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VORTEX SCHEMES FOR VISCOUS FLOW

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Abstract. We describe several vortex schemes that simulate viscous incompressible flow. We represent a deterministic scheme for the linearized Navier-Stokes equations, for which we outline the convergence proof in the discrete L_2 norm. This scheme approximates the linearized Navier-Stokes by first formulating them along particle trajectories and then approximating the viscous term via a discrete convolution of the vorticity with the Laplacian of a cutoff function. In the last section we introduce cutoff functions that satisfy moments conditions for semi-infinite domains.

1. Introduction

Vortex methods for nonviscous flow are based on Euler's equations in their vorticity formulation. One tracks particle trajectories, along which the vorticity is evolved. One also invokes the velocity-vorticity relation, the Biot-Savart law, which expresses the velocity in terms of the vorticity for incompressible flow.

Several ways were suggested to extend vortex methods for viscous flow, i.e., for the Navier-Stokes equations rather than Euler's equations. One of them is to change the size of the cutoff parameter [11],[4] to allow diffusion of vorticity. In fact, the exact solution of the heat equation via its Green function formulation was adopted. It was proven by Greengard [10] that this process approximates the wrong equations rather than the Navier-

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Stokes equations. However, if one redistribute the vorticity via its discrete convolution with the heat kernel together with time-splitting of the Navier-Stokes equations to Euler's and to the heat equations, the resulting scheme converges to the Navier-Stokes equations [6]. Chorin [5] suggested to simulate diffusion by adding a random walk to each of the particles. This is a robust algorithm, which does not require any spatial differentiation. This process was proved to converge to the exact solution [12].

In [8] we suggested a deterministic vortex method that extends naturally vortex methods for viscous flow. We formulate the Navier-Stokes equations along particle trajectories. In this formulation, a viscous term appears in the time-evolution of the vorticity. This term is approximated by first convolving the vorticity with a cutoff function and then by an analytic differentiation of this function, together with its discrete convolution with the vorticity.

A related method was suggested in [7]; in the latter the Laplacian operator was approximated by an integral type one. The analysis was carried out in two cases. In the first, the viscosity $1/R$ satisfies $1/R \leq C\delta^2$, where R is the Reynolds number and δ is the cutoff parameter of the kernel. In the second, the kernel of the integral operator which approximated the Laplacian is assumed to be positive.

We outline the convergence proof of the scheme for the linearized Navier-Stokes equation in two-dimensions. We first prove the convergence of a similar scheme for the heat equation, and then the consistency, the stability and the convergence of our scheme for the linearized Navier-Stokes equations. Finally we combine consistency and stability to prove the convergence of the scheme. One of the new features of this proof is the energy estimates we use for the vorticity. In order to insure the stability of the scheme, we assume that the cutoff function has a non-negative continuous Fourier transform. It was verified in [8] that this condition is satisfied for several cutoff functions that are commonly used with vortex methods. In [9] we also derive a stability condition for the time-discretized scheme for the linearized Navier-Stokes equation. We prove that if the cutoff function has a non-negative (continuous) Fourier transform and the time step is of order δ^2, then the scheme is stable.

In the last section we refer to one-sided cutoff functions. We suggest moment conditions for semi-infinite domains which ensure that the error committed by approximating a function via its one-sided convolution with a cutoff function is of order δ^d for any positive d that we pick. This approach may as well be useful for the approximation of discontinuous functions, where one wishes to construct an approximation based on points which are contained in regions for which the function is smooth.

2. A Vortex Scheme for Viscous Flows

In [8] we proposed a convolution-type vortex scheme for viscous flows. The two-dimensional Navier Stokes equations, formulated for the vorticity ξ are given below.

$$\partial_t \xi + (\mathbf{u} \cdot \nabla)\xi = R^{-1}\Delta\xi,$$

$$\text{div } \mathbf{u} = 0,$$

where $\xi = \text{curl } \mathbf{u}$, $\mathbf{u} = (u, v)$ is the velocity vector, $\Delta = \nabla^2$ is the Laplace operator and R is the Reynolds number. We formulate the Navier-Stokes equations along particle-trajectories together with the Biot-Savart law $\mathbf{u}(\mathbf{x}, t) = \int K(\mathbf{x} - \mathbf{x}')\xi(\mathbf{x}', t)d\mathbf{x}'$ where $K(x, y) = (-y, x)/2\pi(x^2 + y^2)$ and find

