

Define

$$
\begin{cases}
q = 12\dfrac{h_2^3 - 4h_2^2h_1 + 4h1^2h_2 - h_1^3}{h_1^3h_2^3},\\[2ex]
q_p = 12\dfrac{h_2(h_1h_2 - h_2^2 + 5h_1^2)}{h_1^3(h_1+h_2)^3},\\[2ex]
q_m = -12\dfrac{h_1(h_1h_2 - h_1^2 + 5h_2^2)}{h_2^3(h_1+h_2)^3},\\[2ex]
q_x = 6\dfrac{h_2^2 - 4h_1h_2 + h_1^2}{h_1^2h_2^2},\\[2ex]
q_{x,p} = -6\dfrac{h_2(2h_1 - h_2)}{h_1^2(h_1+h_2)^2},\\[2ex]
q_{x,m} = -6\dfrac{h_1(2h_2 - h_1)}{h_2^2(h_1+h_2)^2}.
\end{cases}
\tag{5.1}
$$

Then, the approximation $\bar{\delta}_x^3\psi_{i,j}$ to $\partial_x^3\psi$ is given by

$$
\begin{aligned}
\bar{\delta}_x^3\psi_{i,j} = {}& q\cdot\psi_{i,j} + q_p\cdot\psi(x_{east}, y_j) + q_m\cdot\psi(x_{west}, y_j)\\
&+ q_x\cdot\psi_x(x_i, y_j) + q_{x,p}\cdot\psi_x(x_{east}, y_j) + q_{x,m}\cdot\psi_x(x_{west}, y_j)).
\end{aligned}
\tag{5.2}
$$

Other pure third-order derivatives may be similarly discretized. Note that in case $h_1 = h_2 = h$ then (5.2) is equivalent to the approximation of $\delta_x^3\psi$ in Eq. (3.29) of [3].

One may show, using Taylor expansion and (4.1), that if the values of ψ_x are chosen as the exact values of the first-order derivative, then the truncation error for the approximation of the $\partial_x^3\psi$ for an irregular element is bounded as follows.

$$|\bar{\delta}_x^3\psi_{i,j} - \partial_x^3\psi| \le Ch^3\|\psi^{(6)}\|_{L^\infty}. \tag{5.3}$$

When we approximated $\partial_x^3\psi$ to fourth-order accuracy on a uniform mesh we need to approximate $\partial_x\psi$ to sixth-order accuracy, so that by using these values for the first-order derivative, one can obtain a fourth-order approximation to $\partial_x^3\psi$. Now we construct the analogue of the sixth-order approximation to ψ_x, derived this time for a non-uniform grid.

Let (x_i, y_j) be a grid point where its nearest two neighbors to the right and to the left are inside the domain, but its next neighbor to the right or to left is on the boundary. The neighbor of (x_i, y_j) to the right is (x_{i+1}, y_j) and its neighbor to the left is (x_{i-1}, y_j). We denote its second neighbor to the right by (x_{east2}, y_j) and its second neighbor to the left by (x_{west2}, y_j). Let $h_1 = x_{east2} - x_{i+1}$, $h_2 = x_{i+1} - x_i = h$, $h_3 = x_i - x_{i-1} = h$ and $h_4 = x_{i-1} - x_{west2}$.

By the requirements we set in Section 2 on a computational point, we find that there exist positive constants, which does not depend on the mesh size, such that

$$\beta \le h_1/h \le 1+\beta, \quad \beta < h_4/h \le 1+\beta, \quad 0 < \beta < 1. \tag{5.4}$$

Let $Q(x)$ be a polynomial of degree less or equal 6.

$$Q(x) = a_0 + a_1(x-x_i) + a_2(x-x_i)^2 + a_3(x-x_i)^3 + a_4(x-x_i)^4 + a_5(x-x_i)^5 + a_6(x-x_i)^6. \tag{5.5}$$

The interpolating data is

$$\psi(x_{west2}, y_j),\ \psi(x_{i-1}, y_j),\ \psi_x(x_{i-1}, y_j),\ \psi(x_i, y_j),\ \psi(x_{i+1}, y_j),\ \psi_x(x_{i+1}, y_j),\ \psi(x_{east2}, y_j). \tag{5.6}$$

Then, a_1 is set as an approximation to $\partial_x\psi$ at (x_i, y_j) and is denoted by $\bar{\psi}_{x,i,j}$.

Define

$$\begin{cases} r = 2\dfrac{-3h_3^2h_2 - 3h_4h_3h_2 - 2h_3^2h_1 - 2h_4h_3h_1 + 3h_3h_2h_1 + 3h_3h_2^2 + 2h_4h_2h_1 + 2h_4h_2^2}{h_3h_2(h_1+h_2)(h_3+h_4)}, \\ r_p = \dfrac{(h_3+h_4)h_3^2(h_1+h_2)(5h_1h_2^2 + 7h_3h_2h_1 + 2h_3^2h_1 + 4h_4h_2h_1 + 2h_4h_3h_1 - h_2^3 - 2h_3h_2^2 - h_3^2h_2 - h_4h_2^2 - h_4h_3h_2)}{h_2(h_2+h_3+h_4)^2(h_2+h_3)^3h_1^2}, \\ r_m = \dfrac{(h_3+h_4)h_2^2(h_1+h_2)(-2h_4h_2h_1 - 2h_4h_2^2 - 7h_4h_3h_2 - 4h_4h_3h_1 - 5h_4h_3^2 + h_3h_2h_1 + h_3h_2^2 + 2h_3^2h_2 + h_3^2h_1 + h_3^3)}{h_3h_4^2(h_2+h_3)^3(h_1+h_2+h_3)^2}, \\ r_{x,p} = \dfrac{(h_1+h_2)(h_3+h_4)h_3^2}{(h_2+h_3+h_4)(h_2+h_3)^2h_1}, \\ r_{x,m} = \dfrac{(h_1+h_2)(h_3+h_4)h_2^2}{(h_1+h_2+h_3)(h_2+h_3)^2h_4}, \\ r_{p2} = \dfrac{(h_3+h_4)h_3^2h_2^2}{(h_1+h_2+h_3+h_4)(h_1+h_2+h_3)^2(h_1+h_2)h_1^2}, \\ r_{m2} = \dfrac{(h_1+h_2)h_3^2h_2^2}{(h_1+h_2+h_3+h_4)(h_2+h_3+h_4)^2(h_3+h_4)h_4^2}. \end{cases} \tag{5.7}$$

