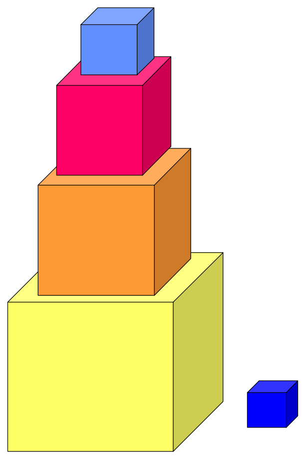
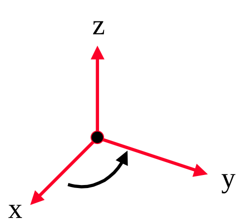
# 3D Geometric Transformation

(Chapt. 5 in FVD, Chapt. 11 in Hearn & Baker)

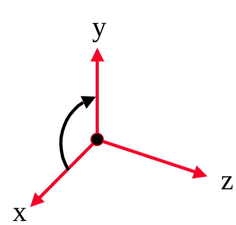


## 3D Coordinate Systems

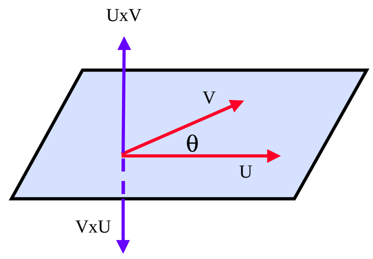
- Right-handed coordinate system:




- Left-handed coordinate system:



## Reminder: Vector Product




$$U \times V = \hat{n}|U||V|\sin\theta$$

$$U \times V = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \begin{bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{bmatrix}$$

## 3D Point

- A 3D point *P* is represented in homogeneous coordinates by a 4-dim. vector:

$$P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- Note, that

$$p = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \equiv \begin{bmatrix} \alpha x \\ \alpha y \\ \alpha z \\ \alpha \end{bmatrix}$$

## 3D Transformations

- In homogeneous coordinates, 3D transformations are represented by 4x4 matrices:

$$\begin{bmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- A point transformation is performed:

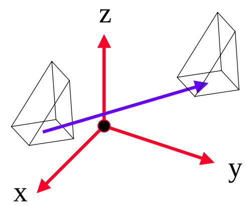
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

## 3D Translation

- P in translated to P' by:

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} x+t_x \\ y+t_y \\ z+t_z \\ 1 \end{bmatrix}$$

Or $T\,P = P'$

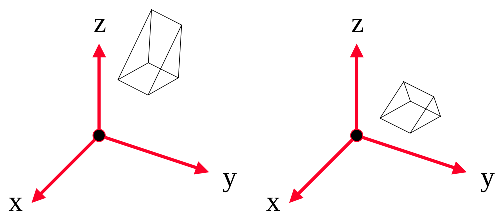



- Inverse translation: $T^{-1}P' = P$

# Scaling

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} ax \\ by \\ cz \\ 1 \end{bmatrix}$$

Or $S\,P = P'$



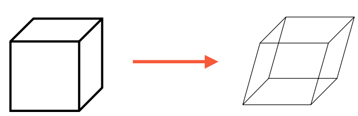

$$S^{-1}P' = P$$

# 3D Shearing

- Shearing:

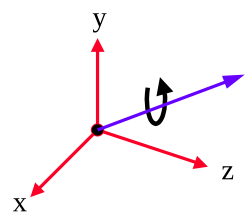
$$\begin{bmatrix} 1 & a & b & 0 \\ c & 1 & d & 0 \\ e & f & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + ay + bz \\ cx + y + dz \\ ex + fy + z \\ 1 \end{bmatrix}$$

- The change in each coordinate is a linear combination of all three.
- Transforms a cube into a general parallelepiped.



## 3D Rotation

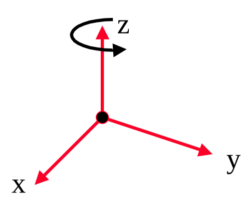
- To generate a rotation in 3D we have to specify:
  - axis of rotation (2 d.o.f)
  - amount of rotation (1 d.o.f)
- Note, the axis passes through the origin.




