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SPECTRAL METHODS FOR THE SMALL DISTURBANCE EQUATION OF TRANSONIC FLOWS*

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Abstract. Spectral methods for the small disturbance equation of transonic flows are developed. Two schemes are presented. One of them is spectral in x and y and of second order in t. The other is spectral in x and of second order in y and t. A method for extracting a highly accurate solution for problems containing a discontinuity is presented. The solution is obtained by fitting the standard spectral approximation to a sum of a step function and a truncated Chebyshev series. An application to the Burgers equation and to the small disturbance equation is described.

Key words. spectral methods, shock waves, transonic flows, Chebyshev polynomials

AMS(MOS) subject classifications. 76H05, 76L05, 33A65, 35L65

1. Introduction. The small disturbance equation describing transonic flows is treated. This equation is a model for describing flow with Mach number close to 1 over a thin body. The steady-state equation is

$$(k\phi_x - \tfrac{1}{2}(\gamma+1)\phi_x^2)_x + 4\phi_{yy} = 0 \tag{1.1}$$

and the time-dependent one is

$$2\phi_{tx} = (k\phi_x - \tfrac{1}{2}(\gamma+1)\phi_x^2)_x + 4\phi_{yy} \tag{1.2}$$

where ϕ is the velocity potential and k and γ are positive constants.

If the Mach number far away from the body is close to 1, the solution of (1.1) or (1.2) contains a shock (see [7], [20]). Moreover, the steady-state equation (1.1) is of mixed type.

These equations were treated previously by E. Murman and J. Cole [20] and by B. Engquist and S. Osher [7] (see also [3], [5], [15]), using finite differencing.

E. Murman and J. Cole [20] proposed a scheme for the steady-state equation which is type dependent, i.e., in the hyperbolic region (supersonic flow) upward differencing was used, while in the elliptic region (subsonic flow) centered differencing was used. The difference equations are solved using relaxation procedure.

B. Engquist and S. Osher [7] modified this scheme in the region of interface between supersonic and subsonic domains. The new scheme is nonlinearly stable and does not admit solutions violating the entropy condition. It is of first order in the x direction and of second order in the y and t directions; at steady state the elliptic domain becomes second order accurate in the x direction as well.

We offer a way of treating the small disturbance equation, using spectral methods. As we are interested in the steady state only, we advance in time via a modified-Euler scheme. In the x direction spectral differencing is used, while for the y variable we choose either spectral differencing or finite differencing, depending on the number of grid points we use in the y-direction. For a coarse grid (8 or 16 points) it is preferable to use spectral differencing; for finer meshes we use finite differencing, due to the

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limiting time step. We may do it without affecting the accuracy too much as changes spread slower in the y-direction.

Thus, we are looking for a scheme which is spectral in the x-direction and is capable of capturing the shock.

In order to stabilize the scheme and capture the shock we add two filters every time step. The first one is a filter offered by A. Majda, J. McDonough and S. Osher [18] which damps high modes in the Fourier space. The second filter is a second order Shuman filter proposed by A. Hartan and H. Tal-Ezer [14]. The results, using these filters, agree with those obtained by the Engquist-Osher (E–O) scheme, except near the region of interface between the subsonic and supersonic regions. In the E–O scheme, we used a grid which is four times finer in the x-direction than the spectral one.

The total time of computations depends on the shape of the airfoil. We have checked two types of airfoils. For one of them the computational time for the spectral method, as compared to the E–O scheme, is reduced by a factor of 1.3 and for the other the factor is 3.3.

Since the Shuman filter may reduce the accuracy of the scheme, we removed the two filters mentioned above for a few iterations (1–10) after reaching a steady state, and applied a spectral filter. The latter fits the solution to a sum of a step function with an unknown location and a smooth part. The smooth part is introduced by a truncated Chebyshev series,

$$u \sim d_2 S(x, x_e) + \sum_{k=0}^{M} b_k T_k(x)$$

where $S(x, x_e)$ is a step function, having a jump at $x = x_e$.

It turns out that we got the location of the shock very accurately, regardless of the number of iterations for which we have removed the two filters applied until reaching a steady state. The location of the shock agrees with that prescribed by the E–O scheme, using a grid which is four times finer in the E–O scheme. Moreover, the results are improved compared to those obtained before using the spectral filter, especially in the neighborhood of the shock.

In § 2 we present the differential problem describing transonic flows and in § 3 we describe the boundary conditions. Two schemes are presented in § 4. One is spectral in x and y and of second order in t and the other is spectral in x and of second order in y and t.

In § 5 we discuss the problem of approximating discontinuous solutions using spectral methods and in § 6 we present a method to extract a highly accurate solution by fitting the standard Fourier approximation to a sum of a saw-tooth function and a truncated Fourier series. In § 7 we develop a similar method for a nonperiodic problem, using a step function instead of a saw-tooth function. An application to the small disturbance equation of transonic flows is described in § 8 and numerical results are presented in § 9.

2. Presentation of the problem. The formulation of the small disturbance problem of transonic flows is as follows:

$$2\phi_{tx} = \left(k\phi_x - \frac{\gamma+1}{2}\phi_x^2\right)_x + 4\phi_{yy}, \tag{2.1}$$

$$\phi(-1, y, t) = 0, \tag{2.2}$$

$$\frac{\partial \phi}{\partial x}(1, y, t) = 0, \tag{2.3}$$

(2.4) $$\frac{\partial \phi}{\partial y}(x, \pm 1, t) = F\pm(x),$$

(2.5) $$\phi(x, y, 0) = \phi_0(x, y).$$

The steady-state equation is

(2.6) $$\left(k\phi_x - \frac{\gamma+1}{2}\phi_x^2\right)_x + 4\phi_{yy} = 0$$

where ϕ is the potential velocity and k and γ are positive constants.

The small disturbance equation of transonic flow is derived by asymptotic expansion procedure applied to the exact equations of gas dynamic. The small parameter of expansion is taken to be the airfoil thickness ration τ, and the flow is presented as small disturbance on a uniform stream. The freestream Mach number M_∞ is considered to approach 1 and $\tau \to 0$, such that the transonic similarity parameter k, $k = (1 - M_\infty^2)/\tau^{2/3}$ is fixed. For more details about the expansion procedure, including high order approximation see [6], [5].

3. Boundary conditions. We consider a bounded spatial domain $-1 \leqq x, y \leqq 1$, in which the airfoil is represented by

$$y(x) = -1 + \tau F(x), \qquad |x| < x_0, \quad x_0 \ll 1.$$

Assume that the boundaries $x = \pm 1$, $y = 1$ can be viewed as far away from the airfoil, so that the disturbed flow there is zero. Hence, we have

$$\phi(-1, y, t) = 0, \quad u(1, y, t) = 0, \quad \phi_y(x, 1, t) = 0.$$

On the airfoil the flow is tangent to the body. Since in our asymptotic expansion τ tends to zero, this condition should be applied at $y = -1$, $|x| < x_0$. Thus

$$\phi_y(x, -1, t) = \begin{cases} F'(x), & |x| < x_0, \\ 0, & |x| > x_0. \end{cases}$$

We should supply initial conditions for (1.1)

$$\phi(x, y, 0) = \phi_0(x, y).$$

For a description of the boundary conditions and the geometry of the problem, see Fig. 8 at the end of § 9.