$$\frac{d\mathbf{x}}{dt} = \int K(\mathbf{x} - \mathbf{x}')\xi(\mathbf{x}', t)d\mathbf{x}', \tag{2.1}$$

$$\frac{d\xi}{dt} = R^{-1}\Delta\xi. \tag{2.2}$$

We set an initial uniform grid $\mathbf{x}_j(0), j = 1, ..., n$ with spacing h_1, h_2 in x, y respectively. For simplicity, we assume $h_1 = h_2 = h$. Let $\mathbf{x}_j^h(t), \xi_j^h(t)$ be the approximate particle locations and the approximate vorticity respectively at time t, then equation (2.1) is discretized by (see [5])

$$\frac{d\mathbf{x}_i^h(t)}{dt} = \sum_{j=1}^{n} K_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\xi_j^h(t)h^2.$$

Here we approximate the singular kernel $K(\mathbf{x})$ by a smoothed one $K_\delta(\mathbf{x})$, where $K_\delta = \phi_\delta * K$ and $\phi_\delta(\mathbf{x}) = (1/\delta^2)\phi(\mathbf{x}/\delta)$ is a cutoff function.

We shall now describe the approximation of the viscous term $R^{-1}\Delta\xi$ of (2.2). We approximate the vorticity by convolving it with a cutoff function, i.e., ξ is approximated by $\phi_\delta * \xi$. We then derive an approximation to the Laplacian of the vorticity by differentiating this convolution, i.e., by $\Delta(\phi_\delta * \xi) = (\Delta\phi_\delta) * \xi$. Finally, we approximate the integrals involved in the convolution by the trapezoid rule, and obtain

$$\frac{d\mathbf{x}_i^h(t)}{dt} = \sum_{j=1}^{n} K_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\xi_j^h(t)h^2, \tag{2.3}$$

$$\frac{d\xi_i^h(t)}{dt} = R^{-1}\sum_{j=1}^{n} \Delta\phi_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\xi_j^h(t)h^2. \tag{2.4}$$

We shall first discuss a convolution-type approximation for the heat equation, being a simplification of the linearized Navier-Stokes equation.

3. The Heat Equation

We prove the consistency and the stability of the scheme for the heat equation; this results in a convergent scheme. We treat the two-dimensional heat equation. The three-dimensional case can be treated similarly.

Consider

$$\frac{\partial \xi}{\partial t} = R^{-1}\Delta\xi,$$

$$\xi(\mathbf{x}, 0) = \xi_0(\mathbf{x}).$$

Suppose the initial vorticity has compact support and that $\mathbf{x}_i, i = 1, ..., n$ are equally distributed points in this region. h_1, h_1 are the spacing in x and y respectively, which for simplicity are assumed to be equal, i.e., $h_1 = h_2 = h$. Consider the scheme

$$\frac{\partial}{\partial t}\xi^h(\mathbf{x}_i, t) = R^{-1}\sum_j \Delta\phi_\delta(\mathbf{x}_i - \mathbf{x}_j)\xi^h(\mathbf{x}_j, t)h^2, \tag{3.1}$$

$$\xi^h(\mathbf{x}_i, 0) = \xi_0(\mathbf{x}_i),$$

where $\phi_\delta(\mathbf{x})$ is a cutoff function which approximates a delta function. We shall first state a theorem on the consistency of the scheme. For the proof of this theorem the reader is referd to [8].

Let $W^{m,p}$ be the Sobolev space which includes all functions for which the function and its derivatives up to order m are in L_p.

Consistency Theorem [8]. Let the cutoff function ϕ satisfy the following conditions.

$$\phi \in W^{m+2,1}(R^2), m \geq 1 \tag{3.2}$$

$$\int_{R^2} \phi(\mathbf{x})d\mathbf{x} = 1, \quad \int_{R^2} \mathbf{x}^\alpha \phi(\mathbf{x})d\mathbf{x} = 0, |\alpha| \leq d-1, \quad \int_{R^2} |\mathbf{x}|^d |\phi(\mathbf{x})| d\mathbf{x} < \infty. \tag{3.3}$$

Assume the initial vorticity has compact support and has m continuous bounded derivatived, where $m \geq d+2$. Let $\mathbf{x}_j, j = 1, ..., n$ be unifomly distributed grid points in this region. Then, there exists a constant C such that

$$|\xi(\mathbf{x},t) - \sum_{j=1}^{n} \Delta\phi_\delta(\mathbf{x}-\mathbf{x}_j)\xi_j h^2| \leq C(\delta^d + \frac{h^m}{\delta^{m+2}}).$$

