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# A COMPACT DIFFERENCE SCHEME FOR THE BIHARMONIC EQUATION IN PLANAR IRREGULAR DOMAINS*

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**Abstract.** We present a finite difference scheme, applicable to general irregular planar domains, to approximate the biharmonic equation. The irregular domain is embedded in a Cartesian grid. In order to approximate $\Delta^2\Phi$ at a grid point we interpolate the data on the (irregular) stencil by a polynomial of degree six. The finite difference scheme is $\Delta^2 Q_\Phi(0,0)$, where $Q_\Phi$ is the interpolation polynomial. The interpolation polynomial is not uniquely determined. We present a method to construct such an interpolation polynomial and prove that our construction is second order accurate. For a regular stencil, [M. Ben-Artzi, J.-P. Croisille, and D. Fishelov, *SIAM J. Sci. Comput.*, 31 (2008), pp. 303–333] shows that the proposed interpolation polynomial is fourth order accurate. We present some suitable numerical examples.

**Key words.** biharmonic problem, irregular domain, high accuracy, compact approximations, finite differences

**AMS subject classifications.** Primary, 65N06; Secondary, 41A05, 65D05, 35A40

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## 1. Introduction.

In this paper we consider a compact finite difference scheme for the Dirichlet problem for the biharmonic equation,

$$(1)\qquad \left.\begin{aligned} \Phi(x,y) &= g_1(x,y) \\ \Phi_n(x,y) &= g_2(x,y) \end{aligned}\right\}, \quad (x,y)\in\partial\Omega,$$
$$\Delta^2\Phi(x,y) = f(x,y), \quad (x,y) \text{ in } \Omega.$$

Basically we propose a generalization to an irregular $\Omega$ of the well-known nine-point Stephenson scheme [31]. Such a scheme serves as the main building block in the approximation of the two dimensional Navier–Stokes system in the pure stream-function formulation [5, 6].

Due to the significance of the biharmonic operator, a large number of methods for discretizing (1) have been proposed. It seems that the majority of these methods are related to the finite elements methodology (see, for example, [3, 10, 14, 15, 29] and the references therein). However, we concentrate here on finite difference methods. In this category it seems that most of the works are limited to the case where $\Omega$ is a rectangular domain. In this case our scheme is actually equivalent to [7], where a fast direct solver is proposed.

### 1.1. Background.

Let us review briefly some of the works which are closer in spirit to the finite difference approach. In order to avoid the need to deal with a

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fourth order differential operator it has been proposed [2, 3, 12, 14, 15, 16, 17, 24] to split (1) into two coupled Poisson equations for $\Phi$ and $\Phi_1$:

$$\begin{aligned} \Delta\Phi(x, y) &= \Phi_1(x, y), \\ \Delta\Phi_1(x, y) &= f(x, y). \end{aligned} \tag{2}$$

The difficulty with this approach is that the boundary conditions for $\Phi$ are overdetermined, while being underdetermined for the new unknown $\Phi_1$. The boundary conditions for $\Phi_1$ need to be approximated from the discrete form of (2). Within the finite element methodology, many works have used this splitting, sometimes including the use of multigrid technique as well. There have been also some finite difference schemes related to (2). Most of them are limited to a rectangular domain (for example, [17, 24]). In [2] this approach was used for nonregular meshes inside a rectangular domain. Recently [12] suggested an algorithm for an irregular domain, which is based on an immersed interface fast Poisson solver.

A different finite difference approach is to discretize (1) on a uniform grid using a 13-point (or 25-point) direct approximation of the biharmonic operator. Such methods were presented in [9, 20] and are applicable only for a rectangular domain. These approximations must always be modified at grid points near the boundary, a modification which can reduce the accuracy of the scheme.

We mention other approaches to the biharmonic problem on a nonrectangular domain. These include methods based on a conformal mapping to a disk [11, 27], integral equations [26], the fast multipole method [19], and orthogonal spline collocation [4].

Another approach is to use a nine-point compact cell. This approach includes discretizing (1) using not just the grid values of $\Phi$ but also the values of the gradients $\Phi_x$ and $\Phi_y$. This method does not require any modifications near the boundary, as the boundary conditions also include the values of $\nabla\Phi$ [31]. Some variations of this approach and a multigrid technique were proposed in [1]. So far, all the algorithms based on the nine-point stencil were applicable only to a rectangular domain.

**1.2. Structure of the paper.** In this paper we develop the idea of using a nine-point compact cell utilizing the grid values of $\Phi_x$ and $\Phi_y$ in addition to $\Phi$. However, it does not require the stencil to be regular, which enables our scheme to discretize an irregular boundary. The finite difference coefficients are calculated using an interpolation polynomial of degree six. This technique enables us to handle an irregular domain without adding unnecessary grid points to the calculation.

The plan of this paper is as follows. In section 2 we describe the construction of the grid, the assignment of the "calculated nodes," and the treatment of the irregular boundary. Section 3 is the core section of this paper. We develop here our polynomial interpolation over compact irregular stencils and define the discretized approximation to the biharmonic operator. This approximation is used for all the calculated nodes, including near-boundary nodes. Our key result, Theorem 3.11, states that under a reasonable structural assumption on the grid, the discrete approximation of the biharmonic operator is second order accurate. The approximate grid values of $\Phi_x, \Phi_y$ are related to those of $\Phi$ by means of a Hermitian form which guarantees fourth order accuracy even in the irregular case. This construction is described in section 4. Some numerical test cases are presented in section 5. Finally, we mention in the conclusion the connection of this work with the Shortley–Weller scheme for the Poisson problem. For a convergence proof of the new scheme, we refer to the forthcoming work [8].

**2. Embedding a Cartesian grid in an irregular domain.** Consider a domain embedded in a large uniform grid of mesh size $h$. A grid point is a point $(ih, jh)$ for $i, j \in \mathbb{Z}$. Most of the nodes, which are interior to $\Omega$, will be designated as "calculated nodes," i.e., nodes where the approximate functional values are actually calculated. These values are $\Phi, \Phi_x, \Phi_y$, which serve as approximate values to the analytical values of $\Phi, \nabla\Phi$ at the nodes. A small number of interior nodes close to $\partial\Omega$ are not calculated nodes and only serve in the construction of the scheme. There are no approximate values associated with these nodes, and we label them as "edge nodes." In Figure 1 these nodes are marked with an "x."

The division of the interior nodes between edge nodes and calculated nodes is a parameter of the scheme. Some exterior nodes which are close to $\partial\Omega$ serve in the geometric phase of the scheme, as is explained below. They do not take part in the calculations, i.e., they do not carry approximate values or serve as ghost points. They are shown as simple dots "$\cdot$" in Figure 1.

The essential step is to determine the nodes which are designated as "edge nodes." They are selected in a way which limits the distortion of irregular polygons. More specifically, only nodes which are sufficiently close to the boundary are marked as "edge nodes," so that all resulting irregular polygons (as constructed in the following paragraphs) satisfy the requirements imposed in (30) and also ensure the boundedness of $\epsilon$ in (48). This is achieved in turn by restricting the edge nodes to those whose distance from the boundary is $O(h)$. In practice, however, we use even stricter requirements, as in Remark 5.1. Furthermore, selecting the edge nodes in a proper fashion is crucial to carry out the convergence analysis; see [8].

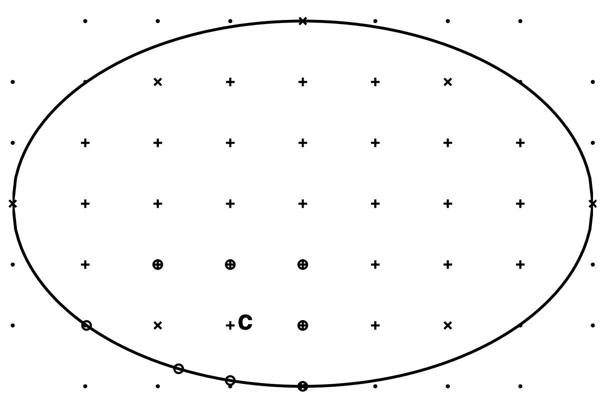


FIG. 1. *An ellipse embedded in the grid.* + *represents calculated nodes,* x *represents edge nodes,* $\cdot$ *represents exterior nodes. The circles are the eight neighbors of the calculated node C.*

Our scheme is a compact scheme; all approximate values of high order derivatives use values of $\Phi, \Phi_x, \Phi_y$ at immediate neighbors. More specifically, given a node $\mathbf{p_0} = (ih, jh)$, we consider the following eight grid points:

$$\tilde{\mathbf{p_1}} = ((i-1)h, (j+1)h), \quad \tilde{\mathbf{p_2}} = (ih, (j+1)h), \quad \tilde{\mathbf{p_3}} = ((i+1)h, (j+1)h),$$
$$\tilde{\mathbf{p_4}} = ((i-1)h, jh), \quad \tilde{\mathbf{p_5}} = ((i+1)h, jh),$$
$$\tilde{\mathbf{p_6}} = ((i-1)h, (j-1)h), \quad \tilde{\mathbf{p_7}} = (ih, (j-1)h), \quad \tilde{\mathbf{p_8}} = ((i+1)h, (j-1)h).$$

Our goal is to construct suitable neighboring points $\mathbf{p_1}, \ldots, \mathbf{p_8}$, which are either calculated nodes or boundary points. The values of $\Phi, \Phi_x, \Phi_y$ at these points are all that is needed in order to calculate the various approximate derivatives at $\mathbf{p_0}$.

Suppose we wish to approximate $\Delta^2\Phi(x_0, y_0)$ at a calculated node $\mathbf{p_0} = (x_0, y_0)$, such that one of its neighbors $\tilde{\mathbf{p_i}} = (x_i, y_i)$ is either an edge node or an exterior node. Imagine the ray that begins at $\mathbf{p_0}$ and goes through $\tilde{\mathbf{p_i}}$. This ray must cross the boundary of the domain. We define the point where the ray first crosses the boundary as $\mathbf{p_i}$ (in the case of a convex domain there is only one point common to the ray and the boundary). The calculation of the approximate value to $\Delta^2\Phi(x_0, y_0)$ relies on the data of $\mathbf{p_i}$ rather $\tilde{\mathbf{p_i}}$. This idea is demonstrated in Figure 1. We have a calculated node designated by C. We construct eight neighboring points $\mathbf{p_1}, \ldots, \mathbf{p_8}$ which carry the data needed for the calculation at C. The four neighbors $\tilde{\mathbf{p_1}}, \tilde{\mathbf{p_2}}, \tilde{\mathbf{p_3}}, \tilde{\mathbf{p_5}}$ are other calculated nodes, so we use them in the calculation, i.e., $\tilde{\mathbf{p_i}} = \mathbf{p_i}$. The other four neighbors $\tilde{\mathbf{p_4}}, \tilde{\mathbf{p_6}}, \tilde{\mathbf{p_7}}, \tilde{\mathbf{p_8}}$ are either edge or exterior nodes, so they are replaced by points on the boundary as described above. We therefore obtain $\mathbf{p_i}$ (the eight circled points), the actual points used in the calculation.