4. Discretization in time and space.

4.1. Discretization in time. As in [7], we split the problem (2.1)-(2.5) into two differential problems. The first one is

(4.1) $$u_t = -(f(u))_x,$$

(4.2) $$u(1, y, t) = 0$$

where $u = \phi_x$, and

$$f(u) = \frac{\gamma+1}{4}u^2 - \frac{k}{2}u.$$

Observe that (4.1) is in conservative form.

The second is

(4.3) $$\phi_{tx} = 2\phi_{yy},$$

(4.4) $$\phi(-1, y, t)=0,$$

(4.5) $$\phi_y(x, \pm 1, t)=F\pm(x).$$

Problems (4.1)-(4.2) and (4.3)-(4.5) must be supplied by initial conditions.

One may present both problems above in the form

$$u_t=G(u).$$

For the first one

$$G(u)=G_1(u)=-\frac{\partial}{\partial x}f(u)$$

and for the second

$$G(u)=G_2(u)=2\int_{-1}^{x}\frac{\partial^2 u}{\partial y^2}\,dx.$$

Since we are interested in the steady-state only, we discretize u_t in (4.1) or (4.3) using a modified-Euler scheme.

$$u^{n+1/2}=u^n+\frac{\Delta t}{2}G(u^n),$$

$$u^{n+1}=u^n+\Delta t G(u^{n+1/2}).$$

Denote by $L_1(\Delta t)$, $L_2(\Delta t)$ the operators acting on u^n to get u^{n+1} for (4.1)-(4.2) and (4.3)-(4.5) respectively by the modified-Euler scheme, and use a Strang-type approximation:

(4.6) $$u^{n+1}=L_1\left(\frac{\Delta t}{2}\right)L_2\left(\frac{\Delta t}{2}\right)L_2\left(\frac{\Delta t}{2}\right)L_1\left(\frac{\Delta t}{2}\right)u^n.$$

According to [9], the above discretization in time is accurate up to order two in the time variable, even in the nonlinear case. One may also consider an implicit time integration. In this work we are concerned essentially with treating the shock using spectral methods. One may modify the spectral scheme to be implicit in time, and compare the results to an implicit scheme using the Murman–Cole switch [3], or the Engquist–Osher one [16].

4.2. Discretization in space. In both problems (4.1)-(4.2) and (4.3)-(4.5) derivatives or integrals with respect to the spatial variables x or y appear. It is sufficient to describe how we discretize $\partial u/\partial x$ and $\int_{-1}^{x} u(\tau)\,d\tau$.

Let $P_N u$ be the Chebyshev-pseudospectral projection of u on the subspace of polynomials of degree N or less.

$$u_N(x, y)=P_N u(x, y)=\sum_{n=0}^{N} a_n(y)T_n(x)$$

where

$$u_N(x_i, y)=u(x_i, y),\quad x_i=\cos\frac{\pi i}{N},\quad 0\leqq i\leqq N.$$

We discretize $\partial/\partial x$ by differentiating $P_N u(x, y)$, i.e.,

$$L_N u=P_N\frac{\partial}{\partial x}P_N u=\sum_{n=0}^{N} a_n(y)T_n'(x)=\sum_{n=0}^{N} b_n(y)T_n(x)$$

where

$$b_N(y)=0, \qquad b_{N-1}(y)=2Na_N(y)$$

and

$$\bar{c}_k b_k(y)=b_{k+2}(y)+2(k+1)a_{k+1}(y), \qquad 0 \leqq k \leqq N-2,$$
$$\bar{c}_0=\bar{c}_N=2$$

and

$$\bar{c}_j=1, \qquad 1 \leqq j \leqq N-1.$$

We apply $L_N u$ for every y_j

$$y_j=\cos\frac{\pi j}{M}, \qquad 0 \leqq j \leqq M.$$

Next, integration is done in a similar way, i.e.,

$$I_N=P_N \int_{-1}^{x} P_N u \, d\tau = P_N \sum_{n=0}^{N} a_n(y) \int_{-1}^{x} T_N(\tau)\, d\tau$$
$$=P_N \sum_{n=0}^{N+1} d_n(y) T_n(x).$$

By integrating the recurrence formula

$$2T_n(x)=\frac{T'_{n+1}(x)}{n+1}-\frac{T'_{n-1}(x)}{n-1},$$

we have

$$d_{N+1}=\frac{a_N}{2(N+2)}, \qquad d_N=\frac{a_{N-1}}{2(N+1)},$$

$$d_n=\frac{1}{2}\frac{a_{n-1}-a_{n+1}}{n}, \qquad 3 \leqq n \leqq N-1,$$

$$d_2=\frac{a_1}{4}-\frac{a_3}{4}, \qquad d_1=a_0-\frac{a_2}{2}$$

and we choose d_0 such that

$$\sum_{n=0}^{N+1} d_n(y) T_n(-1)=0.$$

Two types of schemes for the discretization of $\partial^2/\partial y^2$ are possible. The first is spectral in y, and the second is a finite-difference one. We may use the latter, since in the transonic problem perturbations spread much slower in the y-direction, in comparison to those in the x-direction. In this way we avoid the stability limited time step

$$\Delta t = O\left(\frac{1}{M^4}\right)$$

for the spectral discretization, where M is the number of points in the y direction.

Using finite differences in the y direction implies

$$\Delta t = O\left(\frac{1}{M^2}\right).$$

In this case we have

$$\frac{\partial^2}{\partial y^2} u \approx D_M(y) u(x, y) = \frac{u(x, y_{j+1}) - 2u(x, y_j) + u(x, y_{j-1})}{(\Delta y)^2}$$

where $\Delta y = 2/M$, $y_j = 1 - (\Delta y) \cdot j$, $1 \leqq j \leqq M-1$.

We apply $D_M(y)$ at

$$x = x_i = \cos\frac{\pi i}{N}, \qquad 0 \leqq i \leqq N-1.$$

Denote by U the approximation to u and by Φ the approximation to ϕ, where

$$\Phi = I_N U.$$

Hence, the semidiscrete approximation to (4.1)-(4.2) is

$$\frac{\partial U}{\partial t} = -L_N(x) f(U), \tag{4.7}$$

$$U(-1, y, t) = 0. \tag{4.8}$$

And for (4.3)-(4.5),

$$\frac{\partial \Phi}{\partial t} = 2 I_N(x) D_M(y) \Phi, \tag{4.9}$$

$$\frac{\Phi(x_i, 1) - \Phi(x_i, 1-\Delta y)}{\Delta y} = F_+(x_i), \qquad 0 \leqq i \leqq N-1, \tag{4.10}$$

$$\frac{\Phi(x_i, -1+\Delta y) - \Phi(x_i, -1)}{\Delta y} = F_-(x_i), \qquad 0 \leqq i \leqq N-1 \tag{4.11}$$

where $\Phi = I_N U$.