Then, the approximation $\bar{\psi}_{x,i,j}$ to $\partial_x\psi$ is given by

$$\begin{aligned} &\bar{\psi}_{x,i,j} + r_{x,p}\cdot\bar{\psi}_{x,i+1,j} + r_{x,m}\cdot\bar{\psi}_{x,i-1,j} \\ &\quad = r\cdot\psi_{i,j} + r_p\cdot\psi_{i+1,j} + r_m\cdot\psi_{i-1,j} + r_{p2}\cdot\psi(x_{east2}, y_j) + r_{m2}\cdot\psi(x_{west2}, y_j) \end{aligned} \tag{5.8}$$

Other pure first-order derivatives may be discretized similarly. It may be shown, using the error of the interpolation problem above and (5.4), that if the values of $\bar{\psi}_{x,i-1,j}$, $\bar{\psi}_{x,i+1,j}$ are chosen as the exact values of $\partial_x\psi$ at (x_{i-1}, y_j), (x_{i+1}, y_j) respectively, then the truncation error in $\bar{\psi}_{x,i,j}$ for an interior point may be bounded by

$$|\bar{\psi}_{x,i,j} - \partial_x\psi| \le Ch^6\|\psi^{(7)}\|_{L^\infty}. \tag{5.9}$$

Suppose that the point (x_i, y_j) is close to the boundary, such that it has only one neighbor to the right (x_{east}, y_j) inside the computational domain or on the boundary, but two neighbors to the left (x_{i-1}, y_j) and (x_{west2}, y_j). In this case we construct a polynomial of degree 5 at most, which interpolates

$$\psi(x_{west2}, y_j),\ \psi(x_{i-1}, y_j),\ \psi_x(x_{i-1}, y_j),\ \psi(x_i, y_j),\ \psi(x_{east}, y_j),\ \psi_x(x_{east}, y_j). \tag{5.10}$$

Let $h_1 = x_{east} - x_i$, $h_2 = x_i - x_{i-1} = h$ and $h_3 = x_{i-1} - x_{west2}$.

By the requirements we set in Section 2 on a computational point, we find that there exist positive constants, which does not depend on the mesh size, such that

$$\beta \le h_1/h \le 1+\beta, \qquad \beta < h_3/h \le 1+\beta, \quad 0<\beta<1. \tag{5.11}$$

Define

$$\begin{cases} s = \dfrac{-2h_2^2 - 2h_3h_2 + 3h_1h_2 + 2h_1h_3}{h_1h_2(h_2+h_3)}, \\ s_p = \dfrac{(h_2+h_3)h_2^2(5h_1^2 + 7h_1h_2 + 2h_2^2 + 4h_1h_3 + 2h_2h_3)}{(h_1+h_2+h_3)^2(h_1+h_2)^3 h_1}, \\ s_m = \dfrac{(h_2+h_3)h_1^2(-2h_1h_3 - 4h_2h_3 + h_1h_2 + h_2^2)}{h_2h_3^2(h_1+h_2)^3}, \\ s_{x,p} = \dfrac{(h_2+h_3)h_2^2}{(h_1+h_2+h_3)(h_1+h_2)^2}, \\ s_{x,m} = \dfrac{(h_2+h_3)h_1^2}{h_3(h_1+h_2)^2}, \\ s_{m2} = -\dfrac{h_1^2h_2^2}{h_3^2(h_2+h_3)(h_1+h_2+h_3)^2}. \end{cases} \tag{5.12}$$

Then, the approximation $\bar{\psi}_{x,i,j}$ to $\partial_x\psi$ is given by

$$\bar{\psi}_{x,i,j} + s_{x,p}\cdot\bar{\psi}_x(x_{east}, y_j) + s_{x,m}\cdot\bar{\psi}_{x,i-1,j} = s\cdot\psi_{i,j} + s_p\cdot\psi_{(x_{east},y_j)} + s_m\cdot\psi_{i-1,j} + s_{m2}\cdot\psi(x_{west2}, y_j) \tag{5.13}$$

It may be shown, using the error of the interpolation problem above and (5.11), that if the value of $\bar{\psi}_{x,i-1,j}$ is chosen as the exact value of $\partial_x\psi$ at (x_{i-1}, y_j), then the truncation error in $\bar{\psi}_{x,i,j}$ for a near-boundary point may be bounded by

$$|\bar{\psi}_{x,i,j} - \partial_x\psi| \le Ch^5\|\psi^{(6)}\|_{L^\infty}. \tag{5.14}$$

The case where the point (x_i, y_j) is close to the boundary, such that it has only one neighbor to the left (x_{west}, y_j) inside the computational domain or on the boundary, but two neighbors to the right (x_{i+1}, y_j) and (x_{east2}, y_j) is treated similarly.

6. Numerical accuracy of the scheme in irregular domains

In order to verify the spatial fourth order accuracy of the scheme, we performed several numerical tests. The time-step was set to $dt = Ch^2$.