Once the eight points $\mathbf{p_i}$ are determined and approximate values $\Phi, \Phi_x, \Phi_y$ are assigned to them, we can proceed to evaluate an approximate value for $\Delta^2\Phi$ at the point $\mathbf{p_0}$. This is described in the following section.

## 3. The biharmonic $\Delta^2_{\vec{\mathbf{h}}}\Phi$ operator.

In this section we define a scheme for the approximation of the biharmonic operator. Figure 2 shows the nine-point irregular stencil used for the approximation of $\Delta^2\Phi$ at $\mathbf{p_0} = (0, 0)$. Each of the nine grid points $\mathbf{p_i}$ carries three values: $\Phi, \Phi_x, \Phi_y$. These are calculated values if $\mathbf{p_i}$ is a calculated node and given boundary data if $\mathbf{p_i}$ is a boundary point.

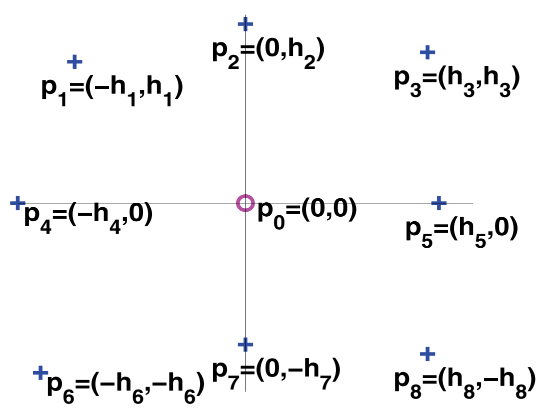


FIG. 2. *The eight neighbors.*

In order to approximate $\Delta^2\Phi$ of a given function $\Phi$ at $\mathbf{p_0}$, we interpolate the data $\Phi, \Phi_x, \Phi_y$ on the stencil $\mathbf{p_0}, \ldots, \mathbf{p_8}$ by a polynomial of degree six. This construction is carried out below. To deal with an irregular stencil, we denote

$$\vec{\mathbf{h}} = (h_1, \ldots, h_8). \tag{3}$$

DEFINITION 3.1. *The finite difference scheme* $\Delta^2_{\vec{\mathbf{h}}}\Phi$ *for the approximation of* $\Delta^2\Phi$ *at* $\mathbf{p_0} = (0, 0)$ *is* $\Delta^2 Q_{\Phi,\vec{\mathbf{h}}}(0, 0)$*, where* $Q_{\Phi,\vec{\mathbf{h}}}$ *is the interpolation polynomial of degree six mentioned above.*

*Remark* 3.2. The actual connection between the values of $\Phi, \Phi_x, \Phi_y$ at all nodes is described in section 4. It uses a fourth order Hermitian form.

**3.1. Approximating the data using a sixth order polynomial.** Let $\mathbb{P}^i$ be the linear space of polynomials in two variables of degree $\leq i$. Let $P^i$ be their single variable counterparts. There are 28 monomials in $\mathbb{P}^6$, so that $\mathbb{P}^6 \cong \mathbb{R}^{28}$.

Let $\mathbf{p_i}$ mark the neighboring points as shown in Figure 2: $\mathbf{p_0} = (0,0), \mathbf{p_1} = (-h_1, h_1), \ldots, \mathbf{p_8} = (h_8, -h_8)$.

Let $Dat_{19}$ be the linear space of the following 19 data:

$$\begin{array}{cc} \Phi(\mathbf{p_i}), & 0 \leq i \leq 8, \\ \Phi_d(\mathbf{p_i}), & 1 \leq i \leq 8, \\ \Phi_x(\mathbf{p_0}), \Phi_y(\mathbf{p_0}), & \end{array} \tag{4}$$

where $\Phi_d$ is the directional derivative, with the direction towards the origin ($d \in \pm\{x, y, \frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\}$). The above 19 data are the only data used to define the interpolation polynomial.

*Remark* 3.3. The literature devoted to polynomial interpolations based on functional values and their gradients, the so-called "Hermite–Birkhoff interpolations," is quite extensive (see [18] and the references therein). In the one dimensional case it is also referred to as a "Lagrangian interpolation problem with repeated arguments." A general formula for this kind of problem is available; see [22]. However, the *unique character* of our scheme (as well as that of [5], [6]) is that it is based *solely* on functional values. The "gradient values" are evaluated as "Hermitian derivatives" and derived from the functional values alone (see section 4).

The goal of this section is to find a polynomial in $\mathbb{P}^6$ which interpolates the 19 data in $Dat_{19}$.

Let $A : \mathbb{P}^6 \to Dat_{19}$ be the linear transformation which is the evaluation operator. That is, given $Q \in \mathbb{P}^6$,

$$A(Q) = \left\{ Q(\mathbf{p_0}), \ldots, Q(\mathbf{p_8}), \frac{\partial Q}{\partial d}(\mathbf{p_1}), \ldots, \frac{\partial Q}{\partial d}(\mathbf{p_8}), \frac{\partial Q}{\partial x}(\mathbf{p_0}), \frac{\partial Q}{\partial y}(\mathbf{p_0}) \right\}.$$

PROPOSITION 3.4. *$A$ is surjective. That is,* $\dim(Im(A)) = 19$ *and* $\dim(\ker(A)) = 9$.

*Proof.* Let $\mathbf{d} \in Dat_{19}$, be a set of data as defined in (4). We will construct a polynomial $Q \in \mathbb{P}^6$ such that $A(Q) = \mathbf{d}$.

Let $Q_1 \in P^5$ be the unique single-variable fifth order polynomial such that $Q_1(x)$ interpolates the six data on the $x$-axis: $\Phi(\mathbf{p_4}), \Phi(\mathbf{p_0}), \Phi(\mathbf{p_5}), \Phi_x(\mathbf{p_4}), \Phi_x(\mathbf{p_0}), \Phi_x(\mathbf{p_5})$.

Let $Q_2 \in P^4$ be the unique polynomial such that $Q_1(x)|_{x=0} + yQ_2(y)$ interpolates the following five data on the $y$-axis: $\Phi(\mathbf{p_2}), \Phi(\mathbf{p_7}), \Phi_y(\mathbf{p_2}), \Phi_y(\mathbf{p_0}), \Phi_y(\mathbf{p_7})$.

In a similar fashion, let $Q_3 \in P^3$ be the unique polynomials such that $xyQ_3(x - y) + yQ_2(y) + Q_1(x)$ interpolates the values along the $y = -x$ diagonal:

$$\Phi(\mathbf{p_1}), \Phi(\mathbf{p_8}), \Phi_{x-y}(\mathbf{p_1}), \Phi_{x-y}(\mathbf{p_8}).$$

Finally, let $Q_4 \in P^3$ be the unique polynomials such that $xy(x + y)Q_4(x + y) + xyQ_3(x - y) + yQ_2(y) + Q_1(x)$ interpolates the values along the $y = x$ diagonal: $\Phi(\mathbf{p_3}), \Phi(\mathbf{p_6}), \Phi_{x+y}(\mathbf{p_3}), \Phi_{x+y}(\mathbf{p_6})$.

Let $Q = Q_1(x) + yQ_2(y) + xyQ_3(x - y) + xy(x + y)Q_4(x + y)$. At each step in the construction above, the interpolations already made aren't modified. Therefore, $Q$ interpolates all the required 19 data: $A(Q) = \mathbf{d}$.

To show that $\dim(\ker(A)) = 9$, we note that $\dim(\ker(A)) = \dim(\mathbb{P}^6) - \dim(Im(A)) = 28 - 19 = 9$. ∎

*Remark* 3.5. Note that by restricting $Q_4 \in P^2$ in the previous proof, we obtain $Q(x,y) \in \mathbb{P}^5$, which can interpolate all but one of the 19 data (leaving out at most one value on the $x = y$ diagonal). This remark will be useful later on, in the proof of Lemma 3.9.

Proposition 3.4 shows that it is always possible to interpolate the data in $Dat_{19}$ using a polynomial in $\mathbb{P}^6$. However, this interpolation is not unique, and therefore, we construct a canonical (unique) interpolation polynomial. To do so, we must first disregard the polynomials in $\ker(A)$ as they are not influenced by the data (even though they might influence the biharmonic operator). Therefore, we will define a polynomial space $\Theta$ such that

$$\mathbb{P}^6 = \ker(A) \oplus \Theta. \tag{5}$$

The linear transformation $A|_\Theta$ is one to one onto $Dat_{19}$. Thus, using $(A|_\Theta)^{-1}$, any set of data will uniquely define an interpolation polynomial in $\Theta \subset \mathbb{P}^6$.

Unfortunately, the equation $\mathbb{P}^6 = \ker(A) \oplus \Theta$ does not uniquely define $\Theta$. Moreover, a different choice of $\Theta$ might produce different results (a different finite difference scheme $\Delta^2_{\mathbf{h}}$ for the biharmonic operator; see Definition 3.1 and Example 3.12). In order to construct $\Theta$, we first characterize $\ker(A)$ using the notion of indifferent polynomials.

DEFINITION 3.6.

(i) *A null-biharmonic polynomial* $Q \in \mathbb{P}^6$ *is a polynomial such that* $\Delta^2 Q(0,0) = 0$.

(ii) *An indifferent polynomial is a null-biharmonic polynomial in* $\ker(A) \subset \mathbb{P}^6$.

(iii) *An indifferent (resp., null-biharmonic) subspace* $L$ *is a linear space such that* $\forall Q \in L$, $Q$ *is indifferent (resp., null-biharmonic).*

The biharmonic operator is simply a linear functional operating on $\mathbb{P}^6$. The kernel of this linear functional is the maximal null-biharmonic subspace. The importance of indifferent polynomials is demonstrated by the following proposition, which shows that $\Theta$ needs to be defined up to an indifferent subspace.

PROPOSITION 3.7. *The finite difference scheme is identical for different* $\Theta$*'s (satisfying* (5)*) which are equivalent up to an indifferent subspace. To be precise, assume* $\mathbb{P}^6 = \ker(A) \oplus \Theta_1$, $\mathbb{P}^6 = \ker(A) \oplus \Theta_2$, $\Theta_2 \subset \Theta_1 \oplus L$, *where* $L$ *is an indifferent subspace, then the finite difference schemes using either* $\Theta_1$ *or* $\Theta_2$ *are identical.*

*Proof.* Let $\mathbf{d} \in Dat_{19}$ be a set of data. Define $Q_1 = A|_{\Theta_1}^{-1}(\mathbf{d})$, $Q_2 = A|_{\Theta_2}^{-1}(\mathbf{d})$, and their difference by $Q = Q_1 - Q_2$. Note that $A(Q_1) = A(Q_2)$; hence $Q \in \ker(A)$. Since $Q \in \Theta_1 \oplus L$ and $L \subset \ker(A)$, where $L$ is an indifferent subspace, one has $Q \in L$, and therefore, $Q$ is an indifferent polynomial. This shows that $\Delta^2 Q(0,0) = 0$, and therefore, $\Delta^2 Q_1(0,0) = \Delta^2 Q_2(0,0)$, proving that the scheme is equivalent for $\Theta_1$, $\Theta_2$. ∎

If $\ker(A)$ were an indifferent subspace, then the proposition shows that any $\Theta$ satisfying (5) would yield an equivalent scheme for the biharmonic operator. However, this is not the case in general.