This scheme has spectral accuracy in the x-variable and is of second order in the y-variable. For further analysis of the schemes above see [8].

5. Spectral methods for problems containing a discontinuity. In § 4 we described two schemes for the small disturbance equation, which have spectral accuracy in x and are of second order in t. One of the schemes has spectral accuracy in y; while the other is of second order in y.

For low Mach numbers no shock appears in the solution; hence we apply the scheme presented in (4.6)-(4.11) and show numerical results in Fig. 1.

When M_∞ begins to approach 1, shocks appear in the solution (see [20] and [7]) and we have to treat the discontinuity. To illustrate the problem caused by the discontinuities, we treat a linear problem, though we shall apply our new method to nonlinear problems as well.

Consider the problem

$$u_t = Lu,$$

$$u(x, 0) = u_0(x)$$

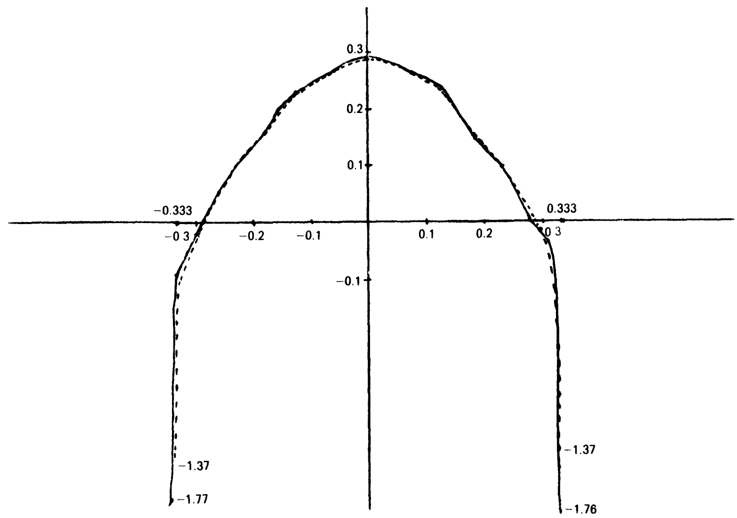

FIG. 1. — *spectral* (49, 17); - - -, E-O (121, 17); *quantity displayed is* $u(x,-1,t)$ *at steady state,* $M_\infty = 0.57$, *airfoil given by* (9.1).

where u belongs to a Hilbert space H, L is a spatial linear operator, x is a scalar or vector spatial variable.

Denote by P_N a projection operator $P_N: H \to B_N$, where B_N is a finite-dimensional subspace $B_N \subset H$.

Let u_N be the solution of the semidiscrete problem

$$\frac{\partial u_N}{\partial t} = P_N L P_N u_N,$$

$$u_N(0) = P_N u_0$$

where $u_N \in B_N$.

Then, by [13] and [4], for spectral methods

(5.1) $$\|u_N(t) - P_N u(t)\|_0 \leqq C N^{-p+1} \|u\|_p, \qquad u \in H^p(\Omega)$$

where $H^p(\Omega)$ is a Sobolev space, for which u and its derivatives up to order p are in $L_2(\Omega)$.

We invoke results in [4]:

(5.2) $$\|P_N u - u\|_0 \leqq C N^{-p} \|u\|_p, \qquad u \in H^p(\Omega).$$

Combining (5.1) and (5.2), we may deduce that

(5.3) $$\|u_N - u\|_0 \leqq C N^{-p+1} \|u\|_p, \qquad u \in H^p(\Omega).$$

From the last inequality it is clear that when the solution u or its derivatives have discontinuities, the rate of convergence of the approximated solution u_N to the exact one u may be very poor.

In fact, it is well known (the Gibbs phenomenon) that for a *piecewise* smooth function

$$|P_N u - u| \sim O\left(\frac{1}{N}\right)$$

away from the discontinuity and $P_N u$ is an oscillatory function.

Can we extract a piecewise C^∞ function from its oscillations? In [19], M. S. Mock and P. D. Lax have argued that for high order schemes moments are preserved within high accuracy (see also [17] for the high resolution of high order schemes). In §§ 6 and 7 we show how we use this idea to deduce pointwise convergence by a postprocessing. We refer the reader to [12] for another kind of postprocessing. We shall first describe the method for a periodic problem, since in this case the theory is more complete.

Our method is based also on the idea of S. Abarbanel and D. Gottlieb appearing in [1] of looking for a solution which is a sum of a step function (or a saw-tooth function in the periodic case) and a smooth function.

For a periodic problem S. Abarbanel and D. Gottlieb [1] minimized

$$H=\sum_{j=0}^{2N-1}\left[u_N(x_j,t)-d_2F_N(x_j,x_l)-\sum_{k=-M}^{M}b_k e^{ikx_j}\right]^2$$

where u_N is a pseudospectral-Fourier approximation to the differential problem. F_N is a pseudospectral-Fourier projection of a saw-tooth function $F(x, x_l)$ onto the subspace spanned by $\{e^{ikx}\}_{k=-N}^{N}$, where

$$F(x,x_l)=\begin{cases}x, & 0\leqq x<x_l,\\ x-2\pi, & x_l\leqq x\leqq 2\pi.\end{cases}\tag{5.4}$$

The jump $2\pi d_2$, its location x_l and b_k are unknowns.

For a nonperiodic problem, instead of a saw-tooth function, they looked for a step function $S(x, x_l)$

$$S(x,x_l)=\begin{cases}0, & -1\leqq x\leqq x_l,\\ 1, & x_l<x\leq 1\end{cases}\tag{5.5}$$

and then minimized

$$H=\sum_{j=0}^{N}\frac{1}{c_j}\left[u_N(x_j,t)-d_2S_N(x_j,x_l)-\sum_{k=0}^{M}b_kT_k(x_j)\right]^2.$$

$|M|<N$ and $S_N(x, x_l)$ is the pseudospectral projection of $S(x, x_l)$ onto the subspace spanned by $\{T_k(x)\}_{k=0}^{N}$. $T_k(x)$ is a Chebyshev polynomial of degree k. Note that l is real, not necessarily an integer.

We also refer the reader to theorems appearing in [2], which show that for spectral Fourier methods moments are preserved within spectral accuracy and then show how to fit the numerical solution to a sum of a step function (or a saw-tooth) and a smooth part, based on preservation of moments.

5.1. Preservation of moments for the Galerkin–Fourier method. We consider first the Fourier–Galerkin method. Define the inner product

$$(u,v)=\int_0^{2\pi}u(x,t)\bar{v}(x,t)\,dx.$$

Let u be a solution of

$$u_t=Lu,\quad 0<x<2\pi,\quad t>0,\tag{5.6}$$

$$u(x,0)=f(x),\tag{5.7}$$

$$u(x,t)=u(x+2\pi,t)\tag{5.8}$$

where L is a linear operator

$$L=a(x)\frac{\partial}{\partial x},\qquad a(x)=a(x+2\pi)\tag{5.9}$$

and $f(x)$ is a piecewise C^∞ function.

