VORTEX METHODS FOR SLIGHTLY VISCOUS THREE DIMENSIONAL FLOW

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Abstract. We represent a three-dimensional vortex scheme which is a natural extension of the two-dimensional ones, in which spatial derivaties are evaluated by exatly differentiating an approximated velocity field . Numerical results are presented for a flow past a semi-infinite plate, and they demonstrate transition to turbulence. We also suggest a new way to treat the viscous term. The idea is to approximate the vorticity by convolving it with with a cutoff function. We then explicitly differentiate the cutoff function to approximate the second order spatial derivatives in the viscous term.

1. Introduction

Vortex methods have been used extensively to simulate incrmpressible flow, especially for two dimensional problems. Though three dimensional vortex methods have been considered inherently difficult, we represent a scheme that involves no elaborate computations and is a natural extension of the two dimensional schemes. We applied this method to a three dimensional flow past a semi-infinite plate at high Reynolds number. Chorin [5] solved the two dimensional problem numerically; he used computational elements, called blobs, with smoothed kernel. He also intoduced a solution to a three dimensional problem using a filament method. Chorin [5] approximates vortex lines by segments and then, using the Biot-Savart law, he updates the endpoints of the segments every time step. Chorin's algorithm involves no elaborate calculations, however it is not highly accurate in space. The purpose of this paper is to modify Chorin's scheme to gain higher spatial accuracy.

Following Beale and Majda [3] and Anderson [1], we achieve higher spatial accuracy by generalizing the two dimensional blobs to three-dimensional ones. Vorticity as well as blob locations must be updated every time step. We applied, for the first time, the method of Anderson, which explicitly differentiates the smoothed kernel to approximate spatial derivatives. The algorithm and its accuracy is then similar to the two-dimensional one. Applying the convergence proofs [2],[4] to our scheme, we show that for smooth cutoff functions second order accuracy in space is gained. Higher order space accuracy can be achieved by using higher order cutoff functions. We were able to resolve three dimensional features of the flow and transition to turbulence. The numerical results are in agreement with experimental results shown in [6], which suggest that at high Reynolds numbers there exist a large number of small hairpins.

We also suggest a new way to treat the viscous term. The idea is to approximate the vorticity by convolving it with a cutoff function, and then approximate the second order derivatives in the viscous term by explicitly differentiating the cutoff the function. Numerical results performed for the Stokes equation demonstrate the accuracy and convergence of the this scheme.

2. Representation of the Problem

The flow is described by the Navier Stokes equations, formulated for the vorticity ξ:

$$\begin{aligned} &\partial_t \xi + (\mathbf{u} \cdot \nabla)\xi - (\xi \cdot \nabla)\mathbf{u} = R^{-1}\Delta\xi, \\ &\text{div } \mathbf{u} = 0, \end{aligned} \tag{2.1}$$

where $\xi = \text{curl } \mathbf{u}$, $\mathbf{u} = (u, v, w)$ is the velocity vector and Δ is the Laplace operator and R is the Reynolds number. We solve the above equations for a flow past a semi-infinite flat plate. Far away from the plate there is a uniform flow. On the plate we impose the no-leak and the no-slip boundary conditions. As was suggested by physical experiments, we assume that the flow is periodic in the spanwise direction.

The Prandtl equations approximate the Navier-Stokes equations near the plate, and are used therefore in a thin layer near the plate. These equations equations admite the two-dimensional steady state solution - the Blasius solution. However, the three-dimensional Navier-Stokes equations are unstable at high Reynolds numbers($R \geq 1000$), i.e., small perturbations in the Blasius solution may cause large perturbations in the solution as time progresses. The experiments of Head and Bandyapodhyay [6] for high Reynolds numbers indicate the existence of large number of vortex pairs or hairpin vortices, extending through at least a substantial part of the boundary-layer thickness.

3. The Numerical Scheme

We split the Navier-Stokes equations into the Euler equations and the heat equation and apply a Strang-type second order scheme to step the Navier-Stokes equations in time. We first describe the scheme for Euler's equations. These equations can be writen as a set of two ordinary differential equations:

$$\begin{aligned} \frac{d\mathbf{x}}{dt}(\alpha,t) &= \mathbf{u}(\mathbf{x}(\alpha,t),t) \quad , \quad \mathbf{x}(\alpha,0)=\alpha, \\ \frac{d\xi}{dt}(\mathbf{x}(\alpha,t),t) &= (\xi(\mathbf{x}(\alpha,t),t)\cdot\nabla)\mathbf{u}(\mathbf{x}(\alpha,t),t), \end{aligned} \tag{3.1}$$

where $\mathbf{x}(\alpha,t)$, where is a particle trajectory. We also use the relation $\mathbf{u}(\mathbf{x},t)=\int K(\mathbf{x}-\mathbf{x}')\xi(\mathbf{x}',t)d\mathbf{x}'$ between vorticity and velocity for incompressible flow to determine the velocity in (3.1). Here K is a matrix valued function given by $K(\mathbf{x})=-\frac{\mathbf{x}}{4\pi|\mathbf{x}|^3}\times$, where $\times$ represent vector product.