Stability Theorem. Let $\phi \in W^{2,1}(R^2)$ and let the Fourier transform of the cutoff function be non-negative,

$$\hat{\phi}(\mathbf{s}) \geq 0, \tag{3.4}$$

and assume the initial vorticity has compact support. Then (3.1) is stable, i.e.,

$$\sum_i (\xi^h(\mathbf{x}_i,t))^2 h^2 \leq \sum_i (\xi^h(\mathbf{x}_i,0))^2 h^2.$$

Proof. Multiplying (3.1) by $\xi^h(\mathbf{x}_i,t)h^2$ and summing over i yields

$$\frac{1}{2}\frac{\partial}{\partial t}\sum_i (\xi^h(\mathbf{x}_i,t))^2 h^2 = R^{-1}\sum_i \xi^h(\mathbf{x}_i,t)h^2 \sum_j \Delta\phi_\delta(\mathbf{x}_i-\mathbf{x}_j)\xi^h(\mathbf{x}_j,t)h^2. \tag{3.5}$$

Expressing $\phi(\mathbf{x})$ via its Fourier transform, one finds $\phi_\delta(\mathbf{x}) = \int \hat{\phi}_\delta(\mathbf{s})e^{i\mathbf{s}\cdot\mathbf{x}}d\mathbf{s}$. Differentiating the last equality with respect to x and y to get the Laplacian of ϕ_δ yields

$$\Delta\phi_\delta(\mathbf{x}) = -\int (\mathbf{s}\cdot\mathbf{s})\hat{\phi}_\delta(\mathbf{s})e^{i\mathbf{s}\cdot\mathbf{x}}d\mathbf{s}. \tag{3.6}$$

We now substitute (3.6) in (3.5) and find

$$\frac{1}{2}\frac{\partial}{\partial t}\sum_i (\xi^h(\mathbf{x}_i,t))^2 h^2 = -R^{-1}\int (\mathbf{s}\cdot\mathbf{s})\sum_i \xi^h(\mathbf{x}_i,t)h^2 \sum_j \hat{\phi}_\delta(\mathbf{s})e^{i\mathbf{s}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\xi^h(\mathbf{x}_j,t)h^2 d\mathbf{s},$$

or

$$\frac{1}{2}\frac{\partial}{\partial t}\sum_i (\xi^h(\mathbf{x}_i,t))^2 h^2 = R^{-1}\int (\mathbf{s}\cdot\mathbf{s})\hat{\phi}_\delta(\mathbf{s})\sum_i \xi^h(\mathbf{x}_i,t)e^{i\mathbf{s}\cdot\mathbf{x}_i}h^2 \sum_j \xi^h(\mathbf{x}_j,t)e^{-i\mathbf{s}\cdot\mathbf{x}_j}h^2 d\mathbf{s}.$$

Since ξ^h is a real function (otherwise we multiply (3.1) by the complex conjugate of ξ^h), we find that

$$\frac{1}{2}\frac{\partial}{\partial t}\sum_i(\xi^h(\mathbf{x}_i,t))^2h^2 = R^{-1}\int(\mathbf{s}\cdot\mathbf{s})\hat{\phi}_\delta(\mathbf{s})\,|\sum_i \xi^h(\mathbf{x}_i,t)e^{i\mathbf{s}\cdot\mathbf{x}_i}h^2|^2 ds. \tag{3.7}$$

The right-hand side of the last equality is non-positive by assumption (3.4), and since $\hat{\phi}_\delta(\mathbf{s}) = \hat{\phi}(\delta\mathbf{s})$. Integrating (3.7) with respect to t completes the proof.

4. The Linearized Navier-Stokes Equations

We shall consider the linearized Navier Stokes equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{a}(\mathbf{x},t), \tag{4.1}$$

$$\frac{d\xi}{dt} = R^{-1}\Delta\xi, \tag{4.2}$$

with $\nabla\cdot\mathbf{a} = 0$.