In Tables 1–10 we present the error, e, and the relative error, where

$$e_2 = \|\psi_{comp} - \psi_{exact}\|_{l_2},$$
$$e_\infty = \|\psi_{comp} - \psi_{exact}\|_{l_\infty}.$$

Similarly,

$$(e_x)_2 = \|(\psi_x)_{comp} - (\psi_x)_{exact}\|_{l_2},$$
$$(e_x)_\infty = \|(\psi_x)_{comp} - (\psi_x)_{exact}\|_{l_\infty}.$$

Here, ψ_{comp}, $(\psi_x)_{comp}$ and ψ_{exact}, $(\psi_x)_{exact}$ are the computed and the exact streamfunction and of ψ and its x-derivative, respectively.

6.0.1. Case 1: $\psi(x,y,t) = e^{x+y-t}$ in a unit circle

Here

$$f(x,y,t) = \partial_t\Delta\psi - \Delta^2\psi, \tag{6.1}$$

where $\psi(x,y,t) = e^{x+y-t}$.

Our aim is to recover $\psi(x,y,t)$ from $f(x,y,t)$. Thus, we resolve numerically

$$\begin{cases} \partial_t\Delta\psi - \Delta^2\psi = f(x,y,t), \quad (x,y)\in\Omega \\ \psi(x,y,0) = e^{x+y}, \quad (x,y)\in\Omega \\ \psi(x,y,t) = e^{x+y-t}, \quad (x,y)\in\partial\Omega \\ \dfrac{\partial\psi(x,y,t)}{\partial n} = \dfrac{\partial e^{x+y-t}}{\partial n}, \quad (x,y)\in\partial\Omega. \end{cases} \tag{6.2}$$

Table 1
Compact scheme with exact solution: $\psi = e^{x+y-t}$ on $x^2 + y^2 \leq 1$. We present e and e_x, the l_2 errors for the streamfunction and for $\partial_x \psi$. Here $\Delta t = 0.25h^2$ and $t = 0.25$.

mesh	9 × 9	Rate	17 × 17	Rate	33 × 33
e_2	1.0366E−04	4.31	5.2399E−06	4.10	3.0546E−07
e_∞	1.6352E−04	4.58	6.8480E−06	3.61	5.5991E−07
$(e_x)_2$	3.2905E−04	4.45	1.5105E−05	3.80	1.0811E−06
$(e_x)_\infty$	7.1337E−04	4.17	3.9506E−05	3.13	4.5110E−06

Table 2
Compact scheme with exact solution: $\psi = (1 - x^2)^3(1 - y^2)^3 e^{-t}$ on $x^2 + y^2 \leq 1$. We present e and e_x, the l_2 errors for the streamfunction and for $\partial_x \psi$. Here $\Delta t = 0.25h^2$ and $t = 0.25$.

mesh	9 × 9	Rate	17 × 17	Rate	33 × 33
e_2	1.103E−02	4.55	4.6820E−04	4.06	2.8002E−05
e_∞	1.201E−02	4.56	5.0703E−04	4.18	2.5776E−05
$(e_x)_2$	2.730E−02	4.29	1.4000E−03	4.65	5.5899E−05
$(e_x)_\infty$	3.7960E−02	4.12	2.1864E−03	4.10	1.2735E−04

Table 3
Compact scheme with exact solution: $\psi = (1 - (x^2 + y^2))^3 e^{-t}/192$ on $x^2 + y^2 \leq 1$. We present e and e_x, the l_2 errors for the streamfunction and for $\partial_x \psi$. Here $\Delta t = 0.25h^2$ and $t = 0.25$.

mesh	9 × 9	Rate	17 × 17	Rate	33 × 33
e_2	3.8146E−05	3.55	3.2473E−06	5.77	5.9640E−08
e_∞	3.7433E−05	3.39	3.5759E−06	6.04	5.4332E−08
$(e_x)_2$	1.2305E−04	4.48	5.5097E−06	3.89	3.7099E−07
$(e_x)_\infty$	1.2827E−04	4.08	7.5821E−06	3.29	7.7447E−07

Table 4
Compact scheme for the Navier–Stokes equation with exact solution: $\psi = e^{x+y-t}$ on $x^2 + y^2 \leq 1$. We present e and e_x, the l_2 errors for the streamfunction and for $\partial_x \psi$. Here $\Delta t = 0.25h^2$ and $t = 0.25$. In Fig. 3 the solution and the error for Case 4 are plotted.

mesh	9 × 9	Rate	17 × 17	Rate	33 × 33
e_2	9.955E−05	4.31	5.0042E−06	4.01	2.998E−07
e_∞	1.6792E−04	4.70	6.4755E−06	3.55	5.5991E−07
$(e_x)_2$	3.0959E−04	4.40	1.4634E−05	3.80	1.0508E−06
$(e_x)_\infty$	6.6237E−04	4.09	3.8936E−05	3.11	4.5138E−06

Table 5
Compact scheme for Navier–Stokes equation with exact solution: $\psi = \frac{1}{288}(1 - x^2)^3(1 - y^2)^3 e^{-t}$ on $x^2 + y^2 \leq 1$. We present e and e_x, the l_2 errors for the streamfunction and for $\partial_x \psi$. Here $\Delta t = 0.25h^2$ and $t = 0.25$. In Fig. 4 the solution and the error for Case 5 are plotted.

mesh	9 × 9	Rate	17 × 17	Rate	33 × 33
e_2	1.1040E−02	4.56	4.6817E−04	4.06	2.8002E−05
e_∞	1.2010E−02	4.57	5.0701E−04	4.30	2.5781E−05
$(e_x)_2$	2.7300E−02	4.33	1.3530E−03	4.60	5.5872E−05
$(e_x)_\infty$	3.7950E−02	4.12	2.1861E−03	4.10	1.2750E−04