*Example* 3.8. In the regular grid (i.e., $h_i = h$, $0 \le i \le 8$) the following two polynomials are in $\ker(A)$ and are not indifferent (not null-biharmonic):

$$\begin{aligned} G_1(x,y) &= x^2(x-h)^2(x+h)^2, \\ G_2(x,y) &= y^2(y-h)^2(y+h)^2. \end{aligned} \tag{6}$$

**3.2. Choice of $\Theta$.** The construction of $\Theta$ begins with the polynomials of degree $\leq 5$. We note that $\mathbb{P}^5 \cong \mathbb{R}^{21}$. Let $\tilde{A} : \mathbb{P}^5 \to Dat_{19}$ be the evaluation operator. That is, $\tilde{A} \cong A|_{\mathbb{P}^5}$.

Let

$$\ker(A) = \mathcal{L}^{\text{ind}} \oplus \mathcal{L}^{\text{res}}, \tag{7}$$

where $\mathcal{L}^{\text{ind}}$ is the maximal indifferent subspace and $\mathcal{L}^{\text{res}}$ is a *residual* subspace satisfying (7). Let $B$ be the principal ideal in $\mathbb{P}^6$ generated by $xy(x-y)(x+y)$:

$$B = \left\{ Q \in \mathbb{P}^6 \mid Q = xy(x-y)(x+y)R\ ,\ R \in \mathbb{P}^2 \right\}. \tag{8}$$

LEMMA 3.9.

(i) *$B$ is a six dimensional indifferent subspace of* $\ker(A)$ *which is independent of* $\vec{\mathbf{h}}$ (see (3)).

(ii) $\ker(\tilde{A}) = B \cap \mathbb{P}^5$ *is an indifferent subspace of dimension* 3*, which is independent of* $\vec{\mathbf{h}}$.

*Proof.* (i) It is easy to see that $B \subset \ker(A)$, and that $\forall Q \in B$, $\Delta^2 Q(0,0) = 0$. Since $\dim(\mathbb{P}^2) = 6$, we have $\dim(B) = 6$.

(ii) Using (8), let

$$\tilde{B} = B \cap \mathbb{P}^5 = \left\{ Q \in \mathbb{P}^5 \mid Q = xy(x-y)(x+y)R\ ,\ R \in \mathbb{P}^1 \right\}. \tag{9}$$

Note that $\tilde{B} \subseteq \ker(\tilde{A})$, $\dim(\tilde{B}) = 3$, thus $\dim(\ker(\tilde{A})) \geq 3$ and $\dim(Im(\tilde{A})) \leq 18$. We show that $\dim(Im(\tilde{A})) \geq 18$, and hence $\ker(\tilde{A}) = \tilde{B}$, which concludes our proof. Refer to the proof of Proposition 3.4 and Remark 3.5. Using a polynomial $Q \in \mathbb{P}^5$ we can interpolate at least 18 of the data. Thus, we know that the image of the polynomials of degree five creates a subspace of dimension $\geq 18$. Altogether we have $\dim(Im(\tilde{A})) = 18$ and $\dim(\ker(\tilde{A})) = 3$, as required. ∎

Note that in the regular case $B \neq \mathcal{L}^{\text{ind}}$, since $xy(x^2+y^2-2h^2)^2 \in \ker(A)$ is an indifferent polynomial.

Let

$$\begin{aligned} \tilde{\Theta} = sp\{&1, x, x^2, x^3, x^4, x^5, y, y^2, y^3, y^4, y^5, xy, xy(x+y); \\ &xy(x+y)^2, xy(x+y)^3, xy(x-y), xy(x-y)^2, xy(x-y)^3\}; \end{aligned} \tag{10}$$

$\tilde{\Theta}$ is spanned by 18 polynomials. Note that $\ker(\tilde{A}) \cap \tilde{\Theta} = 0$. Indeed, if $Q(x,y) \in \ker(\tilde{A}) \cap \tilde{\Theta} \subset \mathbb{P}^5$, it should be divisible by $xy(x+y)(x-y)$. It is readily seen, from the definition of $\tilde{\Theta}$ in (10), that no nontrivial $Q$ in $\tilde{\Theta}$ is divisible by $xy(x+y)(x-y)$. Therefore, we can conclude that

$$\mathbb{P}^5 = \ker(\tilde{A}) \oplus \tilde{\Theta} = \tilde{B} \oplus \tilde{\Theta}. \tag{11}$$

We construct $\Theta$ by adding one more polynomial of degree six to $\tilde{\Theta}$. This yields the desired 19 dimensional space. The choice of $\Theta$ determines the finite difference scheme (as presented in Definition 3.1).

We note that Proposition 3.7 and Lemma 3.9 show that any choice of $\tilde{\Theta}$ satisfying (11) would result in an equivalent scheme.

The polynomial that should be added to $\tilde{\Theta}$ must not be in $\tilde{\Theta} \oplus \ker(A)$. In the regular case $h_i = h,\ 0 \leq i \leq 8$, we have

$$\mathcal{L}^{\text{ind}} = sp\{B, xy(x^2+y^2-2h^2)^2\}, \tag{12}$$

$$\mathcal{L}^{\text{res}} = sp\{G_1, G_2\}, \tag{13}$$

where $\mathcal{L}^{\text{ind}}$, $\mathcal{L}^{\text{res}}$ were defined in (7) and $G_1, G_2$ were defined in (6). It can then be easily verified that

$$\{x^6, x^5y, x^3y^3, xy^5, y^6\} \subset \tilde{\Theta} \oplus \mathcal{L}^{\text{ind}} \oplus \mathcal{L}^{\text{res}}. \tag{14}$$

This leaves just two monomials of degree six: $x^4y^2, x^2y^4$. Note that $x^4y^2 - x^2y^4 = x^2y^2(x-y)(x+y) \in B$ is an indifferent polynomial. Thus, Proposition 3.7 tells us that adding $x^2y^4$ or $x^4y^2$ to $\tilde{\Theta}$ yields an equivalent scheme. Therefore, the scheme is essentially determined for the regular case. For the sake of symmetry we define

$$\Theta = sp\{\tilde{\Theta}, x^2y^2(x^2+y^2)\}. \tag{15}$$

PROPOSITION 3.10. *The subspace $\Theta$ as defined in* (15) *satisfies our requirement in* (5)*, namely the direct sum $\Theta \oplus \ker(A) = \mathbb{P}^6$.*

*Proof.* For the regular case the claim follows from the preceding discussion. One must show that $x^2y^2(x^2+y^2) \notin sp\{\tilde{\Theta}, \ker(A)\}$. The comment after (14) shows that it is equivalent to show that $x^4y^2 \notin sp\{\tilde{\Theta}, \ker(A)\}$.

Assume the contrary:

$$x^4y^2 = Q_1 + Q_2, \quad Q_1 \in \tilde{\Theta}, \quad Q_2 \in \ker(A). \tag{16}$$

We have $A(x^4y^2) = A(Q_1)$. Consider the restrictions of these polynomials on the $x = y$ diagonal. The six data on the $x = y$ diagonal, $\Phi(\mathbf{p_0}), \Phi(\mathbf{p_3}), \Phi(\mathbf{p_6}), \Phi_{x+y}(\mathbf{p_0}), \Phi_{x+y}(\mathbf{p_3}), \Phi_{x+y}(\mathbf{p_6})$, are identical. Therefore, the polynomial $x^6 - Q_1(x,x)$ has three double zeros and must be of the form $Cx^2(x-h_3)^2(x+h_6)^2$, where $C$ is an arbitrary constant. By comparing the $x^6$ coefficient one has $C = 1$:

$$x^6 - Q_1(x,x) = x^2(x-h_3)^2(x+h_6)^2. \tag{17}$$

A similar argument on the $x = -y$ diagonal gives us

$$x^6 - Q_1(x,-x) = x^2(x-h_8)^2(x+h_1)^2. \tag{18}$$

Consider the restriction of (16) to the $x$ axis. Applying $A$ to both sides we have $A(Q_1(x,0)) = 0$. There are six data on the $x$ axis, and therefore, $Q_1(x,0)$ has three double zeros . Since $Q_1(x,0)$ is a polynomial of degree five we have $Q_1(x,0) = 0$. A similar argument shows that $Q_1(0,y) = 0$. In particular, there are no pure powers of $x$ or $y$ in $Q_1(x,y)$. Thus, the only second order term in $Q_1(x,y)$ is $\alpha xy$. However, (17) (resp., (18)) shows that $\alpha$ is positive (resp., negative); hence 0. This is a contradiction to (17) and (18), which contain a nonzero coefficient of $xy$ in $Q_1$. ∎

In the regular case $h_i = h,\ 0 \le i \le 8$, the coefficients of the finite difference scheme for $\Delta^2\Phi$ are equivalent to the fourth order scheme presented in [7]:

$$\begin{aligned}
&\Delta^2_{\mathbf{h}}\Phi(0,0) = \Delta^2 Q_{\Phi,\vec{\mathbf{h}}}(0,0)\\
&= \frac{6}{h^4}\{12\Phi(0,0) + \Phi(-h,h) - 4\Phi(0,h) + \Phi(h,h) - 4\Phi(-h,0)\\
&- 4\Phi(h,0) + \Phi(-h,-h) - 4\Phi(0,-h) + \Phi(h,-h)\}\\
&+ \frac{1}{h^3}\{\Phi_x(-h,h) - \Phi_x(h,h) - 8\Phi_x(-h,0) + 8\Phi_x(h,0) + \Phi_x(-h,-h)\\
&- \Phi_x(h,-h) - \Phi_y(-h,h) + 8\Phi_y(0,h) - \Phi_y(h,h) + \Phi_y(-h,-h)\\
&- 8\Phi_y(0,-h) + \Phi_y(h,-h)\}.
\end{aligned} \tag{19}$$

In the irregular case we are still left with a degree of freedom in the definition of $\Theta$. However, we proceed to show that taking $\Theta$ as presented in (15) also for the irregular case yields second order accuracy. Proposition 3.10 assures that the requirement (5) is conserved.