The proposed scheme for the linearized equations is

$$\frac{d\mathbf{x}_i(t)}{dt} = \mathbf{a}(\mathbf{x}_i,t), \tag{4.3}$$

$$\frac{d\xi_i^h(t)}{dt} = R^{-1}\sum_{j=1}^n \Delta\phi_\delta(\mathbf{x}_i(t)-\mathbf{x}_j(t))\xi_j^h(t)h^2, \tag{4.4}$$

where $\xi_i^h(t) = \xi^h(\mathbf{x}_i(t),t)$ is the approximated vorticity. Here we have used the incompressibility condition $\nabla\cdot\mathbf{a} = 0$ to assure that an area-element h^2 at the initial time remains the same for all times. We shall now outline the convergence of our scheme (4.3)-(4.4) to the linearized Navier-Stokes equations (4.1)-(4.2). Expressing (4.1) and (4.2) at $\mathbf{x}_i$, we find

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{a}(\mathbf{x}_i,t). \tag{4.5}$$

$$\frac{d\xi_i}{dt} = R^{-1}\Delta\xi_i. \tag{4.6}$$

Note that there is no error in particle locations, and we can therefore proceed to the error in the vorticity at the time-dependent grid points. Thus, subtracting (4.6) from (4.4), we find

$$\frac{d(\xi_i^h(t)-\xi_i(t))}{dt} = R^{-1}\sum_{j=1}^n \Delta\phi_\delta(\mathbf{x}_i(t)-\mathbf{x}_j(t))\xi_j^h(t)h^2 - R^{-1}\Delta\xi_i(t). \tag{4.7}$$

Since we want to use energy-type estimates, we multiply and (4.7) by $(\xi_i^h(t)-\xi_i(t))h^2$ and sum over i. We assume the initial vorticity has compact support, and therefore finite number of particles approximates the initial vorticity. For the sake of simplicity we omit (t) from now on.

$$\frac{1}{2}\frac{d}{dt}\sum_i(\xi_i^h-\xi_i)^2h^2 = R^{-1}\sum_i(\xi_i^h-\xi_i)h^2(\sum_j \Delta\phi_\delta(\mathbf{x}_i-\mathbf{x}_j)\xi_j^h h^2 - \Delta\xi_i). \tag{4.8}$$

Rewrite (4.8) as

$$\frac{1}{2}\frac{d}{dt}\sum_i(\xi_i^h-\xi_i)^2h^2 = S1 + C1, \tag{4.9}$$

where

$$S1 = R^{-1}\sum_i(\xi_i^h-\xi_i)h^2(\sum_j \Delta\phi_\delta(\mathbf{x}_i-\mathbf{x}_j)\xi_j^h h^2 - \sum_j \Delta\phi_\delta(\mathbf{x}_i-\mathbf{x}_j)\xi_j h^2) \tag{4.10}$$

and

$$C1 = R^{-1}\sum_i(\xi_i^h-\xi_i)h^2(\sum_j \Delta\phi_\delta(\mathbf{x}_i-\mathbf{x}_j)\xi_j h^2 - \Delta\xi_i). \tag{4.11}$$

We shall now bound $C1$, which is related with the truncation error. Later, in the stability lemma, we shall bound $S1$.

4. Consistency, Stability and Convergence for the Linearized Navier-Stokes

Define the discrete norm $\|f\|_{0,2,h}^2 = \sum_i f^2(\mathbf{x}_i)h^2$.

Consistency Lemma for the Linearized Navier-Stokes Equations [9]. Let the cutoff function ϕ satisfy the following conditions.

$$\phi \in W^{m+2,1}(R^2), m \geq 1 \tag{5.1}$$

$$\int_{R^2}\phi(\mathbf{x})d\mathbf{x} = 1, \quad \int_{R^2}\mathbf{x}^\alpha\phi(\mathbf{x})d\mathbf{x} = 0, |\alpha| \leq d-1, \quad \int_{R^2}|\mathbf{x}|^d|\phi(\mathbf{x})|d\mathbf{x} < \infty. \tag{5.2}$$

Assume the initial vorticity has compact support and has m continuous bounded derivatived, where $m \geq d+2$. Let $\mathbf{x}_j(0), j = 1, ..., n$ be unifomly distributed grid points in this region. Then, there exist a constant C such that

$$C1 \leq CR^{-1}\|\xi^h - \xi\|_{0,2,h}(\delta^d + \frac{h^m}{\delta^{m+2}}). \tag{5.3}$$

Stability Lemma for the Linearized Navier-Stokes Equations [9]. Let

$$A^2 = \sum_i (\xi_i^h - \xi_i)^2 h^2,$$

and let $\phi \in W^{2,1}(R^2)$. Assume further that the Fourier transform of the cutoff function be non-negative, i.e., $\hat{\phi}(s) \geq 0$. Then

$$S1 \leq 0. \tag{5.4}$$

We shall now combine the consistency and the stability theorems to prove the convergence of the scheme.