Table 6
Compact scheme for Navier–Stokes equation with exact solution: $\psi = (1 - (x^2 + y^2))^3 e^{-t}/192$ on $x^2 + y^2 \leq 1$. We present e and e_x, the l_2 errors for the streamfunction and for $\partial_x \psi$. Here $\Delta t = 0.25h^2$ and $t = 0.25$. In Fig. 5 the solution and the error for Case 6 are plotted.

mesh	9 × 9	Rate	17 × 17	Rate	33 × 33
e_2	3.8146E−05	3.55	3.2473E−06	5.77	5.9640E−08
e_∞	3.7433E−05	3.39	3.5759E−06	6.04	5.4332E−08
$(e_x)_2$	1.2305E−04	4.48	5.5097E−06	3.89	3.7099E−07
$(e_x)_\infty$	1.2827E−04	4.08	7.5821E−06	3.29	7.7447E−07

6.0.2. Case 2: $\psi(x, y, t) = (1 - x^2)^3(1 - y^2)^3 e^{-t}$ on a unit circle

Here

$$f(x, y, t) = \partial_t \Delta \psi - \Delta^2 \psi, \tag{6.3}$$

Table 7
Compact scheme for Navier–Stokes equation with exact solution: $\psi = (0.81 - (x^2 + y^2)^2)e^{-t}/64$ on D. We present e and e_x, the l_2 errors for the streamfunction and for $\partial_x\psi$. Here $\Delta t = 0.25h^2$ and $t = 0.16$. In Fig. 6 the solution and the error for Case 7 are plotted.

mesh	11 × 11	Rate	21 × 21	Rate	41 × 41
e_2	6.7927E−09	3.97	4.3433E−10	3.98	2.7595E−11
e_∞	1.1452E−08	3.99	7.2237E−10	3.97	4.6112E−11
$(e_x)_2$	1.8151E−08	3.95	1.1759E−09	3.99	7.3986E−11
$(e_x)_\infty$	3.12494E−08	3.88	2.1241E−09	3.99	1.3374E−10

Table 8
Compact scheme for Navier–Stokes equation with exact solution: $\psi = e^{x+y-t}$ on Ω. We present e and e_x, the l_2 errors for the streamfunction and for $\partial_x\psi$. Here $\Delta t = 0.25h^2$ and $t = 0.16$. In Fig. 7 the solution and the error for Case 8 are plotted.

mesh	11 × 11	Rate	21 × 21	Rate	41 × 41
e_2	9.5923E−06	4.05	5.5957E−07	4.85	1.9380E−08
e_∞	1.9296E−05	3.95	1.2704e-06	5.08	1.9380E−08
$(e_x)_2$	6.1783E−05	4.29	3.1745e-06	4.59	1.3202E−07
$(e_x)_\infty$	1.3602e-04	3.14	1.5512e-05	4.01	9.6381E−07

Table 9
Compact scheme for Navier–Stokes equation with exact solution: $\psi = (1/64)e^{-t}((x^2+y^2)^2 + \cos(x)\cdot\cos(y))$ on Ω. We present e and e_x, the l_2 errors for the streamfunction and for $\partial_x\psi$. Here $\Delta t = 0.25h^2$ and $t = 0.16$. In Fig. 8 the solution and the error for Case 9 are plotted.

mesh	11 × 11	Rate	21 × 21	Rate	41 × 41
e_2	5.1468E−08	4.02	3.1644E−09	4.68	1.2311E−10
e_∞	1.0385E−07	3.94	6.7449E−09	4.80	2.4166E−10
$(e_x)_2$	3.1049E−07	4.28	1.5965E−08	4.61	6.5458E−10
$(e_x)_\infty$	7.1207E−07	4.25	3.7389E−08	3.85	2.5906E−09

Table 10
Compact scheme for Navier–Stokes equation with exact solution: $\psi = (1/64)e^{-t}((x^2+y^2)^2 + e^x\cos(y))$ on Ω. We present e and e_x, the l_2 errors for the streamfunction and for $\partial_x\psi$. Here $\Delta t = 0.25h^2$ and $t = 0.16$. In Fig. 9 the solution and the error for Case 10 are plotted.

mesh	11 × 11	Rate	21 × 21	Rate	41 × 41
e_2	3.0809E−08	4.02	1.8993E−09	4.33	9.4105E−11
e_∞	9.6878E−08	4.21	5.2525E−09	4.25	2.7563E−10
$(e_x)_2$	2.8732E−07	4.17	1.5968E−08	4.16	8.9395E−10
$(e_x)_\infty$	5.6380E−07	4.28	2.8971E−08	3.63	2.3323E−09

where $\psi(x,y,t) = \frac{1}{288}(1-x^2)^3(1-y^2)^3e^{-t}$. Our aim is to recover $\psi(x,y,t)$ from $f(x,y,t)$. Thus, we resolve numerically

$$\begin{cases} \partial_t\Delta\psi - \Delta^2\psi = f(x,y,t), & (x,y)\in\Omega \\ \psi(x,y,0) = \dfrac{1}{288}(1-x^2)^3(1-y^2)^3, & (x,y)\in\Omega \\ \psi(x,y,t) = \dfrac{1}{288}(1-x^2)^3(1-y^2)^2e^{-t}, & (x,y)\in\partial\Omega \\ \dfrac{\partial\psi(x,y,t)}{\partial n} = \dfrac{1}{288}\dfrac{\partial(1-x^2)^3(1-y^2)^2e^{-t}}{\partial n}, & (x,y)\in\partial\Omega. \end{cases} \tag{6.4}$$

6.0.3. *Case 3:* $\psi(x,y,t) = \frac{1}{192}(1-(x^2+y^2))^3e^{-t}$ *on a unit circle*