**3.3. Accuracy of the finite difference scheme.** For the accuracy analysis of the finite difference scheme $\Delta^2_{\vec{\mathbf{h}}}$ (Definition 3.1), we assume $\Phi$ is a regular function, differentiable as much as needed. Using the seventh degree Taylor formula around $(0,0)$,

$$\Phi(x,y) = \sum_{0\le|\alpha|\le 6} \frac{D^\alpha \Phi(0,0)}{\alpha!} x^{\alpha_1} y^{\alpha_2} + O(h^7)E(x,y), \quad \alpha = (\alpha_1, \alpha_2), \tag{20}$$

where

$$|E(x,y)| \le C \sup |\Phi^{(7)}|.$$

Combining (5) and (7),

$$\mathbb{P}^6 = \Theta \oplus \mathcal{L}^{\text{ind}} \oplus \mathcal{L}^{\text{res}}. \tag{21}$$

We assume all the derivatives of $\Phi$ of degree $\le 7$ are bounded with a universal bound. Decomposing the polynomial part of (20) by the direct sum (21),

$$\Phi(x,y) = \tilde{\Phi} + O(h^7)E(x,y),\ \tilde{\Phi} \in \mathbb{P}^6, \tag{22}$$

where

$$\tilde{\Phi} = \Phi^\Theta + \Phi^{\text{ind}} + \Phi^{\text{res}}, \quad \Phi^\Theta \in \Theta, \quad \Phi^{\text{ind}} \in \mathcal{L}^{\text{ind}}, \quad \Phi^{\text{res}} \in \mathcal{L}^{\text{res}}. \tag{23}$$

Let $Q_{\Phi,\vec{\mathbf{h}}} \in \Theta$ be the interpolating polynomial of $\Phi$ using the 19 data in $Dat_{19}$ on the $\vec{\mathbf{h}}$ stencil (see (3), (4)). Recall (Definition 3.1) that $\Delta^2 Q_{\Phi,\vec{\mathbf{h}}}(0,0)$ is the finite difference scheme for the biharmonic operator. Note that $\Phi^\Theta - Q_{\Phi,\vec{\mathbf{h}}}$ is not identical to zero since it is the polynomial interpolating the 19 data of the term $O(h^7)E(x,y)$.

Let $K_{\vec{\mathbf{h}}}$ be the approximation (truncation) error for the biharmonic operator:

$$K_{\vec{\mathbf{h}}}(\Phi) = \Delta^2\Phi(0,0) - \Delta^2 Q_{\Phi,\vec{\mathbf{h}}}(0,0). \tag{24}$$

We note that

$$S \in \Theta \Rightarrow K_{\vec{\mathbf{h}}}(S) = 0, \tag{25}$$

$$R \in \mathcal{L}^{\text{ind}} \Rightarrow K_{\vec{\mathbf{h}}}(R) = 0. \tag{26}$$

Indeed, if $S \in \Theta$, then $S = Q_{S,\vec{\mathbf{h}}}$ by definition. If $R \in \mathcal{L}^{\text{ind}}$, then $R \in \ker(A)$ is an indifferent polynomial, and therefore, $Q_{R,\vec{\mathbf{h}}} = 0$ and $\Delta^2 R(0,0) = 0$.

Thus we are left with

$$K_{\vec{\mathbf{h}}}(\Phi) = K_{\vec{\mathbf{h}}}(\Phi^{\text{res}}) + O(h^7)K_{\vec{\mathbf{h}}}(E). \tag{27}$$

This equation shows us something we already suspected: the error of the scheme results from polynomials which are in $\ker(A)$ and are not indifferent (in addition to the negligible error resulting from the Taylor remainder term of degree $\ge 7$; see (20)).

For the irregular mesh analysis we continue to use a *grid parameter* $h > 0$ (see section 2) and define the ratios,

$$c_i = \frac{h_i}{h},\ \ 0 < c_i, \tag{28}$$

$$\vec{\mathbf{c}} = \{c_1, \dots, c_8\}. \tag{29}$$

Note that $c_i$ can be greater than, less than, or equal to 1. This can be seen in Figure 1 where the boundary points used in the calculation of the scheme can be closer or farther apart than the neighboring grid point in the same direction.

Assume that there exists a constant $M \geq 1$ such that

$$\frac{1}{M} \leq c_i \leq M,\ 1 \leq i \leq 8. \tag{30}$$

Our fundamental result is the following theorem.

THEOREM 3.11. *The finite difference scheme $\Delta^2_{\vec{\mathbf{h}}}$ for the biharmonic operator using* (15) *is second order accurate.*

*Proof.* The proof is given in several steps.

*Step* I. We claim that the coefficients of the $\Phi$ terms in $\Delta^2_{\vec{\mathbf{h}}}\Phi$ are $O(\frac{1}{h^4})$, while the coefficients of the $\Phi_x$, $\Phi_y$ terms in $\Delta^2_{\vec{\mathbf{h}}}$ are $O(\frac{1}{h^3})$.

This claim is obvious in the regular case; see (19). For an irregular stencil the proof is obtained by a scaling argument. The details are given in the appendix.

*Step* II. We claim that

$$K_{\vec{\mathbf{h}}}(\Phi) = \Delta^2 \Phi^{\mathrm{res}}(0,0) + O(h^3). \tag{31}$$

Calculating the error,

$$K_{\vec{\mathbf{h}}}(E) = \Delta^2 E(0,0) - \Delta^2 Q_{E,\vec{\mathbf{h}}}(0,0).$$

Using Step I,

$$\begin{aligned}\Delta^2 Q_{E,\vec{\mathbf{h}}}(0,0) &= \sum_i O\left(\frac{1}{h^4}\right) E(\mathbf{p_i}) + \sum_i O\left(\frac{1}{h^3}\right) E_x(\mathbf{p_i}) + \sum_i O\left(\frac{1}{h^3}\right) E_y(\mathbf{p_i}) \\ &= O\left(\frac{1}{h^4}\right).\end{aligned}$$

Since $\Delta^2 E(0,0)$ is a constant, altogether we have

$$K_{\vec{\mathbf{h}}}(E) = O\left(\frac{1}{h^4}\right).$$

Equation (31) now follows from (27).

*Step* III. We construct a smooth, with respect to $\vec{\mathbf{h}}$, basis $\{G_1(\vec{\mathbf{h}}; x, y), \ldots, G_9(\vec{\mathbf{h}}; x, y)\}$ of $\ker(A_{\vec{\mathbf{h}}})$.

*Note.* We add the $\vec{\mathbf{h}}$ subscript to $A$ to clarify the stencil in which the calculation is done.

To this end we let

$$\begin{aligned}\Theta^c &= \tilde{B} \oplus sp\{x^6, x^5y, x^4y^2 - x^2y^4, x^3y^3, xy^5, y^6\} \\ &= sp\{x^3y - xy^3, x^4y - x^2y^3, x^3y^2 - xy^4, x^6, x^5y, x^4y^2 - x^2y^4, x^3y^3, xy^5, y^6\}.\end{aligned} \tag{32}$$

In view of (15),

$$\mathbb{P}^6 = \Theta \oplus \Theta^c. \tag{33}$$

Indeed, $\mathbb{P}^5 = \tilde{\Theta} \oplus \tilde{B}$ (see (11)) and $\Theta$ includes only one polynomial of degree six, while the other six are in $\Theta^c$.

Let

$$F : \mathbb{R}^8_+ \times \Theta^c \times \Theta \to Dat_{19},$$
$$H : \mathbb{R}^8_+ \times \Theta^c \to \Theta$$

be defined as follows:

$$F(\vec{\mathbf{h}}, T_1, T_2) = A_{\vec{\mathbf{h}}}(T_1 + T_2), \qquad \vec{\mathbf{h}} \in \mathbb{R}^8_+,\ T_1 \in \Theta^c,\ T_2 \in \Theta, \tag{34}$$

$$H(\vec{\mathbf{h}}, T_1) = -Q_{T_1,\vec{\mathbf{h}}}, \qquad \vec{\mathbf{h}} \in \mathbb{R}^8_+,\ T_1 \in \Theta^c, \tag{35}$$

where $Q_{T_1,\vec{\mathbf{h}}} \in \Theta$ is the interpolation polynomial of $T_1$ with respect to the $\vec{\mathbf{h}}$ stencil. Recalling Proposition 3.10, we know that this interpolation polynomial is uniquely determined in $\Theta$. Moreover, $\forall\ \vec{\mathbf{h}} \in R^8_+,\ T_1 \in \Theta^c,\ T_2 \in \Theta$,

$$H(\vec{\mathbf{h}}, T_1) = T_2 \quad \Leftrightarrow \quad F(\vec{\mathbf{h}}, T_1, T_2) = 0. \tag{36}$$

Using the implicit function theorem it is readily seen that $H(\vec{\mathbf{h}}, T)$ is continuously differentiable with respect to all its variables.

From (36) it follows that

$$\ker(A_{\vec{\mathbf{h}}}) = \{T_1 + H(\vec{\mathbf{h}}, T_1) \mid T_1 \in \Theta^c\}. \tag{37}$$

Let $e_i(x, y)$ $1 \le i \le 9$ be the nine homogeneous polynomials spanning $\Theta^c$. We define

$$\begin{aligned} G_i(\vec{\mathbf{h}}; x, y) &= e_i(x, y) + H(\vec{\mathbf{h}}, e_i(x, y)), \\ &= e_i(x, y) - Q_{e_i,\vec{\mathbf{h}}}(x, y), \quad 1 \le i \le 9, \end{aligned} \tag{38}$$

and note that these polynomials constitute the desired smooth basis of $\ker(A_{\vec{\mathbf{h}}})$. Indeed, the linear independence of the $G_i$'s follows from the fact that the $e_i$'s form a basis for $\Theta^c$ while $Q_{e_i,\vec{\mathbf{h}}} \in \Theta$ is the complementary subspace.

*Step* IV. We can now conclude the proof of the theorem.

Combining (5) and (20),

$$\begin{aligned} \Phi(x, y) &= \Phi^\Theta + \sum_{i=1}^{9} b_i G_i(\vec{\mathbf{h}}) + O(h^7)E(x, y),\ \Phi^\Theta \in \Theta,\ G_i(\vec{\mathbf{h}}) \in \ker(A_{\vec{\mathbf{h}}}) \\ &= \Phi^\Theta + \sum_{i=1}^{9} b_i H(\vec{\mathbf{h}}, e_i) + \sum_{i=1}^{9} b_i e_i + O(h^7)E(x, y). \end{aligned}$$

Note that $\Phi^\Theta + \sum_{i=1}^{9} b_i H(\vec{\mathbf{h}}, e_i) \in \Theta$. In terms of the decomposition (33), the $\Theta^c$ component of $\Phi(x, y)$ is $\sum_{i=1}^{9} b_i e_i$. Since $\{e_i,\ 1 \le i \le 9\}$ are fixed polynomials (independent of $\vec{\mathbf{h}}$), the coefficients $\{b_i,\ 1 \le i \le 9\}$ are constants which depend only on the derivatives of $\Phi$ of degree $\le 6$.

Using (31),

$$K_{\vec{\mathbf{h}}}(\Phi) = \sum_{i=1}^{9} b_i\ \Delta^2 G_i(\vec{\mathbf{h}}; x, y)\Big|_{x=0,y=0} + O(h^3). \tag{39}$$

Recalling the assumption (30), we now show that

$$\left| \Delta^2 G_i(\vec{\mathbf{h}}; x, y)\Big|_{(0,0)} \right| \le h^2 J, \quad \vec{\mathbf{h}} \in \mathbb{R}^8_+,\ \ 1 \le i \le 9, \tag{40}$$

where

$$J = \sup_{1\le i\le 9} \left| \Delta^2 G_i(\vec{\mathbf{c}}; x, y)\big|_{(0,0)} \right|, \quad \vec{\mathbf{c}} \in \left[\frac{1}{M}, M\right]^8 \subset \mathbb{R}^8. \tag{41}$$

Since $\Delta^2 G_i(\vec{\mathbf{h}}; x, y)\big|_{(0,0)}$ is a continuous function of $\vec{\mathbf{h}}$, it attains a maximum in the compact cube $\left[\frac{1}{M}, M\right]^8 \subset \mathbb{R}^8$.

