A CONVERGENT PARTICLE SCHEME FOR CONVECTION-DIFFUSION EQUATION

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Abstract. In this paper we prove the convergence of the convolution-type vortex scheme [15] for the convection-diffusion equation in two dimensions. This scheme approximates the convection-diffusion equation by first formulating it along particle trajectories and then approximating the viscous term via a discrete convolution of the vorticity with the Laplacian of a cutoff function. We also derive stability condition for the time-discretized scheme and prove its convergence.

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AMS(MOS) Subject Classifications: 65D25,76D05, 35Q10, 35K05.

1. Introduction

Vortex methods are numerical schemes for the simulation of incompressible Euler and Navier-Stokes equations. In 1932 Rosenhead [30] suggested to track the evolution of vorticity along particle-trajectories for Euler's equations. In this formulation the velocity of the particles is expressed by the Biot-Savart law, which relates the velocity and the vorticity via a singular kernel. Chorin [8] (1973) proposed a smoothing of this singular kernel to stabilize the numerical scheme for a longer time-interval. Since then, a large body of numerical and theoretical work related to vortex methods has been carried out (see for example [26], [28] for a review). The smoothing of the singular kernel was generalized by Hald and Del Prete [19] via the convolution of the kernel with a cutoff function, which approximates a delta function in the sense of moments.

The convergence of vortex methods for Euler's equations was first proved by Hald and Del Prete [19], and subsequently generalized by Beale and Majda [4],[5] to more general cutoff functions and three-dimensional problems. Another proof for the convergence of two-dimensional vortex methods was presented by Raviart [29], and by Cottet [10] for two and three-dimensions. Anderson and Greengard [1] suggested a three-dimensional vortex scheme which evolves the vorticity along particle trajectories via an explicit differentiation of the smoothed kernel. The convergence of this scheme was proved in [2] and [10]; numerical tests were performed in [14]. The convergence of the point-vortex method, which leaves the singular kernel unmodified, was proved in [17]. Hou [23] also suggested a new desingularization of the kernel in order to stabilize the method for longer time intervals.

Several ways were suggested to extend vortex methods to viscous flows, i.e., for the Navier-Stokes equations rather than Euler's equations. One of them is to change the size of the cutoff parameter [26] to allow diffusion of vorticity. In fact, the exact solution of the heat equation via its Green function formulation was adopted. It was proved by Greengard [18] that this process approximates the wrong equations rather than the Navier-Stokes equations. However, if one redistributes the vorticity via its discrete convolution with the heat kernel together with time-splitting of the Navier-Stokes equations to Euler's and to the heat equation, the resulting scheme converges to the Navier-Stokes equations [11]. Chorin [8] suggested to simulate diffusion by adding a random walk to each of the particles; this process was also proved to converge to the exact solution [16],[27]. It was proved in [27] that for the two-dimensional case, the error in the random vortex method is of order $h|\ln h|$, where h is the initial spacing between two neighboring points. Goodman [16] had shown that the error tends to zero as the Reynolds number R tends to infinity, the rate being of order $R^{-1/2}$. The random process has also the advantage that it is easy to apply near boundaries.

In [15] we suggested a deterministic vortex method that extends naturally vortex methods for viscous flow. We formulate the Navier-Stokes vorticity equations along particle trajectories, in which a viscous term appears in the time-evolution of the vorticity. This term is approximated by first convolving the vorticity with a cutoff function and then by an analytic differentiation of this function, together with its discrete convolution with the vorticity. A related method was suggested in [12]; in the latter the Laplacian operator was approximated by an integral type one. The analysis in [12] was carried out for two cases. In the first case the viscosity $1/R$ satisfies $1/R \le C\delta^2$, where R is the Reynolds number and δ is the cutoff parameter of the kernel. In the second case the kernel of the integral operator which approximates the Laplacian is assumed to be positive. In [15] we have proven the consistency of the scheme for the heat and the Navier-Stokes equations. The stability of the scheme in its spatial-continuous form was proved for the heat equation.

The purpose of this paper is to establish a convergence proof for the scheme for the convection-diffusion equation in two-dimensions. Unlike the proof in [15], we consider here the fully spatially discretized scheme, whereas in [15] we analyzed a simplified case where the convolutions involved in the scheme are represented in a continuous fashion. The convergence proof here is based on the consistency and the stability of the scheme; we have used moment properties of the cutoff function for the consistency of the scheme and the non-negativeness of its continuous Fourier transform for the stability proof. This condition is indeed satisfied for several cutoff functions which are commonly used in the context of vortex methods; for examples see [15]. One of the new features of this proof is the energy-type estimates we have used for the vorticity, since with the energy norm we could prove the non-positiveness of the operator associated with the viscous term.

We also formulate a stability condition for the time-discretized scheme, where we have shown that the time step has to be of order $R\delta^2$; here R is the Reynolds number and δ is the cutoff parameter. The convergence of the time discretized scheme is proved as well, using an asymptotic expansion of the particles location error.

The paper is organized as follows. In section 2 the convolution-type scheme for the Navier-Stokes equations is presented. In section 3 we formulate our scheme for the convection-diffusion equation, and decompose the error in the vorticity to the consistency and stability errors. In sections 4 and 5 we prove the consistency, the stability and the convergence of the proposed scheme to the convection-diffusion equation. In section 6 we derive a stability condition for the Euler's-type time-discretization and show that the time step is of order of $R\delta^2$; in section 7 we prove the convergence of the time discretized scheme.

2. A Vortex Scheme for Viscous Flows

In [15] we proposed a convolution-type vortex scheme for viscous flows. We shall review the scheme for the two-dimensional case in this section. The two-dimensional Navier Stokes equations, formulated for the vorticity ξ are given below.

$$\frac{\partial \xi}{\partial t} + (\mathbf{u} \cdot \nabla)\xi = R^{-1}\Delta\xi,$$

$$\text{div } \mathbf{u} = 0,$$

where $\xi = \text{curl } \mathbf{u}$, $\mathbf{u} = (u, v)$ is the velocity vector, Δ is the Laplace operator and R is the Reynolds number. We follow the characteristic lines

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}, \tag{2.1}$$

along which the vorticity evolution is described by the following differential equation

$$\frac{d\xi}{dt} = R^{-1}\Delta\xi. \tag{2.2}$$

Here $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ denotes the total derivative, i.e., the derivative along particle trajectories. In addition, if the velocity vanishes at infinity, then for an incompressible flow we may write it in terms of the vorticity [9] via the Biot-Savart law; the latter holds in case the vorticity has compact support, or in turn, in two dimensions it decayed as $|\mathbf{x}|^{-2}$ as $|\mathbf{x}|$ tends to infinity. Thus,

$$\mathbf{u}(\mathbf{x}, t) = \int K(\mathbf{x} - \mathbf{x}')\xi(\mathbf{x}', t)d\mathbf{x}', \tag{2.3}$$

where for the two-dimensional case $K(x, y) = (-y, x)/2\pi r^2$, and $r^2 = x^2+y^2$. Substituting (2.3) in (2.1), one obtains the following system of ordinary differential equations.

$$\frac{d\mathbf{x}}{dt} = \int K(\mathbf{x} - \mathbf{x}')\xi(\mathbf{x}', t)d\mathbf{x}'. \tag{2.4}$$

$$\frac{d\xi}{dt} = R^{-1}\Delta\xi. \tag{2.5}$$

We set an initial uniform grid $\mathbf{x}_j(0)$ which cover R^2 with spacing h_1, h_2 in x, y respectively; for simplicity, we assume $h_1 = h_2 = h$. In practice we assume that the exact vorticity decays exponentially as $|\mathbf{x}|$ tends to infinity, thus a finite computational domain suffices to approximate the vorticity. Let $\mathbf{x}_j^h(t), \xi_j^h(t)$ be the approximate particle locations and the approximate vorticity respectively at time t, then equation (2.4) is discretized by (see [8],[7])

$$\frac{d\mathbf{x}_i^h(t)}{dt} = \sum_j K_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\xi_j^h(t)h^2.$$

Here we approximate the singular kernel $K(\mathbf{x})$ by a smoothed one $K_\delta(\mathbf{x})$, where $K_\delta = \phi_\delta * K$ and $\phi_\delta(\mathbf{x}) = (1/\delta^2)\phi(\mathbf{x}/\delta)$; $\phi(\mathbf{x})$ is called a cutoff function.