To discretize (3.1), we set $\xi=\sum_j\xi_j$, where the ξ_j are functions of small support. Let κ_j be the intensity of the $j-th$ particle, i.e., $\kappa_j=\int\xi_j dxdydz$. Then we obtain the following set of ordinary differential equations for the approximate locations of the particles $\tilde{\mathbf{x}}_i$ and the approximate intensities $\tilde{\kappa}_i$.

$$\begin{aligned} \frac{d\tilde{\mathbf{x}}_i}{dt}(t) &= \sum_{j=1}^{n} K_\delta(\tilde{\mathbf{x}}_i(t)-\tilde{\mathbf{x}}_j(t))\tilde{\kappa}_j(t) \\ \frac{d\tilde{\kappa}_i}{dt} &= \sum_{j=1}^{n}(\tilde{\kappa}_i\cdot\nabla\mathbf{x})K_\delta(\tilde{\mathbf{x}}_i(t)-\tilde{\mathbf{x}}_j(t))\tilde{\kappa}_j(t), \end{aligned} \tag{3.2}$$

where $\phi:R^3\to R$, $\phi_\delta=\frac{1}{\delta^3}\phi(\mathbf{x}/\delta)$ is the cutoff function, and $K_\delta=K*\phi_\delta$ is a smoothed kernel. In our calculations we chose $K_\delta(\mathbf{x})=K(\mathbf{x})f_\delta(\mathbf{x})$, where $f_\delta(\mathbf{x})=f(\mathbf{x}/\delta)$. $f(x)$ is radially symmetric and given by $f(r)=1$ for $r\geq 1$ and $f(r)=2.5r^3-1.5r^5$ for $r<1$. This function is continuous with its first derivative at $r=1$.

The second equation to solve is the heat equation $\frac{\partial\xi}{\partial t}=R^{-1}\Delta\xi$. Following Chorin [5] we use the random-walk method to step the heat equation in time, i.e., we move the blobs according to $\tilde{\mathbf{x}}_i^{n+1}=\tilde{\mathbf{x}}_i^n+\eta(\Delta t)$, where $\eta(\Delta t)=(\eta_1(\Delta t),\eta_2(\Delta t),\eta_3(\Delta t))$ and η_1,η_2,η_3 are Gaussian random variables with mean zero and variance $2\Delta t/R$, chosen independently of each other. The Prandtl Equations used in a thin layer $0\leq z\leq z_0$ above the plate, were solved numerically by the tile method, which is the three-dimensional extention of the sheet method (see [5]). This was done to evaluate the boundary conditions on the plate, since blobs did not accurately represrnt the velocity field near the plate.

4. Convergence

Convergence for three-dimensional blob-vortex methods was first proved by Beale[2], and then, using a different approach, by Cottet[4]. Let us first define the Sobolev spaces

$W^{m,p} = \{f, \partial^\alpha f \in L^p(R^n), |\alpha| \le m\}$ and the norm $\|f\|^p_{m,p} = \sum_{0\le|\alpha|\le m} \|\partial^\alpha f\|^p_{0,p}$. Cottet proved that if $\phi \in W^{m,\infty}(R^3) \cap W^{m,1}(R^3)$, $\forall m > 0$,

$$\int_{R^3} \phi(\mathbf{x})d\mathbf{x} = 1, \int_{R^3} \mathbf{x}^\alpha \phi(\mathbf{x})d\mathbf{x} = 0, |\alpha| \le d-1, \int_{R^3} |\mathbf{x}|^d \phi(\mathbf{x})dx < \infty, \qquad (4.1)$$

and if there exist constants C and $\beta > 1$ such that $h \le C\delta^\beta$, then for h and δ small enough

$$\|\tilde{u} - u\|_{0,p} \le C\delta^d, \quad p \in (3/2, \infty], \quad t \in [0, \tau].$$

We now apply this theorem to our scheme. Using the relation $\phi(r) = f'(r)/4\pi r^2$, we find that $\phi(r)$ satisfies (4.1) with $d = 2$. In addition, if one chooses the cutoff function ϕ to be infinitely smooth, second order accuracy is achieved. In case that the the cutoff function lies in a Sobolev space for finite m only, Raviat [7, pp. 315] have proved for a two-dimensional problem that if (4.1) holds and $\phi \in W^{m-1,\infty}(R^2)$ for $m \ge 2$ and has compact support, then for all arbitrarily small $s > 0$ there exists a constant C_s, such that $\|\tilde{\mathbf{u}} - \mathbf{u}\|_{0,\infty} \le C_s\delta^{-s}(\delta^d + h^m/\delta^{m-1})$, provided that $c_2^{-1}\delta^\alpha \le h \le c_1\delta^\beta$ with $\alpha \ge \beta > 1$. In our case $\phi \in W^{1,\infty}(R^3)$ and has compact support, then for $\delta = Ch^{2/3}$ the error is at most of order $h^{4/3}$. This can be improved by choosing an infinitely smooth cutoff function. The accuracy of the random-walk algorithm was estimated by Hald for a one-dimensional problem. He proved that the L_2 error decayes like $N^{-1/2}$, where N is the number of tiles.