Using the definition (32) of $\Theta^c$,

$$\Theta^c \cap \mathbb{P}^5 = \tilde{B}.$$

Let $1 \le i \le 9$. If $e_i \in \Theta^c \cap \mathbb{P}^5$, then $e_i \in \tilde{B} \subset \ker(A_{\vec{\mathbf{h}}})$. Hence, $H(\vec{\mathbf{h}}, e_i) = 0$ and $G_i(\vec{\mathbf{h}}; x, y) = e_i$. It follows that $\Delta^2 G_i(\vec{\mathbf{h}}; x, y)\big|_{(0,0)} = 0$ (since $\tilde{B}$ is an indifferent subspace). Otherwise, $e_i$ is one of the six homogeneous polynomials of degree six in (32). Consider the decomposition (33) of the polynomial $R(x, y) = h^6 G_i(\vec{\mathbf{c}}; \frac{x}{h}, \frac{y}{h}) = h^6 \left(e_i(\frac{x}{h}, \frac{y}{h}) + H(\vec{\mathbf{c}}, e_i(\frac{x}{h}, \frac{y}{h}))\right)$. Its $\Theta^c$ component is $h^6 \left(e_i(\frac{x}{h}, \frac{y}{h})\right) = e_i(x, y)$. Also, notice that $R$ is 0 on all the data on the $\vec{\mathbf{h}}$ stencil. Hence, $R \in \ker(A_{\vec{\mathbf{h}}})$ and $R = e_i - Q_{e_i, \vec{\mathbf{h}}} = G_i(\vec{\mathbf{h}})$, using (38). Altogether,

$$G_i(\vec{\mathbf{h}}; x, y) = h^6 G_i\left(\vec{\mathbf{c}};\ \frac{x}{h},\ \frac{y}{h}\right).$$

Hence,

$$\left| \Delta^2 G_i(\vec{\mathbf{h}}; x, y)\Big|_{(0,0)} \right| = \left| \frac{h^6}{h^4}\, \Delta^2 G_i(\vec{\mathbf{c}}; x, y)\big|_{(0,0)} \right| \le h^2 J.$$

It now follows from (39) and the last estimate that

$$|K_{\vec{\mathbf{h}}}(\Phi)| \le \sum_{i=1}^{9} |b_i| h^2 J + O(h^3) = O(h^2). \qquad \square \tag{42}$$

*Example* 3.12. We calculate $\Delta^2 \Phi^{\text{res}}(0, 0)$ for a regular grid.

Using $G_1, G_2$ as in (6), we have $\mathcal{L}^{\text{res}} = sp\{G_1, G_2\}$ (see (13)). Using our choice (15) of $\Theta$, there are no polynomials containing the $x^6$ or $y^6$ monomials in $\Theta$. Also, there are no polynomials containing these monomials in $\mathcal{L}^{\text{ind}}$ (see (12)). Comparing the coefficients of $x^6$ and $y^6$ in $\mathcal{L}^{\text{res}}$ and in (20), we have

$$\Phi^{\text{res}} = \frac{1}{6!}\left(\frac{\partial^6 \Phi}{\partial x^6}(0, 0) G_1 + \frac{\partial^6 \Phi}{\partial y^6}(0, 0) G_2\right),$$

and therefore,

$$\begin{aligned} |\Delta^2 \Phi^{\text{res}}(0, 0)| &\le \frac{1}{6!}\left(48 \sup |\Phi^{(6)}| h^2 + 48 \sup |\Phi^{(6)}| h^2\right) \\ &= O(h^2). \end{aligned}$$

A different choice of $\Theta$ could give us different results.

Let $\overline{\Theta} = (\Theta \cup \{y^6\}) \setminus \{y^4\}$. Using $G_2 \in \ker(A_{\vec{\mathbf{h}}})$, we know that $y^4 - \frac{y^6}{2h^2} - \frac{h^2 y^2}{2} \in \ker(A_{\vec{\mathbf{h}}})$. Therefore, $\overline{\Theta}$ satisfies (5). We rewrite (23), using $\overline{\Theta}$,

$$\tilde{\Phi} = \Phi^{\overline{\Theta}} + \Phi^{\overline{\text{ind}}} + \Phi^{\overline{\text{res}}}, \quad \Phi^{\overline{\Theta}} \in \overline{\Theta}, \quad \Phi^{\overline{\text{ind}}} \in \mathcal{L}^{\text{ind}}, \quad \Phi^{\overline{\text{res}}} \in \mathcal{L}^{\text{res}}.$$

In this case, we must compare the coefficients of $x^6$ and $y^4$ in both $\mathcal{L}^{\mathrm{res}}$ and in (20):

$$\begin{aligned}
\Phi^{\overline{\mathrm{res}}} &= \frac{1}{6!}\frac{\partial^6\Phi}{\partial x^6}(0,0)G_1 - \frac{1}{4!}\frac{\partial^4\Phi}{\partial y^4}(0,0)\frac{G_2}{2h^2},\\
|\Delta^2\Phi^{\overline{\mathrm{res}}}(0,0)| &\le \frac{48}{6!}\sup|\Phi^{(6)}|h^2 + \frac{24}{4!}\sup|\Phi^{(4)}|\\
&= O(h^2) + O(1)\\
&= O(1).
\end{aligned}$$

*Summary.* Our goal is to approximate the biharmonic operator at $(0,0)$ given the data from our nine-point irregular stencil (Figure 2). We use 19 data from the stencil, as defined in (4). We first approximate the data by a polynomial in $\Theta \subset P^6$, which is defined in (10), (15). Proposition 3.10 shows that there exists a unique polynomial, $Q \in \Theta$, which interpolates our 19 data. $\Delta^2 Q(0,0)$ is the finite difference approximation of the biharmonic operator at $(0,0)$. The discussion preceding Proposition 3.10 explains the logic in the construction of $\Theta$. Theorem 3.11 shows the resulting scheme and yields the optimal second order accuracy.

Finally, note that we can interpret Stephenson's scheme on a regular stencil [5, 31] using our polynomial approach. Let $\hat{A} : \mathbb{P}^4 \to Dat_{13}$, where $Dat_{13}$ is the 13 dimensional linear space spanned by the 13 data used in the Stephenson scheme. It is easy to show that $\ker(\hat{A}) = sp\{xy(x-y)(x+y), xy(x-h)(x+h)\}$, showing that $\ker(\hat{A})$ is an indifferent subspace (we are limiting the investigation to a regular stencil). Using Proposition 3.7 we know that any space $\Theta_{13}$, such that $\mathbb{P}^4 = \ker(A) \oplus \Theta_{13}$, yields an equivalent scheme, e.g., one can define $\Theta_{13}$ by omitting the following two polynomials from $\mathbb{P}^4$: $x^3y$ and $xy$, obtaining an equivalent scheme to the one presented in the paper [31] (where $x^3y$ and $xy^3$ were omitted).

**4. The Hermitian connection between $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi_x}$, $\boldsymbol{\Phi_y}$.** Given a smooth function $\Phi$ in $\Omega$, we described in section 3 our finite difference approximation of $\Delta^2\Phi$ at grid points. This approximation is based on the knowledge of the values of $\Phi$, $\Phi_x$, and $\Phi_y$ at these points. The grid values for $\Phi$ are those coming from the function $\Phi$ evaluated at the grid points or suitable approximation given to these values. The values of $\Phi_x$, $\Phi_y$ are supposed to approximate values of the gradient of $\Phi$. However, they are not obtained independently. Instead, they are obtained by an interpolation procedure from the values of the grid function $\Phi$ itself.

In this section, we describe a Hermitian interpolation procedure, which connects these values to the values of $\Phi$ in a linear nonlocal connection. The values $\Phi_x$ aligned parallel to the $x$ axis will be connected only to values of $\Phi$ along this line. A similar treatment is granted to the values of $\Phi_y$ aligned parallel to the $y$ axis. This is in agreement with the classical Hermitian methods and serves as a generalization of the method presented in [5, 6]. Other generalizations related to high order connections of discretized values of a function and its derivative can be found in [18, 30].

Assume a one dimensional function $f : [0,1] \to \mathbb{R}$, differentiable as much as needed. Fix $n+1$ points $0 = x_0 < x_1 < \cdots < x_n = 1$. We are given the following data:

$$\begin{cases} \Phi_i = f(x_i), & 0 \le i \le n,\\ q_0 = f'(x_0), & q_n = f'(x_n). \end{cases} \tag{43}$$

Using these data to find values $\{q_i,\ 1 \le i \le n-1\}$, approximating the derivative values $\{f'(x_i),\ 1 \le i \le n-1\}$ is in fact a classical problem tractable, for example, by spline theory [28]. Here, we describe how it can be handled by a finite difference analysis, using a so-called Hermitian approach.

We consider for each three-point stencil $x_{i-1}, x_i, x_{i+1}$ the fourth order interpolation polynomial

$$g_i(x) = a_{4,i}(x - x_i)^4 + a_{3,i}(x - x_i)^3 + a_{2,i}(x - x_i)^2 + a_{1,i}(x - x_i) + a_{0,i},$$

defined by the following five interpolation conditions:

$$g_i(x_{i-1}) = \Phi_{i-1},\ g_i(x_i) = \Phi_i,\ g_i(x_{i+1}) = \Phi_{i+1},\ g_i'(x_{i-1}) = q_{i-1},\ g_i'(x_{i+1}) = q_{i+1}. \tag{44}$$

See Remark 3.3 concerning the class of such interpolation problems.