Here

$$f(x,y,t) = \partial_t\Delta\psi - \Delta^2\psi, \tag{6.5}$$

where $\psi(x,y,t) = \frac{1}{192}(1-(x^2+y^2))^3e^{-t}$. Our aim is to recover $\psi(x,y,t)$ from $f(x,y,t)$. Thus, we resolve numerically

$$\begin{cases} \partial_t\Delta\psi - \Delta^2\psi = f(x,y,t), & (x,y)\in\Omega \\ \psi(x,y,0) = \dfrac{1}{192}(1-(x^2+y^2))^3, & (x,y)\in\Omega \\ \psi(x,y,t) = 0, \quad \dfrac{\partial\psi(x,y,t)}{\partial n} = 0, & (x,y)\in\partial\Omega. \end{cases} \tag{6.6}$$

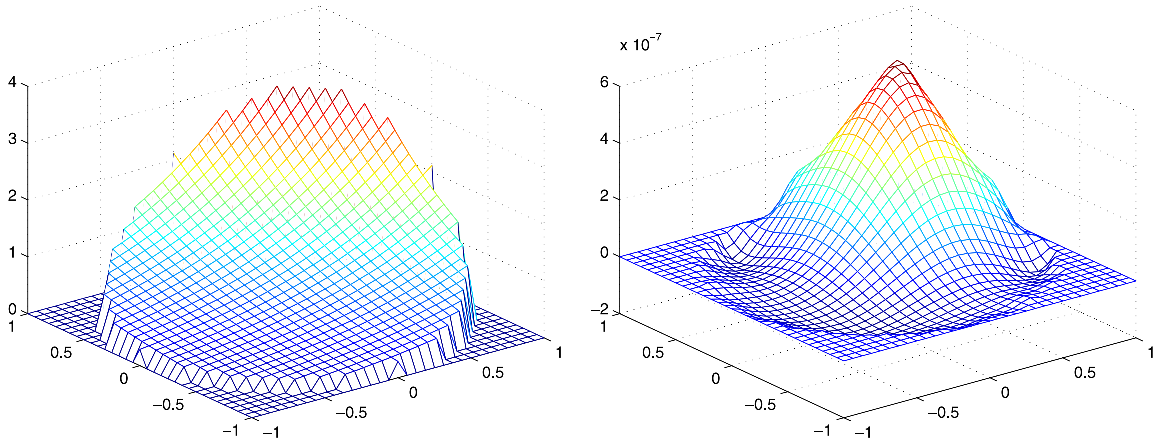


Fig. 3. Case 4. Left: Approximation for $\psi = e^{x+y-t}$. Right: The error.

6.0.4. Case 4: Navier–Stokes with exact solution $\psi(x, y, t) = e^{x+y-t}$ in a unit circle

Here

$$f(x, y, t) = \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi, \tag{6.7}$$

where $\psi(x, y, t) = e^{x+y-t}$.

Our aim is to recover $\psi(x, y, t)$ from $f(x, y, t)$. Thus, we resolve numerically

$$\begin{cases} \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi = f(x, y, t), & (x, y) \in \Omega \\ \psi(x, y, 0) = e^{x+y}, & (x, y) \in \Omega \\ \psi(x, y, t) = e^{x+y-t}, & (x, y) \in \partial\Omega \\ \dfrac{\partial\psi(x, y, t)}{\partial n} = \dfrac{\partial e^{x+y-t}}{\partial n}, & (x, y) \in \partial\Omega. \end{cases} \tag{6.8}$$

6.0.5. Case 5: Navier–Stokes equation with exact solution $\psi(x, y, t) = (1 - x^2)^3(1 - y^2)^3 e^{-t}$ on a unit circle

Here

$$f(x, y, t) = \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi, \tag{6.9}$$

where $\psi(x, y, t) = \frac{1}{288}(1 - x^2)^3(1 - y^2)^3 e^{-t}$. Our aim is to recover $\psi(x, y, t)$ from $f(x, y, t)$. Thus, we resolve numerically

$$\begin{cases} \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi = f(x, y, t), & (x, y) \in \Omega \\ \psi(x, y, 0) = \dfrac{1}{288}(1 - x^2)^3(1 - y^2)^3, & (x, y) \in \Omega \\ \psi(x, y, t) = \dfrac{1}{288}(1 - x^2)^3(1 - y^2)^2 e^{-t}, & (x, y) \in \partial\Omega \\ \dfrac{\partial\psi(x, y, t)}{\partial n} = \dfrac{1}{288}\dfrac{\partial(1 - x^2)^3(1 - y^2)^2 e^{-t}}{\partial n}, & (x, y) \in \partial\Omega. \end{cases} \tag{6.10}$$

6.0.6. Case 6: Navier–Stokes equation with exact solution $\psi(x, y, t) = \frac{1}{192}(1 - (x^2 + y^2))^3 e^{-t}$ on a unit circle

Here

$$f(x, y, t) = \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi, \tag{6.11}$$

where $\psi(x, y, t) = \frac{1}{192}(1 - (x^2 + y^2))^3 e^{-t}$. Our aim is to recover $\psi(x, y, t)$ from $f(x, y, t)$. Thus, we resolve numerically

$$\begin{cases} \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi = f(x, y, t), & (x, y) \in \Omega \\ \psi(x, y, 0) = \dfrac{1}{192}(1 - (x^2 + y^2))^3, & (x, y) \in \Omega \\ \psi(x, y, t) = 0, \quad \dfrac{\partial\psi(x, y, t)}{\partial n} = 0, & (x, y) \in \partial\Omega. \end{cases} \tag{6.12}$$

