The plan for today

- What is triangle mesh
- What is parameterization and what is it good for:
  - Texture mapping
  - Remeshing
- Parameterization
  - Convex mapping
  - Harmonic mapping

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Surface Parametrization

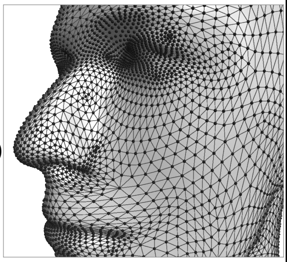
Most slides courtesy of Pierre Alliez and Craig Gotsman

Triangle mesh

- Geometry:
  - Vertex coordinates

    (x_1, y_1, z_1)
    (x_2, y_2, z_2)
    . . .
    (x_n, y_n, z_n)
- Connectivity (the graph)
  - List of triangles

    (i_1, j_1, k_1)
    (i_2, j_2, k_2)
    . . .
    (i_m, j_m, k_m)



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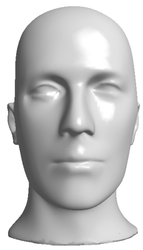
Triangle mesh

- Discrete surface representation
- Piecewise linear surface (made of triangles)



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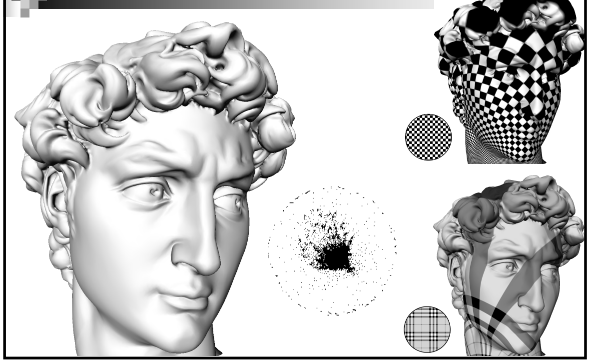
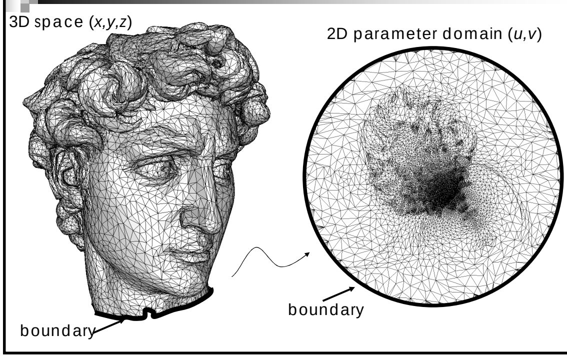
Application - Texture mapping



2D parameterization



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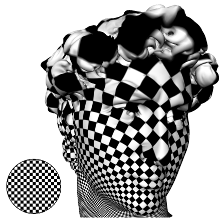
Requirements

- Bijective (1-1 and onto): No triangles fold over.
- Minimal “distortion”
  - Preserve 3D angles
  - Preserve 3D distances
  - Preserve 3D areas
  - No “stretch”



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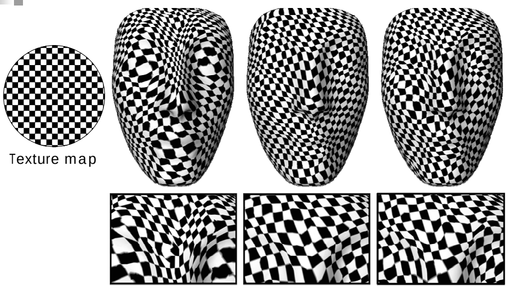
Distortion minimization



Kent et al ‘92 Floater 97 Sander et al ‘01

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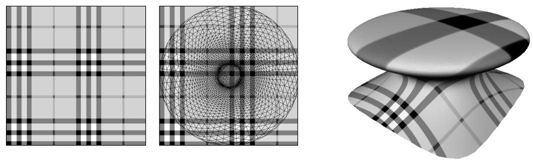
More texture mapping



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Applications

- Texture Mapping
- Remeshing
- Surface Reconstruction
- Morphing
- Compression

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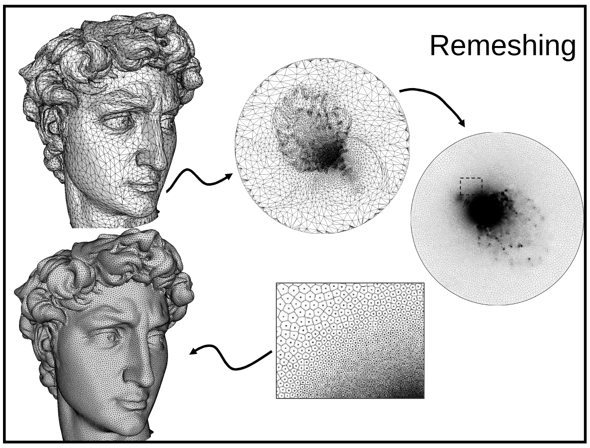
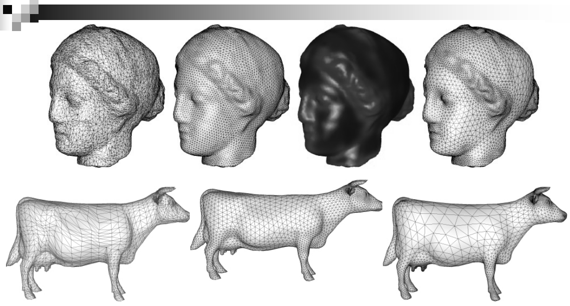
Remeshing