Let u_N be the Galerkin–Fourier approximation of u satisfying (5.6)–(5.9), i.e., u_N satisfies

$$(u_N)_t = L_N u_N, \tag{5.10}$$

$$u_N(x, 0) = P_N f(x), \tag{5.11}$$

$$u_N(x, t) = u_N(x + 2\pi, t) \tag{5.12}$$

where

$$L_N = P_N L P_N$$

and P_N is the Galerkin–Fourier projection defined in [11] and [13].

THEOREM 5.1 (S. Abarbanel, D. Gottlieb and E. Tadmor [2]). *Let $u(t)$ satisfy* (5.6)–(5.9) *and let $u_N(t)$ satisfy* (5.10)–(5.12) *where $f(x)$ is a piecewise C^∞; then*

$$(u_N(T), v(T)) = (u(T), v(T)) + E \tag{5.13}$$

for every $v \in H^p(0, 2\pi)$ and E satisfies

$$|E| \leqq CN^{-p+1} \|v\|_p.$$

5.2. Preservation of moments for the Fourier-pseudospectral method. Consider now the Fourier-pseudospectral method.

Define the discrete inner product

$$(u, v)_N = \frac{\pi}{N} \sum_{j=0}^{2N-1} u(x_j) \bar{v}(x_j)$$

where $x_j = \pi j / N$, $0 \leqq j \leqq 2N - 1$.

Let u_N be a pseudospectral-Fourier approximation to (5.6)–(5.9), i.e., u_N satisfies

$$(u_N)_t = L_N u_N, \quad 0 < x < 2\pi, \quad t > 0, \tag{5.14}$$

$$u_N(x, 0) = P_N^G f(x), \tag{5.15}$$

$$u_N(x, t) = u_N(x + 2\pi, t) \tag{5.16}$$

where

$$L_N = P_N L P_N$$

and P_N^G and P_N are the Galerkin and pseudospectral-Fourier projection respectively defined in [13].

THEOREM 5.2 (S. Abarbanel, D. Gottlieb and E. Tadmor [2]). *Let $u(t)$ be the solution of* (5.6)–(5.9) *and let $u_N(t)$ be a solution of* (5.14)–(5.16).

Assume that (5.14)–(5.16) *is stable. Then*

$$(u(T), v(T)) = (u_N(T), v(T))_N + E \tag{5.17}$$

where

$$|E| \leqq CN^{-p+1} \|v\|_p.$$

6. Fitting the approximated solution to a saw-tooth function (periodic problem). In the previous section we have quoted theorems stating that spectral-Fourier methods, applied to linear problems, preserve moments within spectral accuracy.

The question is how to extract pointwise convergence from that property.

For a periodic problem (5.6)–(5.9), we assume that the nonsmooth part of the problem is a saw-tooth function and approximate the smooth part by a truncated

Fourier series, i.e.,

$$u(x,t) \sim d_2 F(x,x_l) + \sum_{|k|=0}^{M} b_k e^{ikx} \tag{6.0}$$

where $F(x, x_l)$ is a saw-tooth function defined in (5.4).

If there are other types of singularities, we may add other singular functions to the sum (6.0), i.e.,

$$u(x,t) \sim \sum_{|k|=0}^{M_1} d_k F_k(x,x_l) + \sum_{|k|=0}^{M_2} b_k e^{ikx}$$

where $F_0(x, x_l) = F(x, x_l)$ and $F_k(x, x_l)$ are periodic functions, they and their derivatives up to order $k-1$ are continuous, and their kth derivative has a discontinuity at x_l. In this paper a representation similar to (6.0) for nonperiodic problems was used, but it is possible to include more nonsmooth terms as suggested in order to improve the results.

In (6.0) the location of the jump x_l, its magnitude $2\pi d_2$ and the coefficients b_k are prescribed using preservation of moments.

For the Galerkin-Fourier method, we substitute (6.0) for (5.13) and choose the smooth functions $v(T)$ in (5.13) to be e^{ijx}, $|j| = 0, \cdots, M+2$ and M such that

$$M \leqq N-2.$$

The following set of equations results:

$$\int_0^{2\pi} \Big(d_2 F(x,x_l) + \sum_{|k|=0}^{M} b_k e^{ikx}\Big) e^{-ijx}\, dx = \int_0^{2\pi} u_N(x)\, e^{-ijx}\, dx \qquad \text{for } |j| = 0, \cdots, M+2.$$

For the pseudospectral-Fourier method, we get (using (5.17) instead of (5.13))

$$\sum_{n=0}^{2N-1} \Big(d_2 F_N(x_n,x_l) + \sum_{|k|=0}^{M} b_k e^{ikx_n}\Big) e^{-ijx_n} = \sum_{n=0}^{2N-1} u_N(x_n)\, e^{-ijx_n} \qquad \text{for } |j| = 0, \cdots, M+2 \tag{6.1}$$

where

$$x_n = \frac{\pi n}{N}, \qquad 0 \leqq n \leqq 2N-1,$$

and $F_N(x, x_l)$ is the pseudospectral-Fourier projection of $F(x, x_l)$, i.e.,

$$F_N(x,x_l) = \sum_{|k|=0}^{N} A_k(x_l)\, e^{ikx}, \qquad A_0 = \frac{\pi}{N}\Big(l - N + \frac{1}{2}\Big), \tag{6.2}$$

$$A_k = \frac{\pi}{2Nc_l} 2\Big(\frac{1-e^{-ik\pi(l+1)/N}}{1-e^{-i\pi l/N}} + \cot\frac{\pi k}{2N} - 1\Big), \qquad 1 \leqq |k| \leqq N, \qquad c_l = 1 \quad \text{for } |l| \leqq N-1, \qquad c_l = 2 \quad \text{for } |l| = N. \tag{6.3}$$

Letting l be real, though not necessarily an integer, enables us to locate the jump within spectral accuracy. It is clear that its profile would be sharp.

Equations (6.1) can be written in a simpler form, but we shall write this simpler form in detail for the nonperiodic case, because of the similarity of these two cases, and since our goal in this work is to apply the method for the nonperiodic small disturbance equation of transonic flow.

7. Fitting the approximated solution to a step function-(nonperiodic problem). We now develop a similar method to the one presented in § 6 for a nonperiodic problem, using Chebyshev polynomials. Assuming that the nonsmooth part of the solution is a step function, we search for a solution which is a sum of a step function and a truncated Chebychev series, i.e.,

$$u(x,t)\sim d_2S(x,x_l)+\sum_{k=0}^{M} b_kT_k(x) \tag{7.0}$$

where $S(x, x_l)$ is a step function defined in (5.5).