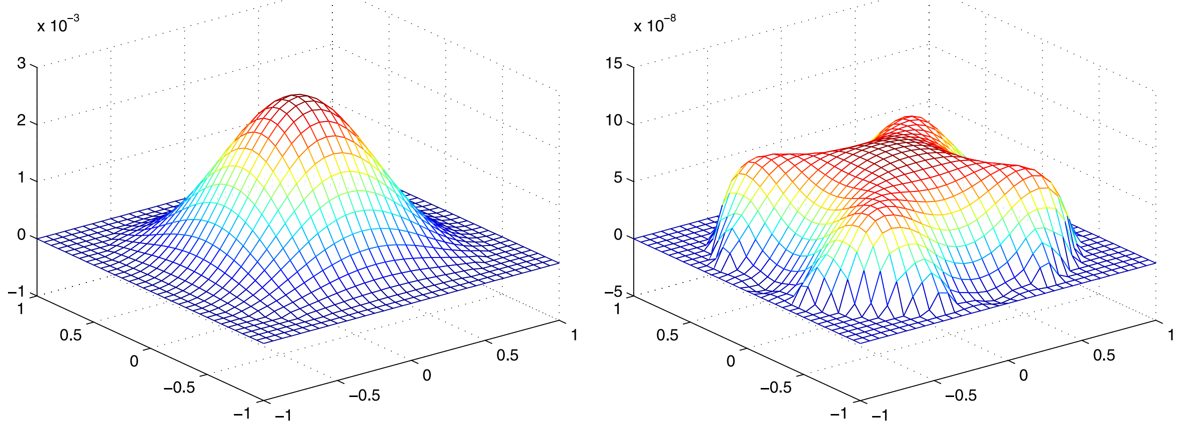


Fig. 4. Case 5. Left: Approximation for $\psi = \frac{1}{288}(1-x^2)^3(1-y^2)^3e^{-t}$. Right: The error.

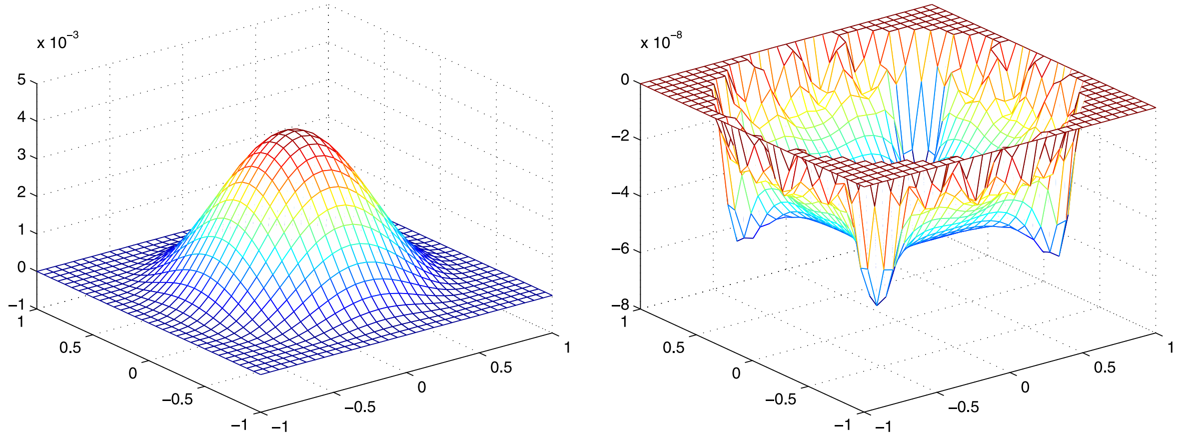


Fig. 5. Case 6. Left: Approximation for $\psi(x, y, t) = \frac{1}{192}(1-(x^2+y^2))^3e^{-t}$. Right: The error.

6.1. *Intersection of two non-concentric circles*

In this subsection the domain Ω is the intersection of two non-concentric circles.

$$\Omega = \{(x, y)|(x-0.4)^2 + y^2 < 0.5\} \cup \{(x, y)|(x+0.4)^2 + y^2 < 0.5\} \tag{6.13}$$

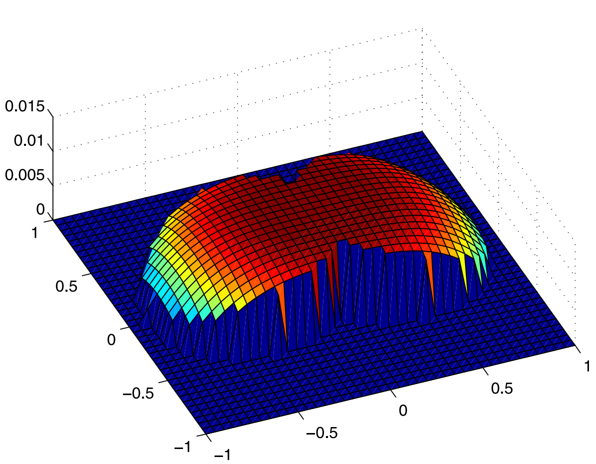
6.1.1. *Case 7: Navier–Stokes equation with exact solution $\psi(x, y, t) = \frac{1}{64}(0.81-(x^2+y^2)^2)e^{-t}$ on two circles intersection*

Here

$$f(x, y, t) = \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi, \tag{6.14}$$

where $\psi(x, y, t) = \frac{1}{64}e^{-t}(0.81-(x^2+y^2)^2)$. Our aim is to recover $\psi(x, y, t)$ from $f(x, y, t)$. Thus, we resolve numerically

$$\begin{cases} \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi = f(x, y, t), & (x, y) \in \Omega \\ \psi(x, y, 0) = \dfrac{1}{64}(0.81-(x^2+y^2)^2), & (x, y) \in \Omega \\ \psi(x, y, t) = \dfrac{1}{64}(0.81-(x^2+y^2)^2)e^{-t}, & (x, y) \in \partial\Omega \\ \dfrac{\partial\psi(x, y, t)}{\partial n} = \dfrac{1}{64}\dfrac{\partial((0.81-(x^2+y^2)^2)e^{-t})}{\partial n}, & (x, y) \in \partial\Omega. \end{cases} \tag{6.15}$$