We shall now describe the approximation of the viscous term $R^{-1}\Delta\xi$ of (2.5). We approximate the vorticity by convolving it with a cutoff function, i.e., ξ is approximated by $\phi_\delta * \xi$. We then derive an approximation to the Laplacian of the vorticity by differentiation of this convolution, i.e., by $\Delta(\phi_\delta * \xi) = (\Delta\phi_\delta) * \xi$. Finally, we approximate the integrals involved in the convolution by the trapezoid rule, and obtain

$$\frac{d\mathbf{x}_i^h(t)}{dt} = \sum_j K_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\xi_j^h(t)h^2, \tag{2.6}$$

$$\frac{d\xi_i^h(t)}{dt} = R^{-1}\sum_j \Delta\phi_\delta(\mathbf{x}_i^h(t) - \mathbf{x}_j^h(t))\xi_j^h(t)h^2. \tag{2.7}$$

This yields a scheme which is similar in nature to vortex schemes for the Euler's equations.

3. The Convection-Diffusion Equation

We shall consider the convection-diffusion equation,

$$\frac{\partial\xi}{\partial t} + (\mathbf{a}\cdot\nabla)\xi = R^{-1}\Delta\xi$$

with div $\mathbf{a} = 0$. Along particle trajectories, it takes the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{a}(\mathbf{x}, t), \qquad \text{div } \mathbf{a} = 0, \tag{3.1}$$

$$\frac{d\xi}{dt} = R^{-1}\Delta\xi, \qquad (3.2),$$

where $d/dt = \partial/\partial t + (\mathbf{a}\cdot\nabla)$. Following the analysis of [24, pp. 227-233], this problem has a solution for which ξ and its derivatives to order two are uniformly bounded and continuous for $0 \le t \le T$, and the solution is unique, provided that $\mathbf{a}$ and the initial conditions are in C^∞. In fact, the continuity and uniform boundedness of all derivatives to order four will suffice (see [24]).

The proposed scheme for the convection-diffusion equation is

$$\frac{d\mathbf{x}_i(t)}{dt} = \mathbf{a}(\mathbf{x}_i, t), \qquad \text{div } \mathbf{a} = 0, \tag{3.3}$$

$$\frac{d\xi_i^h(t)}{dt} = R^{-1}\sum_j \Delta\phi_\delta(\mathbf{x}_i(t) - \mathbf{x}_j(t))\xi_j^h(t)h^2, \tag{3.4}$$

where $\xi_i^h(t) = \xi^h(\mathbf{x}_i(t), t)$ is the approximated vorticity. In the consistency theorem we shall use the incompressibility condition $\nabla \cdot \mathbf{a} = 0$ to assure that an area-element h^2 at the initial time remains the same for all times. We shall now prove the convergence of our scheme (3.3)-(3.4) to the convection-diffusion equation (3.1)-(3.2). Expressing (3.1) and (3.2) at $\mathbf{x}_i$, we find

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{a}(\mathbf{x}_i, t). \tag{3.5}$$

$$\frac{d\xi_i}{dt} = R^{-1}\Delta\xi_i, \tag{3.6}$$

where $\xi_i = \xi(\mathbf{x}_i, t)$ and $\Delta\xi_i = (\Delta\xi)(\mathbf{x}_i, t)$. Note that there is no error in particle locations, and we can therefore proceed to the error in the vorticity at the time-dependent grid points. Thus, subtracting (3.6) from (3.4), we find

$$\frac{d(\xi_i^h(t) - \xi_i(t))}{dt} = R^{-1}\sum_j \Delta\phi_\delta(\mathbf{x}_i(t) - \mathbf{x}_j(t))\xi_j^h(t)h^2 - R^{-1}\Delta\xi_i(t). \tag{3.7}$$

Since we want to use energy-type estimates, we multiply and (3.7) by $(\xi_i^h(t) - \xi_i(t))h^2$ and sum over i. For the sake of simplicity we omit (t) from now on.

$$\frac{1}{2}\frac{d}{dt}\sum_i(\xi_i^h - \xi_i)^2h^2 = R^{-1}\sum_i(\xi_i^h - \xi_i)h^2(\sum_j \Delta\phi_\delta(\mathbf{x}_i - \mathbf{x}_j)\xi_j^h h^2 - \Delta\xi_i). \tag{3.8}$$

Rewrite (3.8) as

$$\frac{1}{2}\frac{d}{dt}\sum_i(\xi_i^h - \xi_i)^2h^2 = S1 + C1, \tag{3.9}$$

where

$$S1 = R^{-1}\sum_i(\xi_i^h - \xi_i)h^2(\sum_j \Delta\phi_\delta(\mathbf{x}_i - \mathbf{x}_j)\xi_j^h h^2 - \sum_j \Delta\phi_\delta(\mathbf{x}_i - \mathbf{x}_j)\xi_j h^2) \tag{3.10}$$

and

$$C1 = R^{-1}\sum_i(\xi_i^h - \xi_i)h^2(\sum_j \Delta\phi_\delta(\mathbf{x}_i - \mathbf{x}_j)\xi_j h^2 - \Delta\xi_i). \tag{3.11}$$

We shall now bound $C1$, which is related with the truncation error. Later, as part of the stability theorem, we shall bound $S1$.

4. Consistency and Stability for the Convection-Diffusion Equation

For $p \in [1, \infty)$ and $m \geq 0$ define the Sobolev space

$$W^{m,p}(R^2) = \{f, \partial^\alpha f \in L^p(R^2), |\alpha| \leq m\}$$

and the norm $\| \cdot \|_{m,p}$

$$\|f\|_{m,p}^p = \sum_{0 \leq |\alpha| \leq m} \|\partial^\alpha f\|_{L_p}^p;$$

for $p = \infty$ we refer to the maximum norm.

Define the discrete norm $\|f\|_{0,2,h}^2 = \sum_i f^2(\mathbf{x}_i)h^2$.

Lemma 4.1. Consistency for the Convection-Diffusion Equation. Let the cutoff function ϕ satisfy the following conditions.

$$\phi \in W^{m+2,1}(R^2), m \geq 3 \tag{4.1}$$

$$\int_{R^2} \phi(\mathbf{x})d\mathbf{x} = 1, \quad \int_{R^2} \mathbf{x}^\alpha \phi(\mathbf{x})d\mathbf{x} = 0, |\alpha| \leq d-1, \quad \int_{R^2} |\mathbf{x}|^d |\phi(\mathbf{x})| d\mathbf{x} < \infty. \tag{4.2}$$

Assume the vorticity is in $W^{m+3,2}(R^2)$, $m \geq d+2$. Let $\mathbf{x}_j(0)$ be unifomly distributed grid points in R^2 and assume that the transformation from R^2 to itself via (3.1) has continuous and uniformly bounded derivatives to order $m+3$. Then, there exist a constant C such that

$$\begin{aligned} |C1| &= R^{-1} |\sum_i (\xi_i^h - \xi_i) h^2 (\sum_j \Delta\phi_\delta(\mathbf{x}_i - \mathbf{x}_j)\xi_j h^2 - \Delta\xi_i)| \\ &\leq CR^{-1}\|\xi^h - \xi\|_{0,2,h}(\delta^d + \frac{h^m}{\delta^{m+2}}). \end{aligned} \tag{4.3}$$

Proof. We write the truncation error in (3.11) as a sum of two terms; the first is associated with the regularization error e_r, and the second is associated with the discretization one e_d.

$$C1 = \sum_i (\xi_i^h - \xi_i) h^2 e_t(\mathbf{x}_i, t),$$

where

$$e_t(\mathbf{x}_i,t) = R^{-1}(\sum_j \Delta\phi_\delta(\mathbf{x}_i-\mathbf{x}_j)\xi_j h^2 - \Delta\xi(\mathbf{x}_i,t)),$$

$$e_t = e_r + e_d,$$

and

$$e_r(\mathbf{x}_i,t) = R^{-1}((\Delta\phi_\delta * \xi)(\mathbf{x}_i,t) - \Delta\xi(\mathbf{x}_i,t)),$$

$$e_d(\mathbf{x}_i,t) = R^{-1}(\sum_j \Delta\phi_\delta(\mathbf{x}_i-\mathbf{x}_j)\xi_j h^2 - (\Delta\phi_\delta * \xi)(\mathbf{x}_i,t)).$$

Let us first treat the regularization error, e_r. Rewrite it in the following form

$$e_r(\mathbf{x}_i,t) = R^{-1}(\phi_\delta * \Delta\xi - \Delta\xi)(\mathbf{x}_i,t).$$