The location of the jump x_l, its magnitude d_2 and the coefficients b_k are prescribed using preservation of moments. For a nonperiodic case, we choose the smooth function $v(T)$ to be $T_j(x)$, $j=0,\cdots,M+2$, and M such that

$$M\leqq N-2.$$

For the Galerkin-Chebyshev method we interpret (5.13) in the following way:

$$\int_{-1}^{1}\left(d_2S(x,x_l)+\sum_{k=0}^{M} b_kT_k(x)\right)T_j(x)(1-x^2)^{-1/2}\,dx=\int_{-1}^{1} u_N(x)T_j(x)(1-x^2)^{-1/2}\,dx$$

$$\text{for } j=0,\cdots,M+2.$$

For the pseudospectral-Chebyshev method, we require that

$$\sum_{n=0}^{N}\frac{1}{c_n}\left(d_2S_N(x_n,x_l)+\sum_{k=0}^{M} b_kT_k(x_n)\right)T_j(x_n)=\sum_{n=0}^{N}\frac{1}{c_n}u_N(x_n)T_j(x_n) \quad \text{for } j=0,\cdots,M+2 \tag{7.1}$$

where

$$c_n=1 \quad \text{for } 1\leqq j\leqq N-1, \qquad c_n=2 \quad \text{for } j=0,N,$$

$$x_n=\cos\frac{\pi n}{N}, \qquad 0\leqq n\leqq N.$$

$S_N(x, x_l)$ is the pseudospectral-Chebyshev projection of $S(x, x_l)$, i.e.,

$$S_N(x,x_l)=\sum_{k=0}^{N} A_k(x_l)T_k(x)$$

where $x_l=\cos(\pi/N)l$

$$A_0=\frac{1}{N}l, \qquad A=\frac{1}{2N}\sin\pi l, \tag{7.2}$$

$$A_k=\frac{1}{N}\sin\frac{k\pi}{N}l/\sin\frac{k\pi}{2N}, \qquad 1\leqq k\leqq N-1. \tag{7.3}$$

Equations (7.1) form a set of $M+3$ equations for the $M+3$ unknowns

$$d_2, x_l, b_0,\cdots,b_M.$$

We shall now write down the system of equations resulting from (7.1).

Define

$$F_k(t) = \frac{2}{c_k N} \sum_{j=0}^{N} \frac{1}{c_j} u(x_j, t) T_k(x_j), \qquad k = 0, \cdots, N. \tag{7.4}$$

We use orthogonality properties

$$\sum_{j=0}^{N} \frac{1}{c_j} T_l(x_j) T_k(x_j) = 0, \qquad k \neq l, \quad 0 \leqq k, l \leqq N$$

and the following system of equations results from (7.1):

$$d_0 A_0 + b_0 = F_0, \tag{7.5}$$

$$d_2 A_k + b_k = F_k, \qquad 1 \leqq k \leqq M, \tag{7.6}$$

$$d_2 A_{M+1} = F_{M+1}, \tag{7.7}$$

$$d_2 A_{M+2} = F_{M+2}. \tag{7.8}$$

There is a solution to the system (7.5)-(7.8) if and only if

$$F_{M+1} A_{M+2} = F_{M+2} A_{M+1}. \tag{7.9}$$

Equation (7.9) is a nonlinear equation for x_l, which is solved iteratively.

Then, we get

$$d_2 = F_{M+1} / A_{M+1},$$
$$b_0 = F_0 - d_2 A_0,$$
$$b_k = F_k - d_2 A_k, \qquad 1 \leqq k \leqq M.$$

Therefore, the position of the jump x_l, its magnitude d_2 and the smooth part of the solution

$$\sum_{k=0}^{\infty} b_k T_k(x)$$

are prescribed within spectral accuracy, provided that the singular part of the solution is a step function.

8. Application to the transonic problem. In our approximations to the solution of the transonic problem, we are interested in the solution in the steady state. Using the scheme presented in § 4 we have a nonstable procedure due to nonlinear instabilities which appears in the presence of a shock for t large enough.

In order to stabilize the procedure we have used two filters ((a) and (b)).

(a) A. Majda, J. McDonough and S. Osher in [18] have offered a procedure for damping high modes in the approximated solution.

If

$$u(x) = \sum_{k=0}^{N} a_k T_k(x), \qquad \tilde{u}(x) = \sum_{k=0}^{N} \tilde{a}_k T_k(x) \tag{8.1}$$

where

$$\tilde{a}_k = \begin{cases} 1, & |k| < k_0, \\ e^{-\alpha(k-k_0)^4}, & |k| \geqq k_0, \end{cases} \tag{8.2}$$

and k_0 is an integer which depends on the strength of the shock, we choose

$$k_0 \sim \tfrac{2}{3}N.$$

This is a very weak filter since there is no change in the low modes.

In [18] it was proved that for *linear* problems, this filter insures stability for the Fourier method. Moreover, if we also smooth the initial data in a certain way, (see [18, preliminary section]) this filter leads to a spectral accurate approximation away from a set of discontinuities of the exact solutions.

(b) The smoothing described in (a) was not sufficient for our *nonlinear* problem. Therefore, we applied every time step a Shuman filter as well.

Denote by u_{jk}^n the values of the approximated velocity $u(x, y)$ in the x-direction at the point (x_j, y_k) at time t_n. The filtered values $\bar{u}_{jk}^n$ are given by

$$\bar{u}_{jk}^n = u_{jk}^n + \Theta_{j+1/2}(u_{j+1,k}^n - u_{jk}^n) + \Theta_{j-1/2}(u_{jk}^n - u_{j-1,k}^n). \tag{8.3}$$

The smoothing factors Θ_j are chosen such that they are small in the smooth part of the solution and become large $(0(1))$ only in the neighborhood of the discontinuities.

Following Harten and Tal-Ezer [14], we choose

$$\Theta_{j+1/2} = \beta \frac{|(u_{j+2,k} - u_{j+1,k}) - 2(u_{j+1,k} - u_{j,k}) + (u_{jk} - u_{j-1,k})|}{|u_{j+2,k} - u_{j+1,k}| + 2|u_{j+1,k} - u_{jk}| + |u_{jk} - u_{j-1,k}|} \tag{8.4}$$

where $0 < \beta < 1$.

We used $\beta = 0.01$ in our calculations. This filter was also used by D. Gottlieb, L. Lustman and S. Orszag [10]. It reduces the order of accuracy of our scheme. But our strategy was first to reach a steady state and afterwards to construct a highly accurate approximation.

After achieving a steady state, we omitted the two filters described above for a few iterations (1-10) and applied the spectral filter presented in (7.5)-(7.8).

To conclude:

(8.5) We first worked out the scheme described in (4.6)-(4.11).

(8.6) At each time step applied the filters described in (8.1)-(8.2) and (8.3)-(8.4).

(8.7) After reaching a steady state we removed the above filters and applied a spectral filter presented in (7.5)-(7.8).