The unknown value $q_i$ is then obtained as $g'(x_i) = a_{1,i}$. Using a suitable Hermitian integration formula, we obtain the following tridiagonal linear system for the collection of unknowns $\{q_i, 1 \le i \le n-1\}$:

$$\alpha_{1,i} q_{i-1} + q_i + \alpha_{2,i} q_{i+1} = \beta_{1,i}\Phi_{i-1} + \beta_{2,i}\Phi_i + \beta_{3,i}\Phi_{i+1}, \quad 1 \le i \le n-1. \tag{45}$$

The five coefficients $\alpha_{1,i}, \alpha_{2,i}, \beta_{1,i}, \beta_{2,i}, \beta_{3,i}$ are

$$\begin{cases} \alpha_{1,i} = \frac{h_{i+1}^2}{(h_{i+1}+h_i)^2}, \quad \alpha_{2,i} = \frac{h_i^2}{(h_{i+1}+h_i)^2}, \\ \beta_{1,i} = -\frac{2h_{i+1}^4 + 4h_{i+1}^3 h_i}{h_{i+1}(h_{i+1}+h_i)^3 h_i}, \quad \beta_{2,i} = \frac{2h_{i+1}^4 + 4h_{i+1}^3 h_i - 4h_{i+1}h_i^3 - 2h_i^4}{h_{i+1}(h_{i+1}+h_i)^3 h_i}, \\ \beta_{3,i} = \frac{2h_i^4 + 4h_{i+1}h_i^3}{h_{i+1}(h_{i+1}+h_i)^3 h_i}. \end{cases} \tag{46}$$

To analyze the accuracy of the scheme, we use the parameters $\bar{h}, \epsilon$, defined by

$$\bar{h} = \max_{1 \le i \le n-1} \{h_i\}, \tag{47}$$

$$\epsilon = \min_{1 \le i \le n-1} \left\{ \frac{h_{i+1}}{h_i}, \frac{h_i}{h_{i+1}} \right\}. \tag{48}$$

LEMMA 4.1. *Assume $f$ to be a regular function on $[0,1]$. Then the truncation error*

$$R_i = \alpha_{1,i} f'(x_{i-1}) + f'(x_i) + \alpha_{2,i} f'(x_{i+1}) - \beta_{1,i}\Phi_{i-1} - \beta_{2,i}\Phi_i - \beta_{3,i}\Phi_{i+1} \tag{49}$$

*satisfies*

$$|R_i| \le C_1 \bar{h}^4 \left\| f^{(5)} \right\|_{\infty,[0,1]}. \tag{50}$$

*Proof.* Fix $1 \le i \le n-1$. Let $H(x)$ be the fourth order Taylor polynomial for $f(x)$ around $x = x_i$. The following five identities hold:

$$\begin{cases} \Phi_{i-1} = f(x_{i-1}) = H(x_{i-1}) - \frac{h_i^5}{5!} f^{(5)}(\xi_{i,1}), \quad \Phi_{i+1} = f(x_{i+1}) = H(x_{i+1}) + \frac{h_{i+1}^5}{5!} f^{(5)}(\xi_{i,2}), \\ \Phi_i = f(x_i) = H(x_i), \\ f'(x_{i-1}) = H'(x_{i-1}) - \frac{h_i^4}{4!} f^{(5)}(\xi_{i,3}), \quad f'(x_{i+1}) = H'(x_{i+1}) + \frac{h_{i+1}^4}{4!} f^{(5)}(\xi_{i,4}), \end{cases} \tag{51}$$

where $\xi_{i,j} \in [x_{i-1}, x_{i+1}]$, $1 \le j \le 4$. Replacing $f(x_{i-1})$, $f(x_i)$, $f(x_{i+1})$, $f'(x_{i-1})$, $f'(x_{i+1})$ by their values in (49) and observing that $R_i = 0$ for any fourth order polynomial (in particular $H(x)$) yields the estimate (50). $\square$

So far we showed that $f$ satisfies the equation up to fourth order accuracy. Following a routine procedure we show that solving the tridiagonal system (45) does not amplify this error.

LEMMA 4.2. *The system* (45) *has a unique solution. Assuming $f$ is a regular function, the solution is "almost" fourth order accurate:*

$$\max_{1\le i\le n-1} |f'(x_i) - q_i| \le C\frac{\bar{h}^4}{\epsilon}\left\|f^{(5)}\right\|_{\infty,[0,1]},$$

*where $\bar{h}, \epsilon$ are as defined in* (47), (48).

*Proof.* Writing the system (45) in matrix notation one has

$$Pv = w, \tag{52}$$

where

$$P = \begin{pmatrix} 1 & \alpha_{2,1} & 0 & \cdots & 0 \\ \alpha_{1,2} & 1 & \alpha_{2,2} & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \alpha_{1,n-2} & 1 & \alpha_{2,n-2} \\ 0 & \cdots & 0 & \alpha_{1,n-1} & 1 \end{pmatrix}, \quad v = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_{n-2} \\ q_{n-1} \end{pmatrix},$$
$$w = \begin{pmatrix} -\alpha_{1,1}q_0 + \beta_{1,1}\Phi_0 + \beta_{2,1}\Phi_1 + \beta_{3,1}\Phi_2 \\ \beta_{1,2}\Phi_1 + \beta_{2,2}\Phi_2 + \beta_{3,2}\Phi_3 \\ \vdots \\ \beta_{1,n-2}\Phi_{n-3} + \beta_{2,n-2}\Phi_{n-2} + \beta_{3,n-2}\Phi_{n-1} \\ -\alpha_{2,1}q_n + \beta_{1,n-1}\Phi_{n-2} + \beta_{2,n-1}\Phi_{n-1} + \beta_{3,n-1}\Phi_n \end{pmatrix}. \tag{53}$$

Note that $\alpha_{1,i} + \alpha_{2,i} < 1$; hence $P$ is diagonally dominant, therefore invertible. This proves that our system has a unique solution.

Lemma 4.1 shows that

$$Pf' = w + \bar{h}^4 K, \tag{54}$$

where

$$f' = \begin{pmatrix} f'(x_1) \\ \vdots \\ f'(x_{n-1}) \end{pmatrix}, \quad K = \begin{pmatrix} -K_1 \\ \vdots \\ -K_{n-1} \end{pmatrix}. \tag{55}$$

Using, finally, the principle of the proof of Lemma 3.1 in [6], we obtain that

$$\max_{1\le i\le n-1} |f'(x_i) - q_i| \le C\frac{\bar{h}^4}{\epsilon}\left\|f^{(5)}\right\|_{\infty,[0,1]}. \qquad \square \tag{56}$$

*Remark* 4.3. In [6] the scheme presented here is shown to be fourth order accurate on the regular grid. On the irregular grid, Lemma 4.2 shows that the result is almost as good, as long as the grid isn't very "irregular" ($\frac{1}{\epsilon}$ is bounded).

TABLE 1
*Recovering $\Phi(x,y) = x^3 ln(1+y) + \frac{y}{1+x}$ from $\Delta^2\Phi$ in the unit square.*

| | $16 \times 16$ | Rate | $32 \times 32$ | Rate | $64 \times 64$ |
|---|---|---|---|---|---|
| Our scheme | $4.9 \cdot 10^{-7}$ | 4.0 | $3.1 \cdot 10^{-8}$ | 4.1 | $1.8 \cdot 10^{-9}$ |
| Fourth order Stephenson scheme [1] | $8.9 \cdot 10^{-8}$ | 3.9 | $5.8 \cdot 10^{-9}$ | 3.8 | $4.1 \cdot 10^{-10}$ |

TABLE 2
*Recovering $\Phi(x,y) = x^3 ln(1+y) + \frac{y}{1+x}$ from $\Delta^2\Phi$ in the ellipse $\frac{x^2}{0.5^2} + \frac{y^2}{0.15^2} \leq 1$.*

Our scheme

| | $h = \frac{1.2}{16}$ | Rate | $h = \frac{1.2}{32}$ | Rate | $h = \frac{1.2}{64}$ |
|---|---|---|---|---|---|
| Error | $3.0 \cdot 10^{-6}$ | 4.8 | $1.1 \cdot 10^{-7}$ | 6.2 | $1.5 \cdot 10^{-9}$ |

IIM [12]

| | $h = \frac{1.2}{64}$ | Rate | $h = \frac{1.2}{128}$ | Rate | $h = \frac{1.2}{256}$ | Rate | $h = \frac{1.2}{512}$ |
|---|---|---|---|---|---|---|---|
| Error | $3.7 \cdot 10^{-4}$ | 1.9 | $9.5 \cdot 10^{-5}$ | 2.2 | $2.1 \cdot 10^{-5}$ | 2.1 | $5.0 \cdot 10^{-6}$ |

**5. Numerical results.** For the biharmonic operator in a rectangular domain our results are identical to the fourth order scheme in [7], and therefore, we mainly focus on irregular domains. In all the following examples we try to recover the values of a certain function $\Phi$ inside a domain $\Omega$ from the knowledge of $\Delta^2\Phi$ inside $\Omega$ and $\Phi, \nabla\Phi$ on the boundary $\partial\Omega$. That is, we try to solve (1) for given $f, g_1, g_2$, and unknown $\Phi$. The given values are those of an exact solution which we try to recover. We observed an average fourth order accuracy throughout the numerical tests.

*Remark* 5.1. In irregular domains we chose an interior node as a calculated node (see section 2) if the distance from the boundary of the ellipse $\geq 2h^{\frac{3}{2}}$. This constant gave us good numerical results throughout our tests.

*Example* 5.2. We try to recover the values of the function

$$\Phi(x,y) = x^3 ln(1+y) + \frac{y}{1+x}, \quad (x,y) \in \Omega, \tag{57}$$

from the knowledge of $\Delta^2\Phi$. The domain is the unit square: $\Omega = [0,1] \times [0,1]$. Table 1 shows the $L^\infty$ error at each level (maximal difference between the computed solution and the exact one). We compare our results with [1]. It seems that in this case the truncation error is slightly smaller in the fourth order Stephenson scheme than in ours. However, the convergence rate is just as good (even slightly better). The fourth order Stephenson scheme requires five values of the source term for each grid point (equation (9) in [1]). This poses a difficulty when using the scheme for the Navier–Stokes system as done in [5].

*Example* 5.3. We recover the same function $\Phi$ (57) from the knowledge of $\Delta^2\Phi$ inside the skinny ellipse suggested in [12]: $\Omega = \{(x,y) \mid \frac{x^2}{0.5^2} + \frac{y^2}{0.15^2} \leq 1\}$. Table 2 shows the resulting errors. We chose the grid size to be multiples of $h = 1.2$ to match the calculated domain from [12], which is $[-0.6, 0.6] \times [-0.3, 0.3]$. The immersed interface method presented in [12] is fast and calculates using a much larger mesh than ours. However, even under a large mesh, with $h = \frac{1.2}{512}$, the error in [12] is $5.0 \cdot 10^{-6}$, which is comparable to a mesh with $h = \frac{1.2}{16}$ using the proposed scheme. Our scheme performs well; the more than expected rate of convergence (4.8), (6.2) is attributed to two reasons:

TABLE 3
*Recovering $\Phi(x,y) = e^{x+y}$ from $\Delta^2\Phi$ in the unit disk $x^2+y^2 \leq 1$.*

Our Scheme

| | $h=\frac{1}{4}$, $GP=29$ | Rate | $h=\frac{1}{8}$, $GP=177$ | Rate | $h=\frac{1}{16}$, $GP=749$ | Rate | $h=\frac{1}{32}$, $GP=3149$ |
|---|---|---|---|---|---|---|---|
| Error | $3.7\cdot 10^{-4}$ | 5.5 | $8.4\cdot 10^{-6}$ | 2.6 | $1.4\cdot 10^{-6}$ | 6.6 | $1.4\cdot 10^{-8}$ |

Scheme of [23]

| | $GP=1024$ | Rate | $GP=2048$ | Rate | $GP=4096$ | Rate |
|---|---|---|---|---|---|---|
| Error | $1.2\cdot 10^{-3}$ | 1.92 | $3.2\cdot 10^{-4}$ | 1.98 | $8.1\cdot 10^{-5}$ | 1.99 |
| | $GP=8192$ | Rate | $GP=16384$ | | | |
| Error | $2.0\cdot 10^{-5}$ | 1.99 | $5.1\cdot 10^{-6}$ | | | |

TABLE 4
*Recovering $\Phi(x,y) = (1-x^2)^2(1-y^2)^2$ from $\Delta^2\Phi$ in the ellipse $\frac{x^2}{1^2}+\frac{y^2}{2^2} \leq 1$.*

| | $h=\frac{1}{4}$ | Rate | $h=\frac{1}{8}$ | Rate | $h=\frac{1}{16}$ | Rate | $h=\frac{1}{32}$ |
|---|---|---|---|---|---|---|---|
| Our scheme | $3.49\cdot 10^{-3}$ | 4.7 | $1.33\cdot 10^{-4}$ | 5.0 | $4.28\cdot 10^{-6}$ | 4.1 | $2.43\cdot 10^{-7}$ |

- The small amount of calculated nodes means the asymptotic behavior is not reached.
- When the mesh size is decreased the amount of calculated points increases beyond the expected amount. This is because the *boundary layer*, in which interior nodes are not calculated nodes, is also decreased (Remark 5.1), i.e., when the grid size is halved the amount of calculated nodes is more than quadrupled.