We want to estimate $(f - f*\phi_\delta)(\mathbf{x}_i)$ in the discrete L_2 norm for a smooth function $f(\mathbf{x})$. The following is a result of Raviart's theorem [29, pp. 262] for the difference between a trapezoidal sum of a function and its integral. Let $g \in W^{m,1}(R^2)$ and let $\alpha_j, j = 1,2,...$ be points on a uniform mesh in R^2 with spacing h, then

$$|\sum_j g(\alpha_j)h^2 - \int_{R^2} g(\alpha)d\alpha| \le Ch^m\|g\|_{m,1}, \quad m \ge 3, \tag{4.4}$$

where C is a constant. Let us apply the theorem to

$$g(\alpha,t) = [(f - f*\phi_\delta)(\mathbf{x}(\alpha,t),t)]^2,$$

with $m=3$, where $\mathbf{x}(\alpha,t)$ are solutions of (3.1) with $\mathbf{x}(\alpha,0)=\alpha$. We find that ([6])

$$\|f - f*\phi_\delta\|^2_{0,2,h} \le \|f - f*\phi_\delta\|^2_{0,2} + Ch^3\|(f - f*\phi_\delta)^2\|_{3,1}, \tag{4.5}$$

where the norm in $W^{3,1}(R^2)$ is for $(f - f*\phi_\delta)^2$ as a function of α. Now, if $f \in W^{3,2}(R^2)$ and $\phi \in W^{3,1}(R^2)$ then by Young's inequality $f - f*\phi_\delta$ is in $W^{3,2}(R^2)$. By Leibnitz rule and the algebraic inequality $2|ab| \le a^2+b^2$, if $g \in W^{m,2}(R^2)$ then g^2 is in $W^{m,1}(R^2)$ and $\|g^2\|_{m,1} \le C\|g\|^2_{m,2}$. Thus, by (4.5),

$$\|f - f*\phi_\delta\|^2_{0,2,h} \le \|f - f*\phi_\delta\|^2_{0,2} + Ch^3\|f - f*\phi_\delta\|^2_{3,2}, \tag{4.6}$$

Note that since the transformation of R^2 to itself via (3.1) has uniformly bounded derivatives to order 3, the norm $\|f - f * \phi_\delta\|_{3,2}$ can be bounded by a constant (which depends on the bounds of all transformation derivatives to order 3) multiplied by the same norm for which $f - f * \phi_\delta$ is viewed as a function of $\mathbf{x}$ rather than α. By Taylor's expansion and moment conditions (4.2), Raviart [29] expressed the difference $f - f * \phi_\delta$ for a smooth function f in the following way.

$$(f - f * \phi_\delta)(\mathbf{x}) =$$

$$= (-1)^d \frac{1}{(d-1)!} \int_0^1 d\theta (1-\theta)^{d-1} \sum_{\beta_1+\beta_2=d} \int_{R^2} \frac{\partial^{\beta_1+\beta_2}}{\partial x_1^{\beta_1} x_2^{\beta_2}} f(\mathbf{x} - \theta \mathbf{y}) y_1^{\beta_1} y_2^{\beta_2} \phi_\delta(\mathbf{y}) d\mathbf{y}, \quad (4.7)$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. Upon substituing $\mathbf{z} = \theta \mathbf{y}$, we find

$$\|f - f * \phi_\delta\|_{0,2}^2 = \| \sum_{\beta_1+\beta_2=d} [\frac{\partial^{\beta_1+\beta_2}}{\partial x_1^{\beta_1} x_2^{\beta_2}} f(\mathbf{x})$$

$$* \frac{(-1)^d}{(d-1)!} \int_0^1 d\theta (1-\theta)^{d-1} x_1^{\beta_1} x_2^{\beta_2} \phi_\delta(\frac{\mathbf{x}}{\theta}) \theta^{-(d+2)} d\theta] \|_{0,2}^2.$$

By Young's inequality and by subtituing back $\mathbf{y} = \mathbf{z}/\theta$, we find

$$\|f - f * \phi_\delta\|_{0,2} \leq C \|f\|_{d,2} \sum_{\beta_1+\beta_2=d} \| \int_0^1 (1-\theta)^{d-1} y_1^{\beta_1} y_2^{\beta_2} \phi_\delta(\mathbf{y}) d\theta \|_{0,1}.$$

Since $\phi_\delta(\mathbf{y}) = \delta^{-2} \phi(\mathbf{y}/\delta)$ and by the subtitution $\mathbf{z} = \mathbf{y}/\delta$, we find, in view of (4.2), that for $\beta_1 + \beta_2 = d$

$$\| \int_0^1 (1-\theta)^{d-1} \phi_\delta(\mathbf{y}) y_1^{\beta_1} y_2^{\beta_2} d\theta \|_{0,1} \leq C \delta^d.$$

Thus,

$$\|f - f * \phi_\delta\|_{0,2}^2 \leq C \delta^{2d} \|f\|_{d,2}^2.$$

To bound $\|f - f * \phi_\delta\|_{3,2}$, we apply (4.7) for derivatives of f to order three. We find that

$$\|f - f * \phi_\delta\|_{3,2}^2 \leq C \delta^{2d} \|f\|_{d+3,2}^2.$$

Collecting the contributions of the two terms in the right hand side of (4.6), and by the algebraic inequality $a^2 + b^2 \leq (|a| + |b|)^2$, we find that

$$\|f - f * \phi_\delta\|_{0,2,h} \leq C\delta^d(\|f\|_{d,2} + h^{3/2}\|f\|_{d+3,2}).$$

Applying the last inequality for $\Delta\xi$, noting that $\xi \in W^{m+3,2}(R^2)$ for $m \geq 3$, we find that

$$\|e_r\|_{0,2,h} \leq CR^{-1}\delta^d(\|\xi\|_{d+2,2} + h^{3/2}\|\xi\|_{d+5,2}). \tag{4.8}$$

Now we want to bound the discretization error e_d in the discrete L_2 norm. Define

$$v(\alpha, t) = \sum_j \Delta\phi_\delta(\mathbf{x}(\alpha, t) - \mathbf{x}(\alpha_j, t))\xi_j h^2 - \int_{R^2} \Delta\phi_\delta(\mathbf{x}(\alpha, t) - \mathbf{x}(\alpha', t))\xi(\mathbf{x}(\alpha', t), t)d\alpha';$$

Note that $e_d(\mathbf{x}, t) = R^{-1}v(\alpha, t)$. Our purpose is to bound v in the discrete L_2 norm. Applying (4.4) for v^2 with $m = 3$, we find

$$\|v\|^2_{0,2,h} \leq \|v\|^2_{0,2} + Ch^3\|v\|^2_{3,2}.$$

Let us first bound $\|v\|_{0,2}$. We apply (4.4) again, this time for

$$g(\alpha', t; \alpha) = \Delta\phi_\delta(\mathbf{x}(\alpha, t) - \mathbf{x}(\alpha', t))\xi(\mathbf{x}(\alpha', t), t),$$

and find that

$$|\sum_j \Delta\phi_\delta(\mathbf{x}(\alpha, t) - \mathbf{x}(\alpha_j, t))\xi_j h^2 - \int_{R^2} \Delta\phi_\delta(\mathbf{x}(\alpha, t) - \mathbf{x}(\alpha', t))\xi(\mathbf{x}(\alpha', t), t)d\alpha'|$$

$$\leq Ch^m \sum_{|\beta| \leq m} \int_{R^2} |\partial^\beta_{\alpha'}[\Delta\phi_\delta(\mathbf{x}(\alpha, t) - \mathbf{x}(\alpha', t))\xi(\mathbf{x}(\alpha', t), t)]|d\alpha',$$

where $\partial^\beta_{\alpha'}$ denotes partial derivative of order $|\beta|$ with respect to α'. Since the transformation (3.1) has uniformly bounded derivatives to order m, derivatives with respect to α' are bounded by a constant multiplied by derivatives with respect to $\mathbf{x}'$. Applying Leibnitz rule and Young's inequality, we find that

$$\|v\|_{0,2} \leq Ch^m\|\Delta\phi_\delta\|_{m,1}\|\xi\|_{m,2}.$$

In the last inequality ϕ and ξ are viewed as functions of $\mathbf{x}$. Now $\|\phi_\delta\|_{m+2,1} \leq C\delta^{-(m+2)}$ (see [29, pp. 275]), thus

$$\|v\|_{0,2} \leq C\frac{h^m}{\delta^{m+2}}\|\xi\|_{m,2}.$$

To bound $\|v\|_{3,2}$, we apply (4.4) for $g(\alpha', t; \alpha) = \partial^\beta_{\alpha'}\{\Delta\phi_\delta(\mathbf{x}(\alpha, t) - \mathbf{x}(\alpha', t))\xi(\mathbf{x}(\alpha', t), t)\}$, for $\beta \leq 3$ and find that

$$\|v\|_{3,2} \leq C\frac{h^m}{\delta^{m+5}}\|\xi\|_{m+3,2}.$$

Thus,

$$\|e_d\|_{0,2,h} \leq CR^{-1}\frac{h^m}{\delta^{m+2}}(\|\xi\|_{m,2} + \frac{h^{3/2}}{\delta^{3/2}}\|\xi\|_{m+3,2}). \tag{4.9}$$

By Shwartz inequality and (4.8)-(4.9) we obtain the desired result (4.3).