5. A New Scheme for Viscous Flows

To increase the accuracy of the approximation for the viscous part, we propose the following scheme. For simplicity, we describe the scheme for the two-dimensional case, though it can be easily applied for a three-dimensional problem as well. We follow the charachteristic lines $\frac{d\mathbf{x}}{dt} = \mathbf{u}$, along which the vorticity evolution is given by $\frac{d\xi}{dt} = R^{-1}\Delta\xi$. We approximate the vorticity by convolving it with a cutoff function, and then explicitely differentiate the cutoff function to approximate the second order derivatives appearing in the viscous term. Finally we approximate the integral involved in the convolution by the trapezoid rule and obtain

$$\frac{d\tilde{\mathbf{x}}_i}{dt}(t) = \sum_{j=1}^{n} K_\delta(\tilde{\mathbf{x}}_i(t) - \tilde{\mathbf{x}}_j(t))\tilde{\kappa}_j(t),$$

$$\frac{d\tilde{\kappa}_i}{dt} = R^{-1}\sum_{j=1}^{n} \Delta\phi_\delta(\tilde{\mathbf{x}}_i(t) - \tilde{\mathbf{x}}_j(t)\tilde{\kappa}_j(t).$$

This eliminates the error caused by the statistical process, and yields a scheme which is similar in nature to that applied to the Euler equations. Numerical results performed for the Stokes equation demonstrate the accuracy of the scheme within three significant digits for a coarse initial grid.

6. Numerical Results

We display all the results at $t = 22.5, R = 10000$. In Figure 1 the x, z components of vorticity at the Lagrangian computational points at $y = q/2$ is displayed, where q is the period in y. One can see that for larger x the intensity of the vorticity increases, which is one of the features of transition to turbulence. In Figure 2 we show the y, z components of vorticity at 1.4. Note that for large x the vorticity is no longer directed in one direction. This is in agreement with the results in [6], which indicate the appearance of small hairpins as the flow develops in the streamwise direction. In figure 3 we show the velocity field at $y = q/2$ and in Figure 4 vorticity contours at $x = 1.4$. The computational time to reach $t = 22.5$ on a CRAY X-MP is two hours and thirteen minutes.

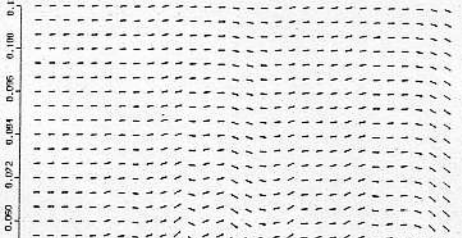

Figure 1. Vorticity in the x, z plane for $y = q/2$, R=10000.

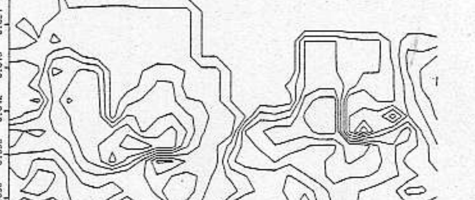

Figure 2. Vorticity in the y, z plane for $x = 1.4$, R=10000.

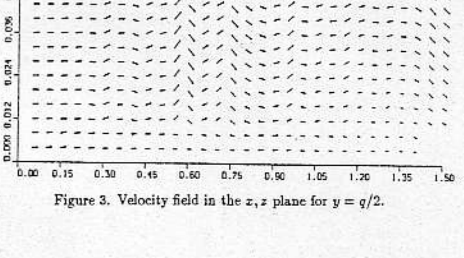

Figure 3. Velocity field in the x, z plane for $y = q/2$.

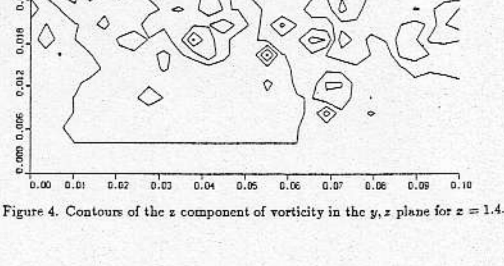

Figure 4. Contours of the x component of vorticity in the y, z plane for $x = 1.4$.

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