*Example* 5.4. We compare our scheme to a numerical example given in [23]. In this case we try to recover the function

$$\Phi(x,y) = e^{x+y}, \;\; (x,y) \in \Omega,$$

from $\Delta^2\Phi$ in the unit disk, $\Omega = \{(x,y) \mid x^2+y^2 \leq 1\}$. Table 3 shows the resulting errors. To compare the results with the polar grid used in [23], we denote by *GP* the number of grid points used in the calculation. The assignment of grid points in [23] is based on an algorithm which sets the points on a certain amount of concentric circles. As in the previous example, our scheme is slower, but compares favorably for much smaller mesh sizes.

*Example* 5.5. We try to recover the function

$$\Phi(x,y) = (1-x^2)^2(1-y^2)^2, \;\; (x,y) \in \Omega,$$

from $\Delta^2\Phi$ in the ellipse: $\Omega = \{(x,y) \mid \frac{x^2}{1^2}+\frac{y^2}{2^2} \leq 1\}$. Table 4 shows the numerical results for this example. The scheme maintains a high numerical convergence rate.

**6. Conclusion.** The problem solved in this paper consists of a natural generalization of the compact scheme of Stephenson [31] to irregular geometries. Contrary to the classical finite element method, which uses a weak form of the biharmonic problem, we stick here to the design of compactly supported finite difference operators using high order interpolating polynomials. This work can also be seen as an extension of the Shortley–Weller scheme for the Laplacian [25] to the biharmonic problem.

As observed in section 3, a significant difficulty in our approach was due to the fact that we used 19 "natural" data (over a stencil of nine points) to be interpolated by a sixth order polynomial. This generated a nontrivial kernel. It was a delicate matter to "optimize" the direct complement to this kernel so as to get good accuracy for the approximation of the biharmonic operator.

Another approach consists of providing sufficiently many points for a given order of the class of polynomials considered. The interpolation problem is then uniquely solved and the error estimates for the approximation of a given differential operation are then based on suitable norms (such as Sobolev norms) of the interpolating polynomial. We refer the reader to [13] for more details. The final setup of the scheme makes use of a Hermitian connection between the primary data $\Phi$ and the gradient data $\Phi_x, \Phi_y$ presented in section 4. The efficiency of the resulting scheme is clearly demonstrated in section 5. While the scheme presented here is shown to be second-order accurate, the problem concerning its convergence is considerably more delicate, and is treated in a forthcoming paper [8]. Contrary to the variational approach of the finite element method, it relies on a careful examination of each kind of point present in the domain, in the spirit of the proofs of the Shortley–Weller scheme for the Poisson problem; see [21, 25] and the references therein.

**7. Appendix.** We prove Step I of Theorem 3.11.

Let

$$Q_{\Phi,\vec{\mathbf{h}}} = Q^0_{\Phi,\vec{\mathbf{h}}} + Q^d_{\Phi,\vec{\mathbf{h}}},$$

where $Q^0_{\Phi,\vec{\mathbf{h}}}$ interpolates the $\Phi$ data, while substituting 0 for all the $\Phi_x$,$\Phi_y$ data. In a similar fashion, $Q^d_{\Phi,\vec{\mathbf{h}}}$ satifies the following (as in (4) after substituting 0 for all the $\Phi$ data):

$$\begin{aligned}
&Q^d_{\Phi,\vec{\mathbf{h}}}(\mathbf{p_i}) = 0, && 0 \le i \le 8,\\
&\frac{\partial Q^d_{\Phi,\vec{\mathbf{h}}}}{\partial d}(\mathbf{p_i}) = \Phi_d(\mathbf{p_i}), && 1 \le i \le 8,\\
&\frac{\partial Q^d_{\Phi,\vec{\mathbf{h}}}}{\partial x}(\mathbf{p_0}) = \Phi_x(\mathbf{p_0}), \frac{\partial Q^d_{\Phi,\vec{\mathbf{h}}}}{\partial y}(\mathbf{p_0}) = \Phi_y(\mathbf{p_0}).
\end{aligned}$$

First, we show that the coefficients of $Q^0_{\Phi,\vec{\mathbf{h}}}$ are $O(\frac{1}{h^4})$.

Let $S_\alpha(x,y)$ be a basis of $\Theta$ consisting of homogeneous polynomials of degree $n(\alpha)$. Let

$$Q^0_{\Phi,\vec{\mathbf{h}}}(x,y) = \sum_{\alpha,i} a^0_{\alpha,i}\left(\vec{\mathbf{h}}\right)\Phi(\mathbf{p_i})S_\alpha(x,y), \tag{58}$$

where the sum is on $i = 0, 1, \dots, 8$.

Proposition 3.10 assures us that there is always a unique representation as presented in (58). Clearly, the $a^0_{\alpha,i}(\vec{\mathbf{h}})$ are continuous functions of $\vec{\mathbf{h}} \in \mathbb{R}^8$. Since the cube $\left[\frac{1}{M}, M\right]^8 \subset \mathbb{R}^8$ is compact, we have a uniform bound:

$$N = \max_{\vec{\mathbf{c}} \in \left[\frac{1}{M}, M\right]^8} \left| a^0_{\alpha,i}\left(\vec{\mathbf{c}}\right)\right|. \tag{59}$$

We proceed by a homogeneity argument to prove that $a^0_{\alpha,i}(\vec{\mathbf{h}})$, the coefficients of $Q^0_{\Phi,\vec{\mathbf{h}}}(x,y)$ are $O\left(\frac{1}{h^4}\right)$. Let

$$\Phi_\lambda = \Phi(\lambda x, \lambda y), \quad \lambda > 0. \tag{60}$$

Evaluate the polynomial $R(x,y) = Q^0_{\Phi_\lambda,\frac{\vec{\mathbf{h}}}{\lambda}}(\frac{x}{\lambda},\frac{y}{\lambda})$ at $\mathbf{p_i}$. Since $Q^0_{\Phi_\lambda,\frac{\vec{\mathbf{h}}}{\lambda}}$ is the interpolation polynomial of $\Phi_\lambda$ on the $\frac{\vec{\mathbf{h}}}{\lambda}$ stencil, we have

$$R(\mathbf{p_i}) = Q^0_{\Phi_\lambda,\frac{\vec{\mathbf{h}}}{\lambda}}\left(\frac{\mathbf{p_i}}{\lambda}\right) = \Phi_\lambda\left(\frac{\mathbf{p_i}}{\lambda}\right) = \Phi(\mathbf{p_i}). \tag{61}$$

Notice also that all the derivative data is 0:

$$\frac{\partial R}{\partial d}(\mathbf{p_i}) = 0,\ 0 \le i \le 8, \tag{62}$$

including both derivatives at $p_0$.

Combining (61) and (62) and using the uniqueness of the interpolation polynomial,

$$R(x,y) = Q^0_{\Phi,\vec{\mathbf{h}}}(x,y).$$

Expanding the above equation using (58),

$$\begin{aligned}\sum_{\alpha,i} a^0_{\alpha,i}\left(\frac{\vec{\mathbf{h}}}{\lambda}\right)\Phi_\lambda\left(\frac{\mathbf{p_i}}{\lambda}\right)S_\alpha\left(\frac{x}{\lambda},\frac{y}{\lambda}\right) &= \sum_{\alpha,i} a^0_{\alpha,i}\left(\vec{\mathbf{h}}\right)\Phi(\mathbf{p_i})S_\alpha(x,y),\\ \sum_{\alpha,i}\left(\frac{1}{\lambda}\right)^{n(\alpha)} a^0_{\alpha,i}\left(\frac{\vec{\mathbf{h}}}{\lambda}\right)\Phi(\mathbf{p_i})S_\alpha(x,y) &= \sum_{\alpha,i} a^0_{\alpha,i}\left(\vec{\mathbf{h}}\right)\Phi(\mathbf{p_i})S_\alpha(x,y).\end{aligned} \tag{63}$$

Thus, we have

$$a^0_{\alpha,i}\left(\vec{\mathbf{h}}\right) = \frac{1}{\lambda^{n(\alpha)}}a^0_{\alpha,i}\left(\frac{\vec{\mathbf{h}}}{\lambda}\right). \tag{64}$$

Choosing $\lambda = h$ and noting (30), (59),

$$\begin{aligned}\left|a^0_{\alpha,i}\left(\vec{\mathbf{h}}\right)\right| &= \frac{1}{h^{n(\alpha)}}\left|a^0_{\alpha,i}\left(\frac{\vec{\mathbf{h}}}{h}\right)\right|\\ &\le \frac{1}{h^{n(\alpha)}}N\\ &= O\left(\frac{1}{h^{n(\alpha)}}\right).\end{aligned}$$

Not all the $a^0_{\alpha,i}(\vec{\mathbf{h}})$ are significant in the calculation of $\Delta^2_{\vec{\mathbf{h}}}$:

$$\begin{aligned}\Delta^2 Q^0_{\Phi,\vec{\mathbf{h}}}(0,0) &= \Delta^2\left(\sum_{\alpha,i} a^0_{\alpha,i}\left(\vec{\mathbf{h}}\right)\Phi(\mathbf{p_i})S_\alpha(x,y)\right)\Bigg|_{x=0,y=0}\\ &= \sum_{\alpha,i} a^0_{\alpha,i}\left(\vec{\mathbf{h}}\right)\Phi(\mathbf{p_i})\ \Delta^2 S_\alpha(x,y)\big|_{x=0,y=0}\\ &= \sum_{n(\alpha)=4,i} b_\alpha a^0_{\alpha,i}\left(\vec{\mathbf{h}}\right)\Phi(\mathbf{p_i}) = \sum_i O\left(\frac{1}{h^4}\right)\Phi(\mathbf{p_i}),\end{aligned} \tag{65}$$

where $b_\alpha = \Delta^2 S_\alpha(x,y)\big|_{x=0,y=0}$.