Remark 4.2 By (4.8) and (4.9) the constant C in (4.3) includes bounds of the type $\|\xi\|_{m,2}$. Bounds in $W^{m,2}(R^2)$ can be derived as follows. Multiplying the convection diffusion equation for ξ

$$\frac{\partial\xi}{\partial t} + (\mathbf{a}\cdot\nabla)\xi = R^{-1}\Delta\xi \tag{4.10}$$

by ξ, we find

$$\frac{1}{2}\frac{d\xi^2}{dt} = R^{-1}\xi\Delta\xi.$$

Let $\Omega_t = R^2$ be the image at time t of the map from α to $\mathbf{x}$ via (3.1), then by integrating the last equation over Ω_t and by the transport theorem [9], we find

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\xi^2 d\mathbf{x} = R^{-1}\int_{\Omega_t}\xi\Delta\xi d\mathbf{x}.$$

Upon integrating the right hand side by parts, we find that

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\xi^2 d\mathbf{x} = -R^{-1}\int_{\Omega_t}|\nabla\xi|^2 d\mathbf{x} \leq 0, \tag{4.11}$$

thus $\|\xi(\mathbf{x}, t)\|_{0,2} \leq \|\xi(\mathbf{x}, 0)\|_{0,2}$.

To derive a bound on $\|\xi\|_{1,2}$, differentiate (4.10) once with respect to x and once with respect to y and find that

$$\frac{\partial \xi_x}{\partial t} + (\mathbf{a}(\mathbf{x},t)\cdot\nabla)\xi_x + (\frac{\partial \mathbf{a}}{\partial x}\cdot\nabla)\xi = R^{-1}\Delta(\xi_x), \tag{4.12a}$$

$$\frac{\partial \xi_y}{\partial t} + (\mathbf{a}(\mathbf{x},t)\cdot\nabla)\xi_y + (\frac{\partial \mathbf{a}}{\partial y}\cdot\nabla)\xi = R^{-1}\Delta(\xi_y), \tag{4.12b}$$

where ξ_x, ξ_y denotes partial derivatives of ξ with respect to x and y respectively. Multiplying (4.12a) and (4.12b) by ξ_x and ξ_y respectively, integrating over Ω_t and summing the resulting equations, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}(\xi_x^2+\xi_y^2)d\mathbf{x} = -\int_{\Omega_t}(\frac{\partial \mathbf{a}}{\partial x}\cdot\nabla)\xi\cdot\xi_x d\mathbf{x} - \int_{\Omega_t}(\frac{\partial \mathbf{a}}{\partial y}\cdot\nabla)\xi\cdot\xi_y d\mathbf{x}$$
$$+R^{-1}\int_{\Omega_t}(\xi_x\Delta\xi_x+\xi_y\Delta\xi_y)d\mathbf{x}.$$

Since first order derivatives of $\mathbf{a}$ can be bounded by γ_1, and by (4.11)

$$\frac{d}{dt}\|\xi\|_{1,2}^2 \leq \gamma_1\|\xi\|_{1,2}^2 - R^{-1}\int_{\Omega_t}(|\nabla\xi_x|^2+|\nabla\xi_y|^2)d\mathbf{x}.$$

The last term is non-positive, thus, in view of Gronwall's inequality

$$\|\xi(\mathbf{x},t)\|_{1,2} \leq e^{\gamma_1 t}\|\xi(\mathbf{x},0)\|_{1,2}.$$

Higher order derivatives can be similarly bounded to obtain

$$\|\xi(\mathbf{x},t)\|_{m,2} \leq C(t)\|\xi(\mathbf{x},0)\|_{m,2},$$

where $C(t)$ includes maximum boundes of derivatives of $\mathbf{a}$ to order m; C depends on t, but is independent of R. Therefore the consistency error decays to zero as R tends to infinity.

Remark 4.3 It was proved in [3, pp. 244] that if the initial vorticity is smooth enough, $\|\xi(\mathbf{x},t;R)\|_{m,2} \leq \|\xi(\mathbf{x},t;R=\infty)\|_{m,2} + CR^{-1}$, for $m \geq 2$, and C depends on the time interval T, $\xi(\mathbf{x},0)$ and m. Here $\xi(\mathbf{x},t;R)$ denotes the solution of the two-dimensional

Navier-Stokes equations with Reynolds number R. Thus the vorticity norms $\|\xi\|_{m,2}$ can be uniformly bounded for $R \geq 1$, and the consistency error in the vorticity evolution decays to zero as R tends to infinity.

We shall now derive bounds for $S1$; this consists of the stability proof of the scheme and was derived with H. Dym [13].

Lemma 4.4. Stability for the Convection-Diffusion Equation. Let $\phi \in W^{2,1}(R^2)$. Assume further that the Fourier transform of the cutoff function be non-negative,

$$\hat{\phi}(\mathbf{s}) \geq 0, \tag{4.13}$$

then

$$S1 = R^{-1} \sum_i (\xi_i^h - \xi_i) h^2 (\sum_j \Delta\phi_\delta(\mathbf{x}_i - \mathbf{x}_j)\xi_j^h h^2 - \sum_j \Delta\phi_\delta(\mathbf{x}_i - \mathbf{x}_j)\xi_j h^2) \leq 0. \tag{4.14}$$

Proof. Let $w_i = \xi_i^h - \xi_i$, thus

$$S1 = R^{-1} \sum_i w_i h^2 \sum_j \Delta\phi_\delta(\mathbf{x}_i - \mathbf{x}_j) w_j h^2. \tag{4.15}$$

Expressing $\phi(\mathbf{x})$ via its Fourier transform, one finds

$$\phi_\delta(\mathbf{x}) = \int \hat{\phi}_\delta(\mathbf{s}) e^{i\mathbf{s}\cdot\mathbf{x}} d\mathbf{s}.$$

Taking the Laplacian of both sides of the last equality yields

$$\Delta\phi_\delta(\mathbf{x}) = -\int (\mathbf{s}\cdot\mathbf{s}) \hat{\phi}_\delta(\mathbf{s}) e^{i\mathbf{s}\cdot\mathbf{x}} d\mathbf{s}. \tag{4.16}$$

We now substitute (4.16) in (4.15) and find

$$S1 = -R^{-1} \int (\mathbf{s}\cdot\mathbf{s}) \sum_i w_i h^2 \sum_j \hat{\phi}_\delta(\mathbf{s}) e^{i\mathbf{s}\cdot(\mathbf{x}_i - \mathbf{x}_j)} w_j h^2 d\mathbf{s},$$

or

$$S1 = -R^{-1} \int (\mathbf{s}\cdot\mathbf{s})\hat{\phi}_\delta(\mathbf{s}) \sum_i w_i e^{i\mathbf{s}\cdot\mathbf{x}_i} h^2 \sum_j w_j e^{-i\mathbf{s}\cdot\mathbf{x}_j} h^2 d\mathbf{s}.$$

Since w is a real function, we find that

$$S1 = -R^{-1} \int (\mathbf{s}\cdot\mathbf{s})\hat{\phi}_\delta(\mathbf{s}) |\sum_i w_i e^{i\mathbf{s}\cdot\mathbf{x}_i} h^2|^2 d\mathbf{s}. \tag{4.17}$$

The right-hand side of the last equality is non-positive by (4.13), and since $\hat{\phi}_\delta(\mathbf{s}) = \hat{\phi}(\delta\mathbf{s})$.

Remark 4.4. I was later notified by Ben-Artzi [6] that this lemma is a direct consequence of Bochner's theorem for sufficient and necessary conditions for a function to be positive definite.

5. Convergence Theorem for the Convection-Diffusion Equation

We shall now combine the consistency and the stability theorems to prove the convergence of the scheme.