Convergence Theorem for the Linearized Navier-Stokes Equations. Let the cutoff function ϕ satisfy the following conditions.

$$\phi \in W^{m+2,1}(R^2), m \geq 1 \tag{5.5}$$

$$\int_{R^2} \phi(\mathrm{x})d\mathrm{x} = 1, \quad \int_{R^2} \mathrm{x}^\alpha \phi(\mathrm{x})d\mathrm{x} = 0, |\alpha| \leq d-1, \quad \int_{R^2} |\mathrm{x}|^d |\phi(\mathrm{x})| d\mathrm{x} < \infty. \tag{5.6}$$

Let the Fourier transform of the cutoff function be non-negative, then $A \leq C(\delta^d + \frac{h^m}{\delta^{m+2}})$

Proof. By (4.9)

$$\frac{1}{2}\frac{dA^2}{dt} = C1 + S1.$$

Using the consistency lemma (5.3), we find that for some bounded range of R^{-1}, i.e., $R \geq R_0$, where R_0 is some positive number,

$$C1 \leq C \cdot A(\delta^d + \frac{h^m}{\delta^{m+2}}). \tag{5.7}$$

By the stability lemma for the the linearized Navier-Stokes equations (5.4) we find $S1 \leq 0$, and therefore

$$\frac{dA^2}{dt} \leq C \cdot A(\frac{h^m}{\delta^{m+2}} + \delta^d). \tag{5.8}$$

Deviding (5.8) by $A = +\sqrt{A^2}$ (if $A = 0$ the proof is trivial) we find $\frac{dA}{dt} \leq C(\delta^d + h^m/\delta^{m+2})$, with the initial condition $A(t=0) = 0$. We therefore conclude that $A \leq C(\delta^d + h^m/\delta^{m+2})$.

In [9] we prove the stability of the time-discretized scheme

$$\mathbf{x}_i^{h,n+1} = \mathbf{x}_i^{h,n} + \Delta t \mathbf{a}(\mathbf{x}_i^{h,n}, t), \tag{5.9}$$

$$\xi_i^{h,n+1} = \xi_i^{h,n} + \Delta t R^{-1} \sum_{j=1}^{n} \Delta\phi_\delta(\mathbf{x}_i^{h,n} - \mathbf{x}_j^{h,n}) \xi_j^{h,n} h^2. \tag{5.10}$$

where $\mathbf{x}_i^{h,n}$ and $\xi_i^{h,n}$ approximate the exact particle location $\mathbf{x}_i^n$ and the exact vorticity ξ_i^n. We show that

$$|\mathbf{x}_i^{h,n} - \mathbf{x}_i^n| \leq C\Delta t, \quad 0 \leq t \leq T.$$

If we further assume that ϕ is a symmetric cutoff that satisfies (3.4), and $\delta = Ch^q$, $0 < q \leq 1$, then (5.9)-(5.10) is stable for $\Delta t \leq CR\delta^2$.

6. Convolution-Type Schemes and Semi-Infinite Domains

In this section, I would like to remark on the application of convolution-type schemes to semi-infinite domains. If the differential equation that we would like to solve holds in a semi-infinite domain, one need to consider the quality of the approximation we make for a given function by convolving it with a cutoff function. In other words, does $\phi_\delta * f$ well approximate the function f in case this function is defined on a semi-infinite domain? The answer is of course that it does not necessarily do. For the sake of simplicity, let us consider the one-dimensional case, i.e., $f(x)$ is defined for $x \geq 0$.

It is possible to formulate moment conditions for the cutoff function ϕ_δ in order to obtain an approximation of order δ^d for any constant d. We wish to approximate $f(x)$ by $\int_x^\infty f(x')\phi_\delta(x - x')dx'$. By substituting $y = x - x'$ we have instead $\int_{-\infty}^0 f(x - y)\phi_\delta(y)dy$. Once the approximation to a function is constructed via one-sided cutoff functions, one may also derive approximations to derivatives of this function. This in turn can be used for approximating the Laplacian of a given function based on one-sided cutoff functions. It may also be applied to problems whose solution contain a shock or a discontinuity, in which it is desirable to avoid differentiations across the discontinuity. We will show the following.