We proceed to show that the coefficients of $Q^d_{\Phi,\vec{\mathbf{h}}}$ are $O(\frac{1}{h^3})$. Let

$$Q^d_{\Phi,\vec{\mathbf{h}}}(x,y) = \sum_{\alpha,i} a^d_{\alpha,i}\left(\vec{\mathbf{h}}\right)\Phi_d(\mathbf{p_i})S_\alpha(x,y), \tag{66}$$

where the sum includes both derivatives at $p_0$, and assume that the approximate values of the derivatives satisfy

$$\left(\Phi_\lambda\right)_d = \left(\Phi(\lambda x,\lambda y)\right)_x = \lambda\Phi_d(\lambda x,\lambda y). \tag{67}$$

This equation is a natural requirement on the connection between $\Phi$ and $\Phi_x$,$\Phi_y$.

Equation (67) follows from

$$\left(\Phi_\lambda\right)_x = \lambda\Phi_x(\lambda x). \tag{68}$$

To verify this relation we must first describe the interval on which $\Phi_\lambda$ is defined. This interval is $[0,\frac{1}{\lambda}]$, where the spaces between the grid points are $\frac{h_i}{\lambda}$. Let $x_i^\lambda = \frac{x_i}{\lambda}$ mark the new grid points. We show that $\left(\Phi_\lambda\right)_x = \lambda\Phi_x(\lambda x)$ is a solution to the system described in (45). Recalling (44) and defining the function $g_{i,\lambda}(x) = g_i(\lambda x)$, we obtain

$$g_{i,\lambda}(x^\lambda_{i-1}) = (\Phi_\lambda)_{i-1}, \quad g_{i,\lambda}(x^\lambda_i) = (\Phi_\lambda)_i, \quad g_i(\lambda x^\lambda_{i+1}) = (\Phi_\lambda)_{i+1},$$
$$g'_{i,\lambda}(x^\lambda_{i-1}) = r_{i-1},\ g'_{i,\lambda}(x^\lambda_{i+1}) = r_{i+1},$$

where $r_j = \lambda q_j$. Thus, $\left(\Phi_\lambda\right)_x = \lambda\Phi_x(\lambda x)$ is a solution to the equation. Lemma 4.2 tells us that the solution is unique, and therefore, we can conclude that

$$\left(\Phi_\lambda\right)_x = \lambda\Phi_x(\lambda x).$$

Returning to the estimate of the coefficients $a^d_{\alpha,i}$ in (66), we evaluate the polynomial $R^d(x,y) = Q^d_{\Phi_\lambda,\frac{\vec{\mathbf{h}}}{\lambda}}(\frac{x}{\lambda},\frac{y}{\lambda})$ at $\mathbf{p_i}$. In a fashion similar to (61),

$$\frac{\partial R^d}{\partial d}(\mathbf{p_i}) = \frac{\partial Q^d_{\Phi_\lambda,\frac{\vec{\mathbf{h}}}{\lambda}}\left(\frac{\mathbf{p_i}}{\lambda}\right)}{\partial d} = \frac{1}{\lambda}\frac{\partial Q^d_{\Phi_\lambda,\frac{\vec{\mathbf{h}}}{\lambda}}}{\partial d}\left(\frac{\mathbf{p_i}}{\lambda}\right) = \frac{1}{\lambda}\left(\Phi_\lambda\right)_d\left(\frac{\mathbf{p_i}}{\lambda}\right) = \Phi_d(\mathbf{p_i}),$$

where the last equality follows from (67). Notice also that

$$R^d(\mathbf{p_i}) = 0,$$

and therefore, by uniqueness,

$$R^d(x,y) = Q^d_{\Phi,\vec{\mathbf{h}}}(x,y).$$

We expand the equality above using (66),

$$\sum_{\alpha,i} a^d_{\alpha,i}\left(\frac{\vec{\mathbf{h}}}{\lambda}\right)\left(\Phi_\lambda\right)_d\left(\frac{\mathbf{p_i}}{\lambda}\right)S_\alpha\left(\frac{d}{\lambda},\frac{y}{\lambda}\right) = \sum_{\alpha,i} a^d_{\alpha,i}\left(\vec{\mathbf{h}}\right)\Phi_d(\mathbf{p_i})S_\alpha(x,y),$$
$$\sum_{\alpha,i}\left(\frac{1}{\lambda}\right)^{n(\alpha)-1} a^d_{\alpha,i}\left(\frac{\vec{\mathbf{h}}}{\lambda}\right)\Phi_d(\mathbf{p_i})S_\alpha(x,y) = \sum_{\alpha,i} a^d_{\alpha,i}\left(\vec{\mathbf{h}}\right)\Phi_d(\mathbf{p_i})S_\alpha(x,y).$$

Repeating calculations (64), (65) using the previous equation,

$$\Delta^2 Q^d_{\Phi,\vec{\mathbf{h}}}(0,0) = \sum_i O\left(\frac{1}{h^3}\right)\Phi_d(\mathbf{p_i}),$$

which completes the required proof.

## REFERENCES


[1] I. Altas, J. Dym, M. M. Gupta, and R. P. Manohar, *Mutigrid solution of automatically generated high-order discretizations for the biharmonic equation*, SIAM J. Sci. Comput., 19 (1998), pp. 1575–1585.

[2] M. Arad, A. Yakhot, and G. Ben-Dor, *A highly accurate numerical solution of a biharmonic equation*, Numer. Methods Partial Differential Equations, 13 (1998), pp. 375–391.

[3] D. N. Arnold and F. Brezzi, *Mixed and nonconforming finite element methods: Implementation, postprocessing and error estimates*, RAIRO Modél. Math. Anal. Numér., 19 (1985), pp. 7–32.

[4] B. Bialecki, *A fast solver for the orthogonal spline collocation solution of the biharmonic Dirichlet problem on rectangles*, J. Comput. Phys., 191 (2003), pp. 601–621.

[5] M. Ben-Artzi, J.-P. Croisille, D. Fishelov, and S. Trachtenberg, *A pure-compact scheme for the streamfunction formulation of Navier-Stokes equations*, J. Comput. Phys., 205 (2005), pp. 640–664.

[6] M. Ben-Artzi, J.-P. Croisille, and D. Fishelov, *Convergence of a compact scheme for the pure streamfunction formulation of the unsteady Navier-Stokes system*, SIAM J. Numer. Anal., 44 (2006), pp. 1997–2024.

[7] M. Ben-Artzi, J.-P. Croisille, and D. Fishelov, *A fast direct solver for the biharmonic problem in a rectangular grid*, SIAM J. Sci. Comput., 31 (2008), pp. 303–333.

[8] M. Ben-Artzi, I. Chorev, J.-P. Croisille, and D. Fishelov, *The biharmonic operator in irregular planar domains: Convergence of a compact scheme*, in preparation, 2009.

[9] P. Bjørstad, *Fast numerical solution of the biharmonic dirichlet problem on rectangles*, SIAM J. Numer. Anal., 20 (1983), pp. 59–71.

[10] S. C. Brenner, *An optimal-order nonconforming multigrid method for the biharmonic equation*, SIAM J. Numer. Anal., 26 (1989), pp. 1124–1138.

[11] R. H. Chan, T. K. DeLillo, and M. A. Horn, *The numerical solution of the biharmonic equation by conformal mapping*, SIAM J. Sci. Comput., 18 (1997), pp. 1571–1582.

[12] G. Chen, Z. Li, and P. Lin, *A fast finite difference method for biharmonic equations on irregular domains and its application to an incompressible Stokes flow*, Adv. Comput. Math., 29 (2008), pp. 113–133.

[13] P. A. Ciarlet and P. G. Raviart, *General Lagrange and Hermite interpolation in $R^n$ with applications to finite element methods*, Arch. Rational Mech. Anal., 46 (1972), pp. 177–199.

[14] P. A. Ciarlet and P. G. Raviart, *A mixed finite element method for the biharmonic equation*, in Mathematical Aspects of Finite Elements in Partial Differential Equations, Academic Press, New York, 1974, pp. 125–145.

[15] C. Davini and I. Pitacco, *An unconstrained mixed method for the biharmonic problem*, SIAM J. Numer. Anal., 38 (2001), pp. 820–836.

[16] E. J. Dean, R. Glowinski, and O. Pironneau, *Iterative solution of the stream function-vorticity formulation of the Stokes problem, applications to the numerical simulation of incompressible viscous flow*, Comput. Methods Appl. Mech. Engrg., 87 (1991), pp. 117–155.

[17] L. W. Ehrlich and M. M. Gupta, *Some difference schemes for the biharmonic equation*, SIAM J. Numer. Anal., 12 (1975), pp. 773–790.

[18] W. F. Finden, *An error term and uniqueness for Hermite–Birkhoff interpolation involving only function values and/or first derivative values*, J. Comput. Appl. Math., 212 (2008), pp. 1–15.

[19] A. Greenbaum, L. Greengard, and A. Mayo, *On the numerical solution of the biharmonic equation in the plane*, Phys. D, 60 (1992) pp. 216–225.

[20] M. M. Gupta and R. P. Manohar, *Direct solution of biharmonic equation using noncoupled approach*, J. Comput. Phys., 33 (1979), pp. 236–248.

[21] Z. Jomaa and C. Macaskill, *The embedded finite difference method for the Poisson equation in a domain with an irregular boundary and Dirichlet boundary conditions*, J. Comput. Phys., 202 (2005), pp. 488–506.

[22] J. Kuntzmann, *Méthodes numériques. Interpolation, dérivées*, Dunod, Paris, 1959.

[23] M.-C. Lai and H.-C. Liu, *Fast direct solver for the biharmonic equation on a disk and its application to incompressible flows*, Appl. Math. Comput., 164 (2005), pp. 679–695.

[24] J. Linden, *A Multigrid Method for Solving the Biharmonic Equation on Rectangular Domains*, Notes Numer. Fluid Mech. 11, Vieweg, Braunschweig, 1985.

[25] N. Matsunaga and T. Yamamoto, *Superconvergence of the Shortley-Weller approximation for Dirichlet problems*, J. Comput. Appl. Math., 116 (2000), pp. 263–273.

[26] A. Mayo, *The fast solution of Poisson's and the biharmonic equations on irregular regions*, SIAM J. Numer. Anal., 21 (1984), pp. 285–299.

[27] S. K. Pandit, *On the use of compact streamfunction-velocity formulation of steady Navier-Stokes equations on geometries beyond rectangular*, J. Sci. Comput., 36 (2008), pp. 219–242.

[28] L. L. Schumaker, *Spline Functions: Basic Theory*, 3rd ed., Cambridge University Press, Cambridge, UK, 2007.

[29] B. Schiff, D. Fishelov, and J. R. Whiteman, *Determination of a stress intensity factor using local mesh refinement*, in Mathematics of Finite Elements and Applications, Academic Press, London-New York, 1979, pp. 55–64.

[30] R. K. Shukla and X. Zhong, *Derivation of high-order compact finite difference schemes for non-uniform grid using polynomial interpolation*, J. Comput. Phys., 204 (2005), pp. 404–429.

[31] J. W. Stephenson, *Single cell discretizations of order two and four for biharmonic problems*, J. Comput. Phys., 55 (1984), pp. 65–80.