Theorem 5.1. Convergence of the time-continuous scheme. Let the cutoff function ϕ satisfy (4.1)-(4.2). Assume that the exact vorticity satisfies the condition specified in the Lemma (4.1). Let the Fourier transform of the cutoff function be non-negative, i.e., it satisfies (4.13). Define $A^2(t) = \sum_i (\xi_i^h(t) - \xi_i(t))^2 h^2$, then

$$A(t) \leq C(t) R^{-1} (\delta^d + \frac{h^m}{\delta^{m+2}}) \tag{5.1}$$

Proof. By (3.9)

$$\frac{1}{2}\frac{dA^2(t)}{dt} = C1 + S1.$$

Using Lemma 4.1, we find that for $R \geq 1$

$$|C1| \leq CR^{-1} A(t) (\delta^d + \frac{h^m}{\delta^{m+2}}).$$

By Lemma 4.4 $S1 \leq 0$, and therefore

$$\frac{dA^2}{dt} \leq CR^{-1}A(\delta^d + \frac{h^m}{\delta^{m+2}}). \tag{5.2}$$

Dividing (5.2) by $A(t) = \sqrt{A^2(t)}$ we find

$$\frac{dA(t)}{dt} \leq CR^{-1}(\delta^d + \frac{h^m}{\delta^{m+2}}),$$

with the initial condition $A(t = 0) = 0$. We therefore conclude that

$$A(t) \leq C(t)R^{-1}(\delta^d + \frac{h^m}{\delta^{m+2}}).$$

In the conditions of Remark 4.2, the constant C above depends on t but is independent of R for $R \geq 1$, therefore the error in ξ, measured in the discrete L_2 norm, decays to zero as R tends to infinity.

6. Stability for the Time-Discretized Convection-Diffusion Equation

In this section we derive a stability condition for the time-discretized scheme for the convection-diffusion equation. Let $t = n\Delta t$ and consider the Euler's scheme for stepping (3.3)-(3.4) in time

$$\mathbf{x}_i^{h,n+1} = \mathbf{x}_i^{h,n} + \Delta t\ \mathbf{a}(\mathbf{x}_i^{h,n}, t), \tag{6.1}$$

$$\xi_i^{h,n+1} = \xi_i^{h,n} + \Delta t R^{-1} \sum_j \Delta\phi_\delta(\mathbf{x}_i^{h,n} - \mathbf{x}_j^{h,n})\xi_j^{h,n} h^2, \tag{6.2}$$

where $\mathbf{x}_i^{h,n}$ and $\xi_i^{h,n}$ approximate the exact particle location $\mathbf{x}_i^n$ and the exact vorticity ξ_i^n respectively. We shall prove the convergence of the approximate particle locations to the exact ones and the stability for the approximate vorticity under the conditions that the continuous Fourier transform of the cutoff function is non-negative and the time-step is of order δ^2. We need the following lemmas.

Lemma 6.1. Assume that $|D^\beta \phi(\mathbf{x})| \le C \ \forall \mathbf{x}$, and that $|D^\beta \phi(\mathbf{x})| \ \le C|\mathbf{x}|^{-2-|\beta|}$ for $|\mathbf{x}| \ge 1$, where D^β denotes a partial derivative of order β with respect to the spatial variables x, y and C may depend on β. Then

$$(a) \quad |D^\beta \phi_\delta(\mathbf{x})| \le C\delta^{-2-|\beta|} \qquad \forall \mathbf{x},$$

$$(b) \quad |D^\beta \phi_\delta(\mathbf{x})| \le C|\mathbf{x}|^{-2-|\beta|} \qquad \forall |\mathbf{x}| \ge \delta.$$

We shall skip the proof as it follows directly from the defenition of ϕ_δ and the conditions of the lemma.

Lemma 6.2 Let $\mathbf{x}_j$ be solutions of (3.3) which are initially uniformly distributed in R^2. Assume $\phi(\mathbf{x})$ satisfies the assumptions of Lemma 6.1 and that $|\mathbf{y}_j| \le C_0\delta$, where $\delta \ge h$. Then

$$\sum_j \max_{|\mathbf{y}_j| \le C_0\delta} |D^\beta \phi_\delta(\mathbf{x} - \mathbf{x}_j + \mathbf{y}_j)| h^2 \le C\delta^{-|\beta|}, \qquad |\beta| \ge 1. \tag{6.3}$$

Proof. The proof is very similar to the one of Lemma 5 in [20] and Lemma 3.2 in [4]. Let B_j be the the quare whose center is $\mathbf{x}_j$ with sides h in the x and y directions. Since $h \le \delta$, $|\mathbf{x}' - \mathbf{x}_j| \le C_1\delta, \ \mathbf{x}' \in B_j$; we may also assume $C_1 \ge C_0$. For fixed $\mathbf{x}$, let

$$J_1 = \{j : |\mathbf{x} - \mathbf{x}_j| \le (3C_1 + 1)\delta\},$$

$$J_2 = \{j : |\mathbf{x} - \mathbf{x}_j| > (3C_1 + 1)\delta\}.$$

We devide the sum in (6.3) into the sum over $j \in J_1$ and the one over $j \in J_2$. For the first, we use Lemma 6.1(a) and find that the corresponding sum is bounded by $C\delta^{-2-|\beta|}$ multiplied by the volume of $\bigcup_{j \in J_1} B_j \subseteq \{\mathbf{x}' : |\mathbf{x}' - \mathbf{x}| \le (4C_1 + 1)\delta\}$. Therefore the sum over J_1 is bounded by

$$C\delta^{-2-|\beta|}\{(4C_1 + 1)\delta\}^2 = C\delta^{-|\beta|}.$$

For $j \in J_2$, we have $|\mathbf{x} - \mathbf{x}_j + \mathbf{y}_j| \geq (2C_1 + 1)\delta \geq \delta$, thus by Lemma 6.1(b)

$$\max_{|\mathbf{y}_j| \leq C_0\delta} |D^\beta \phi_\delta(\mathbf{x} - \mathbf{x}_j + \mathbf{y}_j)| \leq C|\mathbf{x} - \mathbf{x}_j + \mathbf{y}_j|^{-2-|\beta|}.$$

We regard the sum (6.3) over J_2 as the integral of a step function with constant values on each B_j. Now for $\mathbf{x}' \in B_j$,

$$|\mathbf{x} - \mathbf{x}_j + \mathbf{y}_j| \geq |\mathbf{x} - \mathbf{x}'| - 2C_1\delta,$$

and thus the integrand can be estimated by

$$\max_{|y_j| \leq C_0\delta} |D^\beta \phi_\delta(\mathbf{x} - \mathbf{x}_j + \mathbf{y}_j)h^2| \leq (|\mathbf{x} - \mathbf{x}'| - 2C_1\delta)^{-2-|\beta|}.$$

Since $|\mathbf{x}-\mathbf{x}'| \geq |\mathbf{x}-\mathbf{x}_j| - |\mathbf{x}_j - \mathbf{x}'| \geq (1+2C_1)\delta$, the sum (6.3) over J_2 can be bounded by

$$C\int_{(1+2C_1)\delta}^{\infty} (r - 2C_1\delta)^{-2-|\beta|} r\,dr = C\int_{\delta}^{\infty} \rho^{-2-|\beta|}(\rho + 2C_1\delta)d\rho.$$

But $\rho + 2C_1\delta \leq (1 + 2C_1)\rho$ for $\rho \geq \delta$, thus the integral is

$$\leq C\int_\delta^\infty \rho^{-1-|\beta|}d\rho \leq C\delta^{-|\beta|},$$

for $\beta \geq 1$.

We turn now to the stability of the scheme for the convection-diffusion equation. We shall prove that the error in the particle-locations is of order $O(\Delta t)$.

Lemma 6.3. Convergence of particle locations. Consider the time-discretized (Euler's) scheme (6.1)-(6.2) for the convection-diffusion equation. Assume that $\mathbf{a}(\mathbf{x}, t)$ has bounded first order derivatives with respect to x and y, where $\mathbf{x} = (x, y)$, and that the derivatives of $\mathbf{x}(t)$ of order two or less are continuous and bounded. Then

$$|\mathbf{x}_i^{h,n} - \mathbf{x}_i^n| \leq C\Delta t, \quad 0 \leq t \leq T,$$

where $\mathbf{x}_i^{h,n}$ and $\xi_i^{h,n}$ approximate the exact particle location $\mathbf{x}_i^n$ and the exact vorticity ξ_i^n.