9. Numerical results. We first show results for the "inviscid" Burgers' equation

$$\begin{aligned} &u_{t^{-1/2}}(u^2)_x = 0, \\ &u(1, t) = 1, \\ &u(-1, t) = -1, \\ &u(x, 0) = x. \end{aligned}$$

This is a nonperiodic problem, for which one can easily verify that a shock appears in the solution at $t = 1$. The exact solution for $t > 1$ is $u(x, t) = -1$ for negative x, and $u(x, t) = 1$ for positive x. In the numerical solution $\partial/\partial t$ is approximated by a modified Euler scheme and $\partial/\partial x$ by a polynomial pseudospectral method described in § 4 with $N = 32$. At every time step we applied the step function filter (7.5)-(7.8) and got the following results at $t = 2.176$ (see Table 1).

Next, we approximated the solution of the small disturbance problem of transonic flows around a symmetric airfoil, described in § 2. The computational domain is $-1 \leqq x$,

TABLE 1

x	|error|
.9952	.51 (−5)
.9239	.14 (−4)
.7730	.90 (−5)
.5556	.53 (−5)
.2903	.39 (−4)
.0000	.37 (−3)
−.2903	.18 (−4)
−.5556	.89 (−5)
−.7730	.85 (−5)
−.9239	.19 (−4)
−.9952	.37 (−5)

$y \leqq 1$. The airfoil is located at $-x_0 \leqq x \leqq x_0$, $y=-1$. We divided the domain into three parts: $-1 \leqq x \leqq -x_0$, $-x_0 \leqq x \leqq x_0$, $x_0 \leqq x \leqq 1$, and approximated u_N in each subdomain by a Chebyshev polynomial. That gives a natural refinement of the grid near the tips of the airfoil. Typically $x_0 = \frac{1}{3}$ in our calculations.

The shape of the airfoil is given by

$$y = -1 + \tau F(x), \qquad |x| \leqq x_0.$$

Note that in the expansion procedure $\tau \to 0$; hence the airfoil is represented by the segment $-x_0 \leqq x \leqq x_0$, $y=-1$. The shape of the airfoil only affects the boundary condition applied to ϕ_y on this segment. For

$$F(x) = k_0 \cos 1.5\pi x, \qquad |x| \leqq x_0, \tag{9.1}$$

where k_0 was chosen to be $(2/3\pi)^2$,

$$\phi_y = \begin{cases} F'(x) = -1.5\pi k_0 \sin 1.5\pi x, & |x| < x_0, \\ 0, & |x| > x_0. \end{cases}$$

Note that ϕ_y is discontinuous at $x = \pm x_0$.

The calculations were continued until steady state was approximated, i.e., until

$$\varepsilon_1 = \frac{\max_{j,k} |u_{jk}^{n+1} - u_{jk}^n|}{\Delta t} \leqq 10^{-3}. \tag{9.2}$$

For all the numerical results displayed for the small disturbance equation (Figs. 1-7) we used second order finite differencing in y (as in the E–O scheme), therefore the number of grid points in the y direction is identical (17) for both schemes. In Figs. 1-7 the quantity presented is $u(x, -1, t)$ as $t \to \infty$, i.e., the steady-state velocity in the x direction on the airfoil.

We first ran the scheme for low Mach numbers. In this case no shock appears, so we were able to apply the Strang-type scheme (4.6) described in § 4 for marching in time, and (4.7)-(4.11) for discretization in the spatial variables. There was no need to add filters.

As an example, Fig. 1 contains the results for $M_\infty = .57$, $\tau = 0.1$, which implies $k = 2.89$. The airfoil is presented by (9.1) and the grid is of (49×17) points. The results are compared to those obtained by the E–O scheme [7]. One should take a grid of (121×17) points in the E–O scheme to get similar results to those obtained by the spectral method, with a grid of (49×17) points.

While increasing M_∞, we were able to use the same scheme up to M_∞ approximately 0.85. For $M_\infty > 0.85$ we added filters to capture the shock. In Figs. 2–5 we present results for $M_\infty = .9$, $\tau = .1$ which implies $k = .822$ for two types of airfoils described in (9.3) and (9.1). As a shock appears in the solution, we have used the procedure described in the previous section ((8.5)–(8.8)).

We have carried out the calculation for two shapes of airfoils. The first is presented by

$$y = -1 + \tau F(x),$$

where

$$F(x) = k_0 \cos^2 1.5\pi x, \qquad k_0 = .8 \cdot \left(\frac{2}{3\pi}\right)^2. \tag{9.3}$$

The second is presented in (9.1). Notice that for (9.3)

$$\phi_y = \begin{cases} F'(x) = 3\pi k_0 \sin 3\pi x, & |x| < x_0, \\ 0, & |x| > x_0 \end{cases}$$

ϕ_y is continuous at $x = \pm x_0$.

The results for this case are presented in Fig. 2. The location of the shock prescribed by the spectral method was

$$x_l = .08127.$$

In the E–O scheme $a(u) = (\gamma+1)u - k$ is positive for $x_2 = .08163$ and is negative for $x_3 = .0918$. According to a one-dimensional analysis done in [7], the shock might be

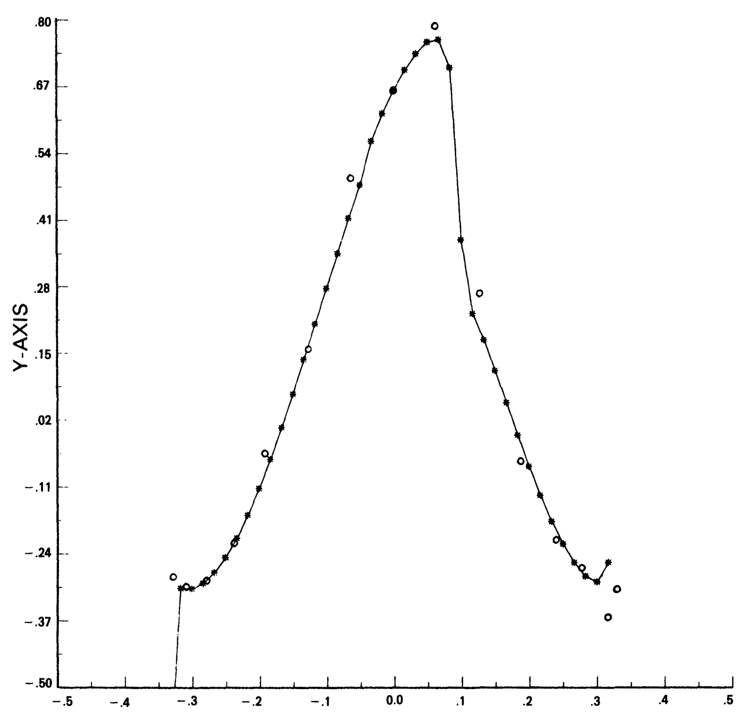

FIG. 2. ○—, *spectral* (49, 17), *—, E–O (121, 17), *quantity displayed is* $u(x, -1, t)$ *at steady state*, $M_\infty = 0.9$, *airfoil given by* (9.3).