Remeshing examples



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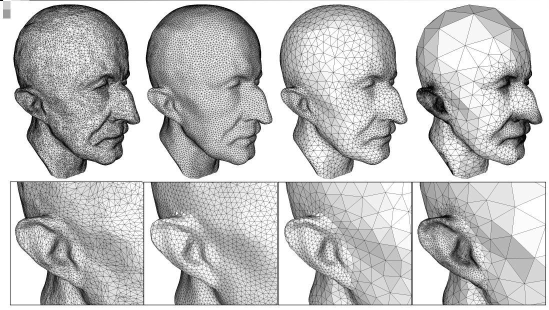
More remeshing examples



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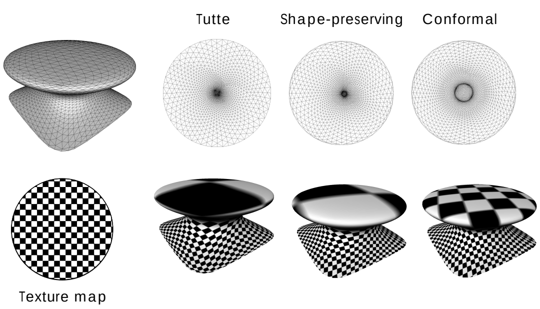
Conformal parametrization



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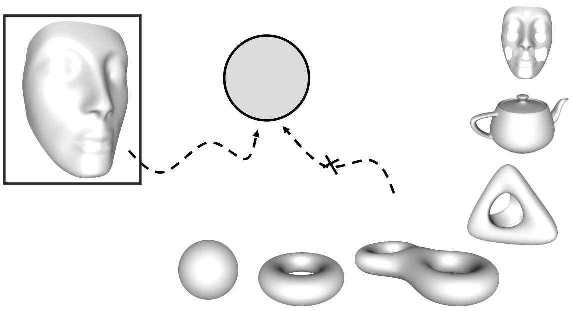
Non-simple domains



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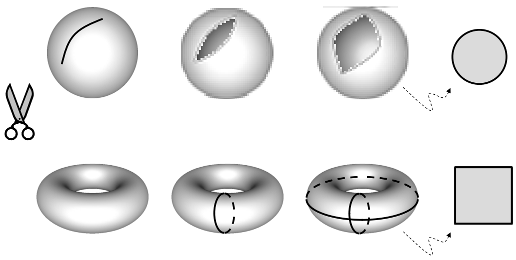
Cutting



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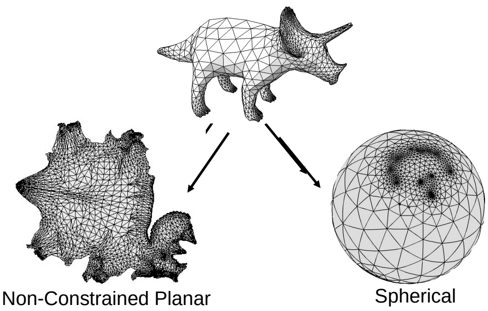
Parameterization of closed genus-0 triangle meshes



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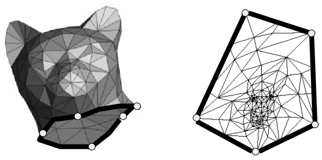
Convex mapping (Tutte, Floater)

- Works for meshes equivalent to a disk
- First, we map the boundary to a convex polygon
- Then we find the inner vertices positions



$\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ – inner vertices; $\boldsymbol{v}_n, \boldsymbol{v}_{n+1}, \ldots, \boldsymbol{v}_N$ – boundary vertices

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Linear system of equations

$$\boldsymbol{v}_i - \sum_{j\in N(i)} \lambda_{i,j}\boldsymbol{v}_j = 0,\quad i = 1, 2, \ldots, n$$

$$\boldsymbol{v}_i - \sum_{j\in N(i)\setminus B} \lambda_{i,j}\boldsymbol{v}_j = \sum_{k\in N(i)\cap B} \lambda_{i,k}\boldsymbol{v}_k\,,\quad i = 1, 2, \ldots, n$$

$$\begin{pmatrix} 1 & & -\lambda_{1,j_1} & & -\lambda_{1,j_{d1}} \\ & 1 & & & \\ & & 1 & & \\ -\lambda_{4,j_1} & & & \ddots & \\ & & -\lambda_{n,j_5} & & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \\ \vdots \\ \boldsymbol{v}_n \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \\ \\ \sigma_n \end{pmatrix}$$