Moment Lemma for Semi-infinite Domain. Let the cutoff function ϕ satisfy the following conditions.

$$\int_{-\infty}^{0} \phi(x)dx = 1 \tag{6.1}$$

$$\int_{-\infty}^{0} x^\alpha \phi(x)dx = 0, \quad 1 \le \alpha \le d-1, \tag{6.2}$$

$$\int_{-\infty}^{0} |x|^d |\phi(x)|dx < \infty. \tag{6.3}$$

Assume that $f(x)$ is defined on $0 \le x < \infty$ and is countinuous and bounded together with its first d derivatives in this domain. Then

$$|f(x) - \int_{-\infty}^{0} f(x-y)\phi_\delta(y)dy| \le C\delta^d,$$

where $\phi_\delta = \frac{1}{\delta}\phi(x/\delta), \quad -\infty < x \le 0.$

Proof. Expanding $f(x-y)$ in Taylor series around x yields

$$f(x-y) = f(x) + \sum_{\alpha=1}^{d-1} \frac{(-1)^\alpha}{\alpha!} f^{(\alpha)}(x)y^\alpha + \frac{(-1)^d}{(d-1)!}\int_0^1 (1-t)^{d-1} f^{(d)}(x-ty)y^d dt.$$

We therefore find that

$$\int_{-\infty}^{0} f(x-y)\phi_\delta(y)dy = f(x)\int_{-\infty}^{0}\phi_\delta(y)dy + \sum_{\alpha=1}^{d-1}\frac{(-1)^\alpha}{\alpha!} f^{(\alpha)}(x)\int_{-\infty}^{0} y^\alpha \phi_\delta(y)dy +$$

$$+\frac{(-1)^d}{(d-1)!}\int_0^1 (1-t)^{d-1}dt \int_{-\infty}^{0} f^{(d)}(x-ty)y^d\phi_\delta(y)dy. \tag{6.4}$$

By (6.1)

$$\int_{-\infty}^{0}\phi_\delta(y)dy = \int_{-\infty}^{0}\phi(z)dz = 1.$$

Similarly, by (6.2)

$$\int_{-\infty}^{0} y^\alpha \phi_\delta(y)dy = 0, \quad 1 \le \alpha \le d-1,$$

We now proceed to the last term in (6.4). Substituting $z = ty$ in this last term yields

$$\frac{(-1)^d}{(d-1)!}\int_0^1 \frac{(1-t)^{d-1}}{t^{d+1}}\int_{-\infty}^{0} f^{(d)}(x-z)z^d\phi_\delta(z/t)dz. \tag{6.5}$$

By assumption $|f^{(d)}(x-z)|$ is bounded for every x and z, and therefore the term above can be bounded by

$$C_d \int_0^1 \frac{1}{t^{d+1}} dt \int_{-\infty}^0 |z|^d |\phi_\delta(z/t)| dz,$$

where C_d is a constant which depends on f and d. Noting that $\phi_\delta(z/t) = (1/\delta)\phi(z/\delta t)$ and substituting $y = z/\delta t$, we find that the last term in (6.4) is bounded by

$$C_d \delta^d \int_{-\infty}^0 |y|^d |\phi(y)| dy.$$

But by (6.3) the last integral is bounded, and therefore

$$|\int_{-\infty}^0 f(x-y)\phi_\delta(y)dy - f(x)| \leq C_d \delta^d.$$

Remark 1. It is possible to prove that the moment error is bounded by $C\delta^d$ in the L_p norm, provided the moments conditions are satisfied for ϕ and the function f is in $W^{d,p}$. For the proof, one should proceed from (6.4) and use Young's inequality $\|g * h\|_{L_p} \leq \|g\|_{L_p}\|h\|_{L_1}$, which can be verified for semi-infinite domains by following the lines of the proof for an infinite domain. We apply Young's inequality to $g(x) = f^{(d)}(x)$ and $h(x) = x^d \phi_\delta(x/t)$, whereas for $\|g\|_{L_p}$ the integral is taken from zero to infinity and for $\|h\|_{L_p}$ from $-\infty$ to zero. We therefore find that

$$\|f(x) - \int_{-\infty}^0 f(x-y)\phi_\delta(y)dy\|_{L_p} \leq C\delta^d \|f\|_{d,p}.$$

Remark 2. One can also construct cutoff functions on the half plane $y \geq 0$ in a similar way. The approximation we use for $f(\mathbf{x})$ is $\int\int_\Omega f(\mathbf{x}-\mathbf{y})\phi_\delta(\mathbf{y})d\mathbf{y}$, where Ω is the set of points for which $\mathbf{y} = (rcos\theta, rsin\theta)$, $\pi \leq \theta \leq 2\pi$, $0 \leq r < \infty$. The moment conditions are very similar to the one-dimensional ones, the difference being that the integrals in (6.1)-(6.3) are taken over Ω.

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