Proof. Since the Euler's scheme is consistent with the differential equation to first order in Δt,

$$\mathbf{x}_i^{n+1} = \mathbf{x}_i^n + \Delta t \ \mathbf{a}(\mathbf{x}_i^n, t_n) + C_1(\Delta t)^2.$$

By substracting (3.3) from (6.1) we find that the error in particle location $\mathbf{e}_i^n = \mathbf{x}_i^{h,n} - \mathbf{x}_i^n$ satisfies the following inequality.

$$|\mathbf{e}_i^{n+1}| \leq |\mathbf{e}_i^n| + \Delta t \ |\mathbf{a}(\mathbf{x}_i^{h,n}, t_n) - \mathbf{a}(\mathbf{x}_i^n, t_n)| + C_1(\Delta t)^2.$$

Using the mean value theorem for $\mathbf{a}$, we have

$$\mathbf{a}(\mathbf{x}_i^{h,n}, t_n) - \mathbf{a}(\mathbf{x}_i^n, t_n) = \int_0^1 D\mathbf{a}(\mathbf{x}_i^n + \theta \mathbf{e}_i^n, t_n) d\theta \cdot \mathbf{e}_i^n,$$

where

$$D\mathbf{a} = \begin{pmatrix} \partial_x a_1 & \partial_y a_1 \\ \partial_x a_2 & \partial_y a_2 \end{pmatrix},$$

and $\mathbf{a} = (a_1, a_2)^T$. We therefore find that

$$|\mathbf{e}_i^{n+1}| \leq (1 + C_2 \Delta t)|\mathbf{e}_i^n| + C_1(\Delta t)^2. \qquad (6.4)$$

Here C_2 is a bound on the induced l_2 norm of $D\mathbf{a}$ over all possible $\mathbf{x}$ and t. Let $F = 1 + C_2 \Delta t$, then using (6.4) succesively we find

$$|\mathbf{e}_i^n| \leq F^n |\mathbf{e}_i^0| + C_1(1 + F + F^2 + ... + F^{n-1})(\Delta t)^2.$$

Since $F^n \leq C_3 e^{TC_2}$ for a small enough time step, and $1 + F + F^2 + ... + F^{n-1} \leq C_3 e^{TC_2}\ n$, $n = CT/\Delta t$, and if we assume that $|\mathbf{e}_i^0|$ is of order Δt (in our scheme $\mathbf{e}_i^0 = 0$), then $|\mathbf{e}_i^n| \leq C \Delta t$.

Lemma 6.4. Stability Condition. Assume $\phi \in W^{2,1}(R^2)$ is a symmetric cutoff $\phi(-\mathbf{x}) = \phi(\mathbf{x})$ that satisfies (4.13) and the conditions of Lemma 6.1. Let $\delta = Ch^q, \quad 0 < q \le 1$ and let $\mathbf{a}$ satisfy the conditions of Lemma 6.3, then (6.1)-(6.2) is stable for $\Delta t \le CR\delta^2$.

Proof. In order to obtain a stability condition which depends on the continuous rather the discrete Fourier transform of ϕ_δ, we square (6.2), multiply the result by h^2 and sum over i. We find

$$\sum_i (\xi_i^{h,n+1})^2 h^2 = \sum_i (\xi_i^{h,n})^2 h^2 + 2R^{-1}\Delta t \sum_i \xi_i^{h,n} h^2 \sum_j \Delta\phi_\delta(\mathbf{x}_i^{h,n} - \mathbf{x}_j^{h,n})\xi_j^{h,n} h^2$$

$$+R^{-2}(\Delta t)^2 \sum_i h^2 (\sum_j \Delta\phi_\delta(\mathbf{x}_i^{h,n} - \mathbf{x}_j^{h,n})\xi_j^{h,n} h^2)^2. \tag{6.5}$$

Let $A = (A_{i,j})$, where $A_{i,j} = \Delta\phi_\delta(\mathbf{x}_i^{h,n} - \mathbf{x}_j^{h,n})$. By (4.13) A is a non positive operator, i.e., $\mathbf{v}^T A \mathbf{v} \le 0$ for every vector $\mathbf{v}$ in the discrete L_2 space.. We can therefore rewrite (6.5) in the following form

$$\sum_i (\xi_i^{h,n+1})^2 h^2 = \sum_i (\xi_i^{h,n})^2 h^2 + 2R^{-1}\Delta t \cdot h^4 (\xi^{h,n})^T A \xi^{h,n} + R^{-2}(\Delta t)^2 h^6 (\xi^{h,n})^T A^T A \xi^{h,n}. \tag{6.6}$$

For strong stability we require

$$\sum_i (\xi_i^{h,n+1})^2 h^2 \le \sum_i (\xi_i^{h,n})^2 h^2,$$

thus

$$\Delta t R^{-1} h^2 A^T A \le -2A.$$

The latter means that $-2A - \Delta t R^{-1} h^2 A^T A$ is a non-negative operator. We assume that the cutoff function ϕ_δ, and therefore A, is symmetric, hence the stability condition is

$$\lambda(-2A - \Delta t R^{-1} h^2 A^2) \ge 0,$$

where $\lambda(B)$ is an the spectrum of B. By the spectral mapping theorem,

$$\lambda(-2A - \Delta t R^{-1} h^2 A^2) = -2\lambda(A) - \Delta t R^{-1} h^2 \lambda^2(A) \geq 0. \tag{6.7}$$

Dividing (6.7) by $\lambda(A)$ which is non-positive, we find

$$-\lambda(A) h^2 \Delta t R^{-1} \leq 2 \ ; \tag{6.8}$$

in case $\lambda(A) = 0$ the stability condition is trivially satisfied. To impose condition (6.8), we look at the spectrum of the finite-dimensional operator $A^{(n)}$, where $(A^{(n)}\mathbf{v})_i = \sum_{j=1,n} A_{i,j} v_j$. Note that $A^{(n)}\mathbf{v}$ tends to $A\mathbf{v}$ as n tends to infinity, thus, since $A^{(n)}$ and A are self-adjoint, each neighborhoud of a point in the spectrum of A contains a point in the spectrum of $A^{(n)}$ for large enough n ([25],[6]). Thus, it suffices to require that (6.8) is satisfied for $A^{(n)}$ with large enough n. By Gershgorin's theorem the eigenvalues of $A^{(n)}$ differ from one of the diagonal elements at most by the sum of the absolute values of the off-diagonal terms. The diagonal elements of $A^{(n)} h^2$ are bounded by $C\delta^{-2}$ for $\delta = Ch^q, \ \ 0 < q \leq 1$. For the off-diagonal elements we use Lemma 6.2 and find that

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} |A_{i,j}| h^2 \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |\Delta\phi_\delta(\mathbf{x}_i^n - \mathbf{x}_j^n)| h^2 +$$

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} \int_0^1 |\nabla\Delta\phi_\delta(\mathbf{x}_i^n - \mathbf{x}_j^n + \theta(\mathbf{e}_i^n - \mathbf{e}_j^n)| d\theta \cdot |\mathbf{e}_i^n - \mathbf{e}_j^n| h^2.$$

Here $\nabla\Delta\phi_\delta$ denotes the gradiant of the Laplacian of the cutoff function. The first term in the last inequality is bounded by $C\delta^{-2}$ by Lemma 6.2. For the second term we bound $|\mathbf{e}_i^n - \mathbf{e}_j^n|$ by $C\Delta t$, using Lemma 6.3. Then $\sum_{j\neq i} |\nabla\Delta\phi_\delta(\mathbf{x}_i^n - \mathbf{x}_j^n + \theta(\mathbf{e}_i^n - \mathbf{e}_j^n))| h^2 \leq C\delta^{-3}$ by Lemma 6.2; the conditions for Lemma 6.2 are indeed satisfied since $|\mathbf{e}_i^n - \mathbf{e}_j^n| \leq C\Delta t \leq CR\delta^2 \leq C\delta$ for $R\delta \leq 1$. We therefore find that $\sum_{j\neq i} |A_{i,j}| h^2 \leq C\delta^{-2}$. By the Gershgorin's theorem, the eigenvalues of $A^{(n)} h^2$ do not exceed $C\delta^{-2}$; thus, if $C\Delta t R^{-1} \delta^{-2} \leq 2$ or equivalently $\Delta t \leq CR\delta^2$, inequality (6.8) is satisfied for $A^{(n)}$, and thus for A.

Now if one chooses δ to be of order $\sqrt{h}$, as suggested for example in [21], the stability condition is $\Delta t \leq CRh$, which is less restrictive then the finite difference-type stability condition $\Delta t \leq CRh^2$. For the convolution-type scheme (6.1)-(6.2), δ plays the role of h in the finite-difference stability condition, since h^2 appears in an-integral type operator of (6.2), which is bounded as a function of h.

7. Convergence of the Time-Discretized Scheme

In this section we prove the convergence of the time-discretized scheme for the convection-diffusion equation. In this proof we use techniques similar to the ones we have used in the previous section, together with asymptotic expansion of the error in particle-locations; the latter was suggested by Hald [22]. We shall first state a lemma on the asymptotic expansion of the error in the particles location.