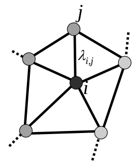
Inner vertices

- We constrain each inner vertex to be a weighted average of its neighbors:

$$\boxed{\boldsymbol{v}_i = \sum_{j\in N(i)} \lambda_{i,j}\boldsymbol{v}_j\,,\quad i = 1, 2, \ldots, n}$$

$$\lambda_{i,j} = \begin{cases} 0 & i, j \text{ are not neighbors} \\ >0 & (i, j) \in E \ \text{(neighbours)} \end{cases}$$

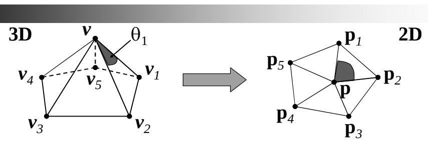
$$\sum_{j\in N(i)} \lambda_{i,j} = 1$$



Linear system of equations

- A unique solution always exists
- Important: the solution is legal (bijective)
- The system is sparse, thus fast numerical solution is possible
- Numerical problems (because the vertices in the middle might get very dense…)

Shape preserving weights



To compute $\lambda_1, \ldots, \lambda_5$, a local embedding of the patch is found:

1) $\| \mathbf{p}_i - \mathbf{p} \| = \| \mathbf{x}_i - \mathbf{x} \|$

2) $angle(\mathbf{p}_i, \mathbf{p}, \mathbf{p}_{i+1}) = (2\pi / \Sigma\theta_i)\ angle(\boldsymbol{v}_i, \boldsymbol{v}, \boldsymbol{v}_{i+1})$

$$\exists\ \lambda_i\,,\ \begin{cases} \mathbf{p} = \Sigma\ \lambda_i\ \mathbf{p}_i \\ \lambda_i > 0 \\ \Sigma\ \lambda_i = 1 \end{cases} \quad \Rightarrow \text{ use these } \lambda \text{ as edge weights.}$$

Energy minimization – least squares

- We want to find such flat positions that the energy is as small as possible.
- Solve the linear least squares problem!

$$\boldsymbol{v}_i = (x_i, y_i)$$

$$E_{harm}(x_1,\ldots,x_n,y_1,\ldots,y_n) = \frac{1}{2}\sum_{(i,j)\in E} k_{i,j}\left\|\boldsymbol{v}_i - \boldsymbol{v}_j\right\|^2 =$$

$$= \frac{1}{2}\sum_{(i,j)\in E} k_{i,j}\left((x_i - x_j)^2 + (y_i - y_j)^2\right).$$

E_{harm} is function of 2n variables

Harmonic mapping

- Another way to find inner vertices
- Strives to preserve angles (conformal)
- We treat the mesh as a system of springs.
- Define spring energy:

$$\boxed{E_{harm} = \frac{1}{2}\sum_{(i,j)\in E} k_{i,j}\left\|\boldsymbol{v}_i - \boldsymbol{v}_j\right\|^2}$$

where $\boldsymbol{v}_i$ are the flat position (remember that the boundary vertices $\boldsymbol{v}_n, \boldsymbol{v}_{n+1}, \ldots, \boldsymbol{v}_N$ are constrained).

Energy minimization – least squares

- To find minimum: $\nabla E_{harm} = 0$

$$\boxed{\begin{aligned} \frac{\partial}{\partial x_i} E_{harm} &= \frac{1}{2} \sum_{j \in N(i)} 2 k_{i,j} (x_i - x_j) = 0 \\ \frac{\partial}{\partial y_i} E_{harm} &= \frac{1}{2} \sum_{j \in N(i)} 2 k_{i,j} (y_i - y_j) = 0 \end{aligned}}$$

- Again, $x_{n+1}, \ldots, x_N$ and $y_{n+1}, \ldots, y_N$ are constrained.

Energy minimization – least squares

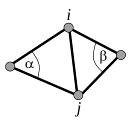
- To find minimum: $\nabla E_{harm} = 0$

$$\boxed{\begin{aligned} \sum_{j \in N(i)} k_{i,j} (x_i - x_j) = 0, \quad i = 1, 2, \ldots, n \\ \sum_{j \in N(i)} k_{i,j} (y_i - y_j) = 0, \quad i = 1, 2, \ldots, n \end{aligned}}$$

- Again, $x_{n+1}, \ldots, x_N$ and $y_{n+1}, \ldots, y_N$ are constrained.

The spring constants $k_{i,j}$

- The weights $k_{i,j}$ are chosen to minimize angles distortion:
  - Look at the edge (i, j) in the 3D mesh
  - Set the weight $\boxed{k_{i,j} = \cot \alpha + \cot \beta}$



3D

Discussion

- The results of harmonic mapping are better than those of convex mapping (local area and angles preservation).
- But: the mapping is not always legal (the weights can be negative for badly-shaped triangles…)
- Both mappings have the problem of fixed boundary – it constrains the minimization and causes distortion.
- There are more advanced methods that do not require boundary conditions.

Thanks