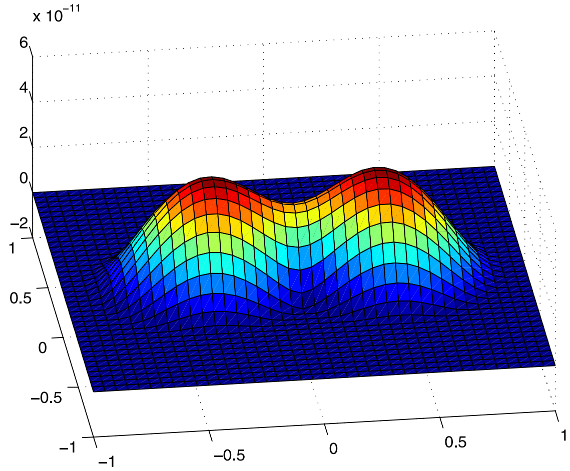


Fig. 6. Case 7. Left: Approximation for $\psi(x, y, t) = \frac{1}{64}(0.81 - (x^2 + y^2)^2)e^{-t}$. Right: The error.

6.1.2. Case 8: Navier–Stokes equation with exact solution $\psi(x, y, t) = e^{x+y-t}$ on two circles intersection

Here

$$f(x, y, t) = \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi, \tag{6.16}$$

where $\psi(x, y, t) = e^{x+y-t}$. Our aim is to recover $\psi(x, y, t)$ from $f(x, y, t)$. Thus, we resolve numerically

$$\begin{cases} \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi = f(x, y, t), & (x, y) \in \Omega \\ \psi(x, y, 0) = e^{x+y}, & (x, y) \in \Omega \\ \psi(x, y, t) = e^{x+y-t}, & (x, y) \in \partial\Omega \\ \dfrac{\partial\psi(x, y, t)}{\partial n} = \dfrac{\partial e^{x+y-t}}{\partial n}, & (x, y) \in \partial\Omega. \end{cases} \tag{6.17}$$

6.1.3. Case 9: Navier–Stokes equation with exact solution $\psi(x, y, t) = (1/64)e^{-t}((x^2 + y^2)^2 + \cos(x) \cdot \cos(y))$ on two circles intersection

Here

$$f(x, y, t) = \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi, \tag{6.18}$$

where $\psi(x, y, t) = (1/64)e^{-t}((x^2 + y^2)^2 + \cos(x) \cdot \cos(y))$. Our aim is to recover $\psi(x, y, t)$ from $f(x, y, t)$. Thus, we resolve numerically

$$\begin{cases} \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi = f(x, y, t), & (x, y) \in \Omega \\ \psi(x, y, 0) = (1/64)((x^2 + y^2)^2 + \cos(x) \cdot \cos(y)), & (x, y) \in \Omega \\ \psi(x, y, t) = (1/64)e^{-t}((x^2 + y^2)^2 + \cos(x) \cdot \cos(y)), & (x, y) \in \partial\Omega \\ \dfrac{\partial\psi(x, y, t)}{\partial n} = \dfrac{\partial(1/64)e^{-t}((x^2 + y^2)^2 + \cos(x) \cdot \cos(y))}{\partial n}, & (x, y) \in \partial\Omega. \end{cases} \tag{6.19}$$

6.1.4. Case 10: Navier–Stokes equation with exact solution $\psi(x, y, t) = (1/64)e^{-t}((x^2 + y^2)^2 + e^x \cos(y))$ on two circles intersection

Here

$$f(x, y, t) = \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi, \tag{6.20}$$

where $\psi(x, y, t) = (1/64)e^{-t}((x^2 + y^2)^2 + e^x \cos(y))$. Our aim is to recover $\psi(x, y, t)$ from $f(x, y, t)$. Thus, we resolve numerically

$$\begin{cases} \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi = f(x, y, t), & (x, y) \in \Omega \\ \psi(x, y, 0) = (1/64)((x^2 + y^2)^2 + e^x \cos(y)), & (x, y) \in \Omega \\ \psi(x, y, t) = (1/64)e^{-t}((x^2 + y^2)^2 + e^x \cos(y)), & (x, y) \in \partial\Omega \\ \dfrac{\partial\psi(x, y, t)}{\partial n} = \dfrac{\partial(1/64)e^{-t}((x^2 + y^2)^2 + e^x \cos(y))}{\partial n}, & (x, y) \in \partial\Omega. \end{cases} \tag{6.21}$$