Lemma 7.1. Asymptotic expansion of the error in particle locations. Consider the time-discretized (Euler's) scheme (6.1)-(6.2) for the convection-diffusion equation. Assume that $\mathbf{a}(\mathbf{x}, t)$ has bounded derivatives of order 2 and less with respect to x and y, where $\mathbf{x} = (x, y)$ and that $\mathbf{x}_i(t)$ has three continuous and bounded derivatives with respect to t. Let the initial error be of order Δt at most, i.e., $\mathbf{e}_i^0 = \Delta t \mathbf{g}_i^0$. Then

$$\mathbf{x}_i^{h,n} - \mathbf{x}_i^n = \Delta t \; \mathbf{g}_i(t_n) + O((\Delta t^2), \quad 0 \leq t \leq T.$$

where $\mathbf{g}_i(t)$ is a vector valued function which satisfies

$$\frac{d\mathbf{g}_i}{dt} = Da(\mathbf{x}_i^n, t_n) \cdot \mathbf{g}_i(t_n) + \frac{1}{2}\frac{d^2\mathbf{x}_i^n}{dt^2}. \tag{7.1}$$

Proof. We expand the exact particle-locations at $(n+1)\Delta t$ around $n\Delta t$ and find that

$$\mathbf{x}_i^{n+1} = \mathbf{x}_i^n + \Delta t \; \mathbf{a}(\mathbf{x}_i^n, t_n) + \frac{\Delta t^2}{2}\frac{d^2\mathbf{x}_i^n}{dt^2} + \frac{\Delta t^3}{6}\frac{d^3}{dt^3}\mathbf{x}_i(t + \theta_i \Delta t), \tag{7.2}$$

for $0 < \theta < 1$. Sustracting (6.1) from (7.2) and using Taylor expansion of $\mathbf{a}(\mathbf{x}_i^{h,n}, t_n)$ around $\mathbf{x}_i^n$ and since

$$\mathbf{a}(\mathbf{x}_i^{h,n}, t_n) - \mathbf{a}(\mathbf{x}_i^n, t_n) = D\mathbf{a}(\mathbf{x}_i^n, t_n) \cdot \mathbf{e}_i^n + \frac{1}{2}\int_0^1 (1-\theta)(\mathbf{e}_i^n)^T D^2\mathbf{a}(\mathbf{x}_i^n + \theta\mathbf{e}_i^n, t_n)\mathbf{e}_i^n d\theta,$$

thus

$$\mathbf{e}_i^{n+1} = [I + \Delta t D\mathbf{a}(\mathbf{x}_i^n, t_n) + O(\Delta t^2)]\mathbf{e}_i^n - \frac{\Delta t^2}{2}\frac{d^2\mathbf{x}_i^n}{dt^2} - \frac{\Delta t^3}{6}\frac{d^3}{dt^3}\mathbf{x}_i(t + \theta_i\Delta t). \tag{7.3}$$

We use (7.1) to expand $\mathbf{g}_i^{n+1}$ around t_n, and find that

$$\mathbf{g}_i^{n+1} = \mathbf{g}_i^n + \Delta t\frac{d\mathbf{g}_i^n}{dt} + O(\Delta t^2) = \mathbf{g}_i^n + \Delta t D\mathbf{a}(\mathbf{x}_i^n, t_n) \cdot \mathbf{g}_i^n - \frac{1}{2}\Delta t^2\frac{d^2\mathbf{x}_i^n}{dt^2} + O(\Delta t^2). \tag{7.4}$$

Define now

$$\mathbf{q}_i^n = \mathbf{e}_i^n - \Delta t\mathbf{g}_i^n,$$

thus by sustructing (7.4) multiplied by Δt from (7.3), we find that

$$\mathbf{q}_i^{n+1} = \{I + \Delta t D\mathbf{a}(\mathbf{x}_i^n, t_n)\}\mathbf{q}_i^n + O(\Delta t^3).$$

Repreating the argument for (6.4) in Lemma 6.3, and since $\mathbf{q}_i^0 = 0$, $\mathbf{q}_i^n = O(\Delta t^2)$, thus

$$\mathbf{e}_i^n = \Delta t\mathbf{g}_i^n + O(\Delta t^2).$$

Theorem 7.2. Convergence of the time-discretized scheme. Assume that (4.1)-(4.3) hold with $d \geq 4$ and that $\Delta t \leq CR\delta^4$, then under the conditions of Lemma 6.2, Lemma 6.3 and Lemma 6.4, equations (6.1)-(6.2) converge to (3.1)-(3.2), and the errors in particles location and the vorticity are of order Δt.

Remark 7.3. If we step the equations for particles locations (3.1) via a second order temporal scheme, we can prove the convergence of the approximated vorticity defined in

(6.2) with $\Delta t \leq R\delta^2$ and $d \geq 2$. The differences in the convergence proof will be indicated throughout the proof below. By numerical evidence [15] the less sever time-step restriction suffices for convergence.

Proof. In Lemma 6.3 we have proved the convergence of particles locations; the object now is to establish the convergence of the approximated vorticity to the exact one. Substructing (6.2) from (3.6) and using the first order accuracy of the Euler's scheme, we find that

$$\xi_i^{h,n+1} - \xi_i^{n+1} = \xi_i^{h,n} - \xi_i^n + \Delta t R^{-1}[\sum_j \Delta\phi_\delta(\mathbf{x}_i^{h,n} - \mathbf{x}_j^{h,n})\xi_j^{h,n}h^2 - \Delta\xi_i] + C_1(\mathbf{x}_i^n, t)(\Delta t)^2,$$

where $C_1(\mathbf{x}_i^n, t) = \frac{1}{2}\int_0^1 (1-\theta)\frac{\partial^2}{\partial t^2}\xi(\mathbf{x}_i^n, t+\theta\Delta t)d\theta$. Denote $\xi_i^{h,n} - \xi_i^n$ by w_i^n, then

$$w_i^{n+1} = w_i^n + \Delta t R^{-1}[\sum_j (\Delta\tilde{\phi_\delta} - \Delta\phi_\delta)w_j^n h^2 + \sum_j \Delta\phi_\delta w_j^n h^2$$

$$+ \sum_j (\Delta\tilde{\phi_\delta} - \Delta\phi_\delta)\xi_j^n h^2 + e_t(\mathbf{x}_i^n, t) + C_1(\mathbf{x}_i^n, t)R\Delta t)], \tag{7.5}$$

where $e_t(\mathbf{x}_i^n, t)$ is the truncation error at $\mathbf{x}_i^n$ and time t. We square (7.5), multiply by h^2 and sum over i; thus using the inequality $(a+b)^2 \leq 2(a^2+b^2)$ we find

$$\sum_i (w_i^{n+1})^2 h^2 \leq \sum_i (w_i^n)^2 h^2 + 2\Delta t R^{-1}\sum_i w_i^n h^2(A_1 + A_2 + A_3 + e_t(\mathbf{x}_i^n, t) + C_1(\mathbf{x}_i^n, t)R\Delta t)$$

$$+ C(\Delta t)^2 R^{-2}\sum_i h^2(A_1^2 + A_2^2 + A_3^2 + e_t^2(\mathbf{x}_i^n, t) + C_1^2(\mathbf{x}_i^n, t)R^2(\Delta t)^2). \tag{7.6}$$

Here $A_1 = \sum_j (\Delta\tilde{\phi_\delta} - \Delta\phi_\delta)w_j^n h^2$, $A_2 = \sum_j \Delta\phi_\delta w_j^n h^2$, and $A_3 = \sum_j (\Delta\tilde{\phi_\delta} - \Delta\phi_\delta)\xi_j^n h^2$. Let us also denote the second to last term in (7.6) by B_1 to B_{10} respectively.