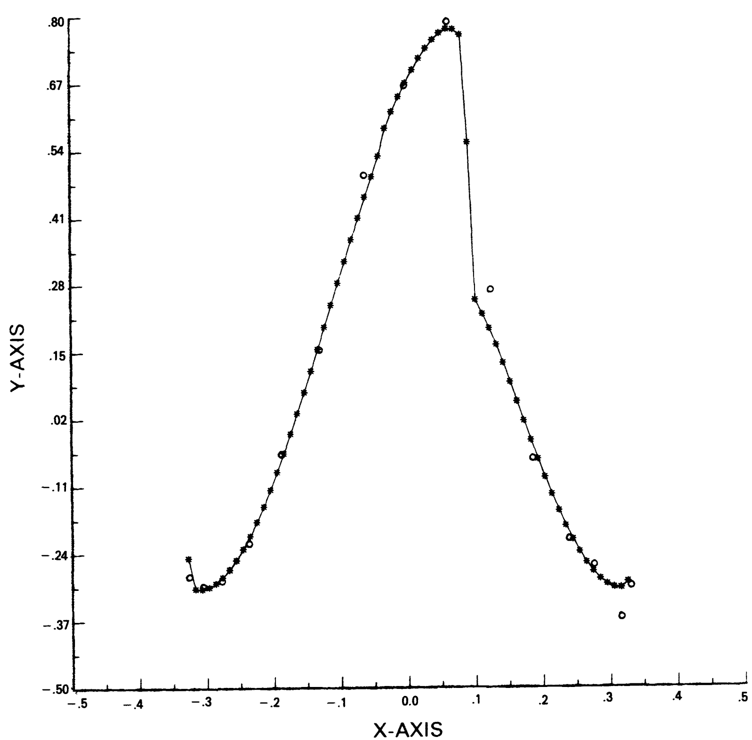

FIG. 3. ○—, *spectral* (49, 17), *—, E-O (197, 17), *quantity displayed is* $u(x, -1, t)$ *at steady state,* $M_\infty = 0.9$, *airfoil given by* (9.3).

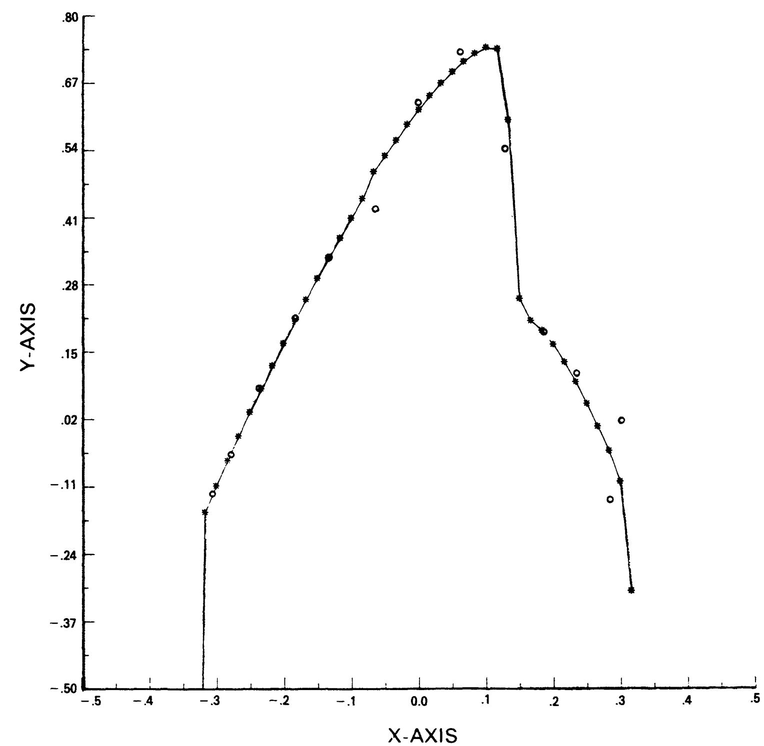

FIG. 4. ○—, *spectral* (49, 17), *—, E-O (121, 17), *quantity displayed is* $u(x, -1, t)$ *at steady state,* $M_\infty = 0.9$, *airfoil given by* (9.1).

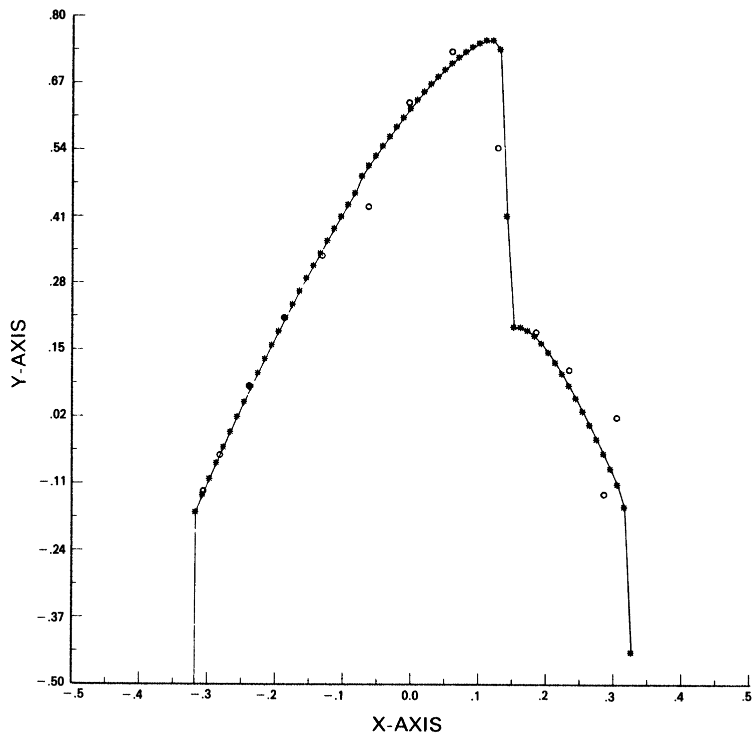

FIG. 5. ∘—, *spectral* (49, 17), *—, E–O (197, 17), *quantity displayed is* $u(x, -1, t)$ *at steady state*, $M_\infty = 0.9$, *airfoil given by* (9.1).

spread over two grid points and therefore might occur between their neighboring points: $x_1 = .07143$ and $x_4 = .102$.

Next, we increased the number of grid points in the E–O scheme to 197×17. Still the spectral location of the shock agrees with that prescribed by the E–O scheme. Moreover, the results are closer to the spectral ones (in comparison to the coarser finite difference grid), especially near the shock. These results are presented in Fig. 3. Results obtained before using the spectral filter are presented in Fig. 6.

In Table 2 we compare the number of iterations NI to reach a steady state by the (197×17) E–O scheme and the spectral one. The total computational time T is compared as well.