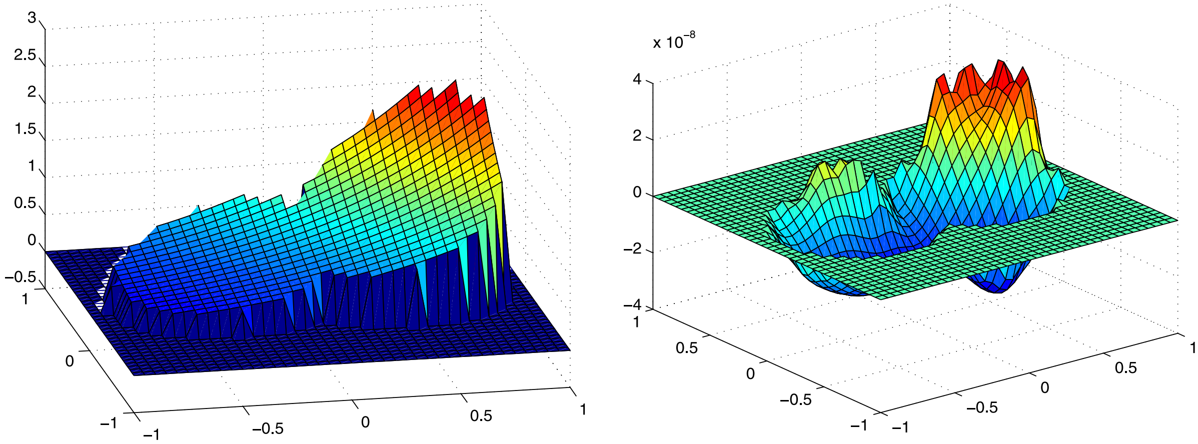


Fig. 7. Case 8. Left: Approximation for $\psi(x, y, t) = e^{x+y-t}$. Right: The error.

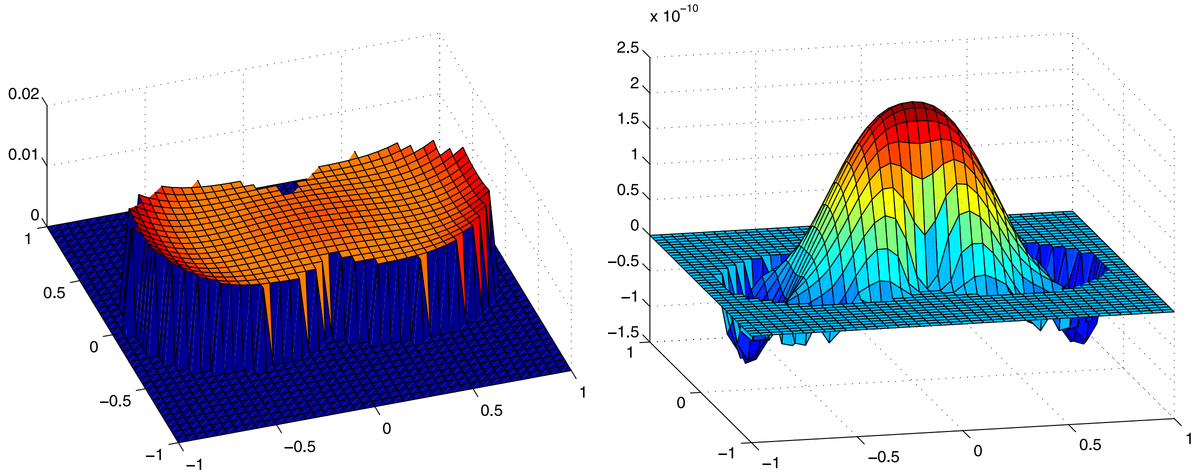


Fig. 8. Case 9. Left: Approximation for $\psi(x, y, t) = (1/64)e^{-t}((x^2 + y^2)^2 + \cos(x) \cdot \cos(y))$. Right: The error.

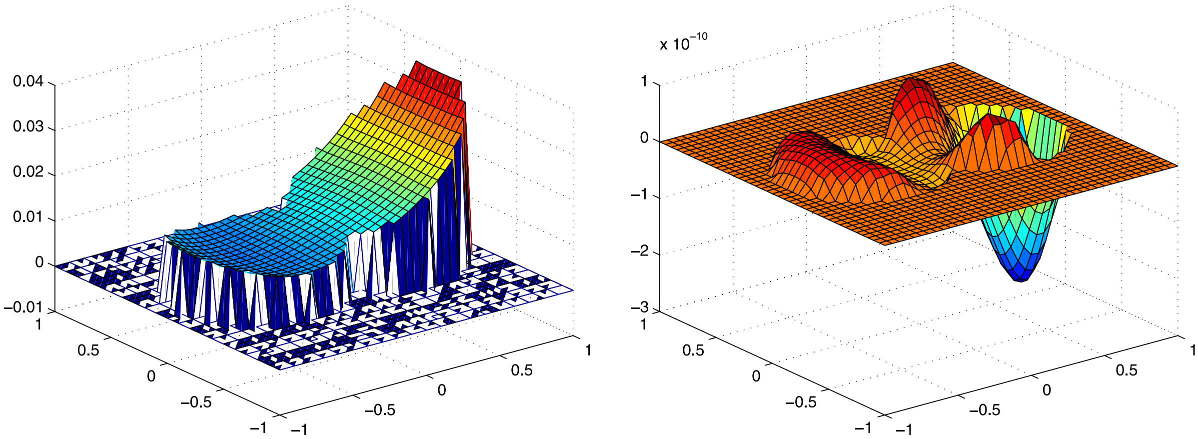


Fig. 9. Case 10. Left: Approximation for $\psi(x, y, t) = (1/64)e^{-t}((x^2 + y^2)^2 + e^x \cos(y))$. Right: The error.

7. Conclusions

We have constructed a new high-order compact scheme for the Navier–Stokes equations in irregular domains. The idea is to express the biharmonic operator via pure fourth-order derivatives along the axis and the diagonals of each computational element. In addition, the third-order derivatives appearing in the convective term as well are written in terms of pure third-order derivatives along the axes and the diagonals. This enable us to approximate the Navier–Stokes operator via one-dimensional discretizations at near boundary points. The truncation errors are analyzed for all one-dimensional operators involved in the representation of the biharmonic and the convective terms. The numerical results demonstrate the fourth-order accuracy of the scheme for all the test cases for which the new scheme was applied. In our future work we plan to test our scheme for additional more complex geometries. It is desirable to have a full error analysis for the new scheme in irregular domains.

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