We first treat $2\Delta t R^{-1}\sum_i w_i^n h^2 A_1$, which we denote by B_1. By the mean value theorem for $\Delta\phi_\delta$ we find that

$$B_1 = 2\Delta t R^{-1}\sum_i w_i^n h^2 \sum_j (\nabla\Delta\phi_\delta(\mathbf{x}_i^n - \mathbf{x}_j^n + \theta(\mathbf{e}_i^n - \mathbf{e}_j^n)) \cdot (\mathbf{e}_i^n - \mathbf{e}_j^n)w_j^n h^2$$

for some $0 \le \theta \le 1$. By the Chaucy-Swartz inequality

$$|B_1| \le 4\Delta t R^{-1} \max_i |\mathbf{e}_i| \|w^n\|_{0,2,h} \| \sum_j \mathcal{K}_{i,j} w_j h^2 \|_{0,2,h},$$

where $\mathcal{K}_{i,j} = \max_{0 \le \theta \le 1} |\nabla \Delta \phi_\delta(\mathbf{x}_i^n - \mathbf{x}_j^n + \theta(\mathbf{e}_i^n - \mathbf{e}_j^n)|$. By Lemma 4.1 of [2]

$$\| \sum_j \mathcal{K}_{i,j} w_j^n h^2 \|_{0,2,h} \le \|\mathcal{K}\| \|w^n\|_{0,2,h},$$

where $\|\mathcal{K}\|$ is the smallest number such that

$$\sum_i |\mathcal{K}_{i,j}| h^2 \le \|\mathcal{K}\|, \quad \sum_j |\mathcal{K}_{i,j}| h^2 \le \|\mathcal{K}\|.$$

Now since $|\mathbf{e}_i^n - \mathbf{e}_j^n| \le C\Delta t \le CR\delta^2$ (here we have also covered the case $m \ge 2$ mentioned in Remark 7.3), then $|\mathbf{e}_i^n - \mathbf{e}_j^n| \le C_0 \delta$ for $R\delta \le 1$, thus by Lemma 6.2 $\|\mathcal{K}\| \le C\delta^{-3}$. Therefore

$$|B_1| \le C\Delta t R^{-1} \|w^n\|_{0,2,h}^2 \frac{\Delta t}{\delta^3}.$$

By assumption $\Delta t \le CR\delta^4$, the latter is bounded by $C\delta^3$ for $R\delta \le 1$, thus

$$|B_1| \le C\Delta t R^{-1} \|w^n\|_{0,2,h}^2. \tag{7.7}$$

Note here (as indicated in Remark 7.3) that for second order time-stepping schemes for particles location $|\mathbf{e}_i^n - \mathbf{e}_j^n| \le C(\Delta t^2)$, and thus we olny have to require that $(\Delta t)^2 \le CR\delta^3$, but this condition is satisfied since $(\Delta t)^2 \le CR^2\delta^4 \le CR\delta^3$ for $R\delta \le 1$. Note that we only lost the factor R^{-1} in the last equality for B_1, and therefore have $B_1 \le C\Delta t \|w^n\|_{0,2,h}^2$.

Under the conditions of Lemma 4.4, we find that

$$B_2 + B_7 = 2\Delta t R^{-1} \sum_i w_i^n A_2 + C(\Delta t)^2 R^{-2} A_2^2 \le 0. \tag{7.8}$$

To estimate the term $2\Delta t R^{-1} \sum_i w_i^n h^2 A_3$ in (7.6), which we denote by B_3, we use Taylor expansion for $\Delta \tilde{\phi}_\delta$ around the exact particle location and Lemma 7.1. We find that

$$B_3 = 2\Delta t R^{-1} \sum_i w_i^n h^2 \sum_j \nabla \Delta \phi_\delta(\mathbf{x}_i^n - \mathbf{x}_j^n)(\mathbf{e}_i^n - \mathbf{e}_j^n)\xi_j h^2$$

$$+\Delta t R^{-1}\sum_i w_i^n h^2 \sum_j (\mathbf{e}_i^n - \mathbf{e}_j^n)^T D^2\Delta\phi_\delta(\mathbf{x}_i^n - \mathbf{x}_j^n + \theta(\mathbf{e}_i^n - \mathbf{e}_j^n))(\mathbf{e}_i^n - \mathbf{e}_j^n)\xi_j h^2,$$

where $D^2\Delta\phi_\delta$ denotes the matrix which contains second order derivatives of $\Delta\phi_\delta$. By Lemma 7.1 we find that $\mathbf{e}_i^n - \mathbf{e}_j^n = \Delta t\ (\mathbf{g}(\mathbf{x}_i,t_n) - \mathbf{g}(\mathbf{x}_j,t_n)) + O((\Delta t)^2)$, where $\mathbf{g}(\mathbf{x},t)$ is a smooth function of $\mathbf{x}$ and t. The first term in B_3 is thus bounded by $C\Delta t R^{-1}\|w^n\|_{0,h,2}(\Delta t + (\Delta t)^2\delta^{-3})$, but $\Delta t \le CR\delta^4 \le C\delta^3$ for $R\delta \le 1$, thus this term is bounded by $C\Delta t R^{-1}\Delta t\|w^n\|_{0,h,2} \le 2C\Delta t R^{-1}(\|w^n\|_{0,h,2}^2 + (\Delta t)^2)$. By Lemmas 6.2 and 6.3 the second term in B_3 can be bounded by $C\Delta t R^{-1}(\Delta t)^2\|w^n\|_{0,h,2}\delta^{-4}$. But since $\Delta t \le CR\delta^4$, this term is bounded by $C(\Delta t)^2\|w^n\|_{0,h,2}$, which is bounded by $C\Delta t(\|w^n\|_{0,h,2}^2 + (\Delta t)^2)$. We therefore conclude that

$$|B_3| \le C\Delta t(\|w^n\|_{0,2,h}^2 + (\Delta t)^2)). \tag{7.9}$$

Note that in case we have used a second order scheme for particles location, $\mathbf{e} = \Delta t^2\mathbf{g} + O(\Delta t^3)$, thus $B_3 \le C\Delta t R^{-1}\|w^n\|_{0,2,h}(\Delta t^2 + \Delta t^3\delta^{-3} + \Delta t^4\delta^{-4})$. It is easy to check that in this case $\Delta t \le R\delta^2$ will suffice to obtain (7.9) with an additional factor R.

Let $B_4 = 2\Delta t R^{-1}\sum_i w_i^n h^2 e_t(\mathbf{x}_i^n, t_n)$, thus

$$|B_4| \le C\Delta t R^{-1}(\|w^n\|_{0,2,h}^2 + \|e_t\|_{0,2,h}^2).$$

But $\|e_t\|_{0,2,h}^2 \le C\delta^{2d}$; if we require that $\delta^d \le C\Delta t$, we find that $\|e_t\|_{0,h,2}^2 \le C(\Delta t)^2$. For our first order Euler scheme $\Delta t = O(R\delta^4)$, thus $\delta^d = O(\Delta t^{d/4}/R^{d/4})$; if $d \ge 4$ the requirement above ($\delta^d \le C\Delta t$) is satisfied. Note that for a second order time-stepping scheme for particles locations $\Delta t = O(R\delta^2)$, thus $\delta^d = O(\Delta t^{d/2}/R^{d/2})$, and the requirement $\delta^d \le C\Delta t$ is satisfied for $d \ge 2$. Thus, in both cases we have

$$|B_4| \le C\Delta t R^{-1}(\|w^n\|_{0,2,h}^2 + (\Delta t)^2). \tag{7.10}$$

The next term B_5 of (7.6) is bounded by

$$|B_5| \le C\Delta t(\|w^n\|_{0,2,h}^2 + (\Delta t)^2)). \tag{7.11}$$

Let B_6 be the next term in (7.6), it can be bounded in as B_1 as follows

$$|B_6| \le C(\Delta t)^2 R^{-2} \|w^n\|^2_{0,2,h} \frac{(\Delta t)^2}{(\delta^3)^2}$$

$$\le C(\Delta t)^2 R^{-2} \|w^n\|^2_{0,2,h}. \qquad (7.12)$$

B_8 can be treated as B_3 to prove that

$$|B_8| = C(\Delta t)^2 R^{-2} \sum_i h^2 A_3^2 \le C(\Delta t)^2 R^{-2} (\Delta t)^2. \qquad (7.13)$$

Similarly B_9 and B_{10} can be treated as B_4 and B_5 respectively, noting that $\frac{\partial^2}{\partial t^2}\xi(\mathbf{x}, t)$ and its spatial derivatives to order 3 are in L_2. Thus,

$$|B_9| \le C\frac{(\Delta t)^2}{R^2} \|e_t\|^2_{0,2,h} \le C(\Delta t)^2 R^{-2} (\Delta t)^2, \qquad (7.14)$$

and

$$|B_{10}| \le C(\Delta t)^2 R^{-2} (\Delta t)^2. \qquad (7.15)$$

Combining inequalities (7.7)-(7.15) for B_1 to B_{10}, we find that

$$\|w^{n+1}\|^2_{0,2,h} \le (1 + C\Delta t)\|w^n\|^2_{0,2,h} + C(\Delta t)^3.$$

Iterating over n, we find that $\|w^n\|^2_{0,2,h} \le C\|w^0\|^2_{0,2,h} + C(\Delta t)^2$, and if $\|w^0\|_{0,2,h} \le C\Delta t$,

$$\|w^n\|_{0,2,h} \le C\Delta t.$$

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