The next example is an airfoil whose shape satisfies (9.1). In this case ϕ_y is discontinuous at $x = \pm x_0$. The results are shown in Fig. 4. The shock location found by the spectral method is

$$x_l = .1291,$$

which is between the two E–O grid points $x_1 = .122$, $x_2 = .132$, corresponding to the 197×17 grid. There are some differences in the results near $x = \pm x_0$, due to the discontinuity of ϕ_y. In order to get better results we should add continuous functions which have discontinuous derivatives to the sum (7.0). The number of grid points taken for the E–O scheme is 121×17 in Fig. 4 and 197×17 in Fig. 5. Note that for this shape of airfoil too there is more agreement with the spectral results in the finer E–O grid, especially near the shock. Results obtained before using the spectral filter are presented in Fig. 7.

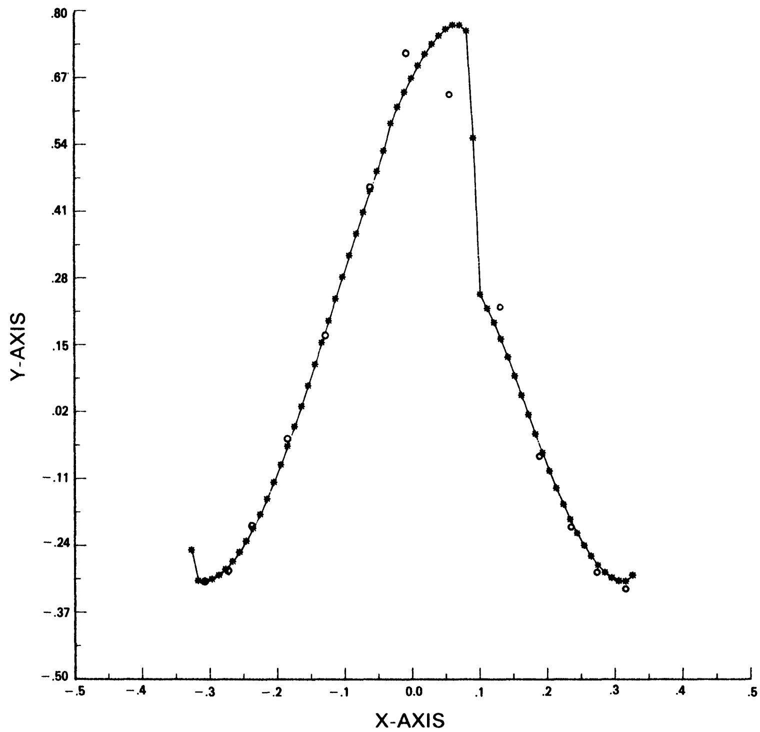

FIG. 6. ○—, *spectral before filtering* (49, 17), ∗—, E–O (197, 17), *quantity displayed is* $u(x, -1, t)$ *at steady state,* $M_\infty = 0.9$, *airfoil given by* (9.3).

TABLE 2

	E–O	Spectral
grid	197 × 17	49 × 17
NI	4,810	6,007
T	6,335	4,938

In Table 3 we compare the same quantities as in Table 2 for the airfoil presented in (9.1). In this case NI and T corresponds to $\varepsilon_1 = 10^{-2}$ in (9.2).

10. Conclusions. Both analytic and computational evidence show that spectral methods can be applied efficiently to the small disturbance equation of transonic flows.

Moreover, the method presented in § 7 for fitting the standard spectral approximation to a sum of a step function and a truncated Chebyshev series is applicable to

TABLE 3

	E–O	Spectral
grid	197 × 17	49 × 17
NI	13,660	6,607
T	17,920	5,432

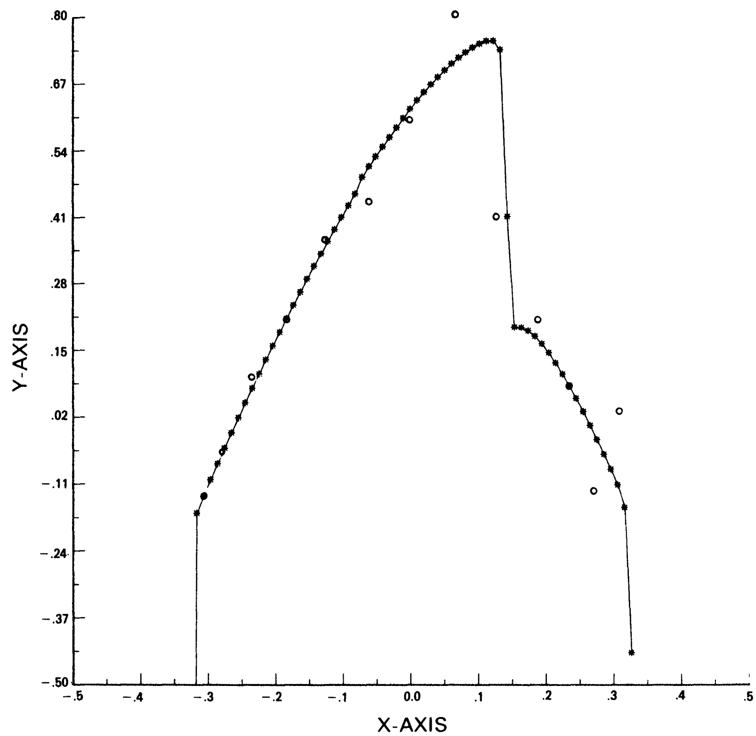

FIG. 7. ○—, *spectral before filtering* (49, 17), ∗—, E–O (197, 17), *quantity displayed is* $u(x, -1, t)$ *at steady state*, $M_\infty = 0.9$, *airfoil given by* (9.1).

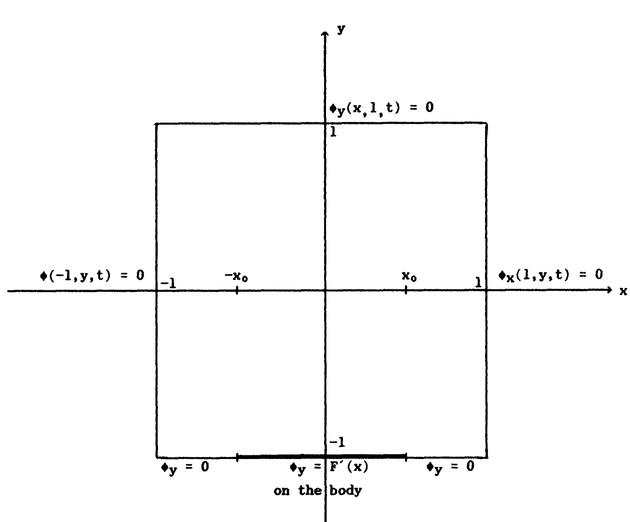

FIG. 8. *Description of boundary conditions. The body is represented by the segment* $y = -1$, $-x_0 \leqq x \leqq x_0$.

other problems, such as Burgers' equation, which contain a discontinuity. If the nonsmooth part of the solution is a step function, the method has spectral accuracy.

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