- A counter-clockwise rotation about the *z*-axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

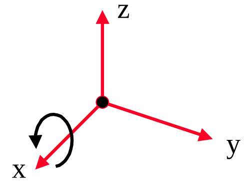
$$p' = R_z(\theta)\, p$$



- A counter-clockwise rotation about the *x*-axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

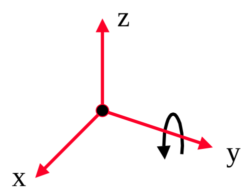
$$p' = R_x(\theta)\, p$$




- A counter-clockwise rotation about the *y*-axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$p' = R_y(\theta)\, p$$



## Inverse Rotation

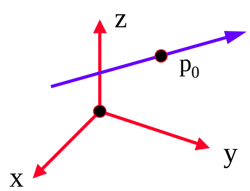
$$p = R^{-1}(\theta)\,p' = R(-\theta)\,p'$$

## Composite Rotations

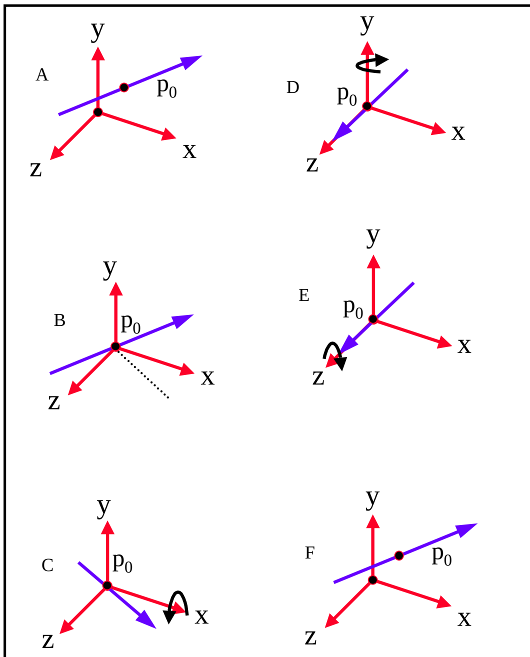
- $R_x$, $R_y$, and $R_z$, can perform *any* rotation about an axis passing through the origin.

## Rotation About an Arbitrary Axis

- Axis of rotation can be located at any point: 6 d.o.f.
- **The idea**: make the axis coincident with one of the coordinate axes (z axis), rotate, and then transform back.
- Assume that the axis passes through the point $p_0$.



- Transformations:
  - Translate $P_0$ to the origin.
  - Make the axis coincident with the *z*-axis (for example):
    - Rotate about the *x*-axis into the *xz* plane.
    - Rotate about the *y*-axis onto the *z*-axis.
    - Rotate as needed about the *z*-axis.
    - Apply inverse rotations about *y* and *x*.
    - Apply inverse translation.



# 3D Reflection

- A reflection through the *xy* plane:

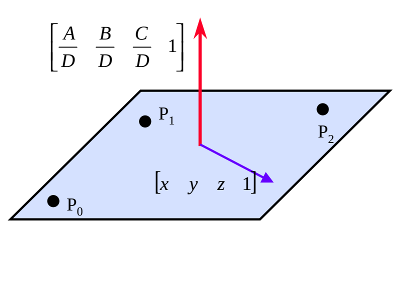
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ -z \\ 1 \end{bmatrix}$$

- Reflections through the xz and the yz planes are defined similarly.
- How can we reflect through some arbitrary plane?

# Transforming Planes

- Plane representation:
  - By three non-collinear points
  - By implicit equation:

$$Ax + By + Cz + D = \begin{bmatrix} A & B & C & D \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$



- One way to transform a plane is by transforming any three non-collinear points on the plane.
- Another way is to transform the plane equation: Given a transformation $T$ that transforms [$x,y,z,1$] to [$x',y',z',1$] find [A',B',C',D'], such that:

$$\begin{bmatrix} A' & B' & C' & D' \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = 0$$

- Note that

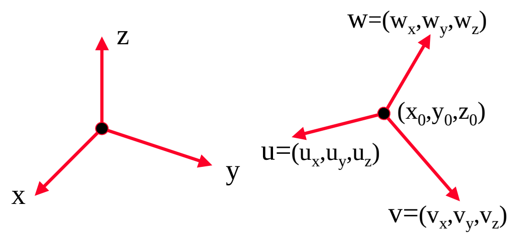
$$\begin{bmatrix} A & B & C & D \end{bmatrix} T^{-1} T \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

- Thus, the transformation that we should apply to the plane equation is:

$$\begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix} = \left(T^{-1}\right)^T \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

## Changing Coordinate Systems

- **Problem**: Given the *XYZ* orthogonal coordinate system, find a transformation, M, that maps *XYZ* to an arbitrary orthogonal system *UVW*.
- This transformation changes a representation from the UVW system to the XYZ system.




- **Solution**: M=RT where T is a translation matrix by $(x_0,y_0,z_0)$, and R is rotation matrix whose columns are *U*,*V*, and *W*:

$$R = \begin{bmatrix} u_x & v_x & w_x & 0 \\ u_y & v_y & w_y & 0 \\ u_z & v_z & w_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

because

$$RX = \begin{bmatrix} u_x & v_x & w_x & 0 \\ u_y & v_y & w_y & 0 \\ u_z & v_z & w_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} u_x \\ u_y \\ u_z \\ 1 \end{bmatrix} = U$$

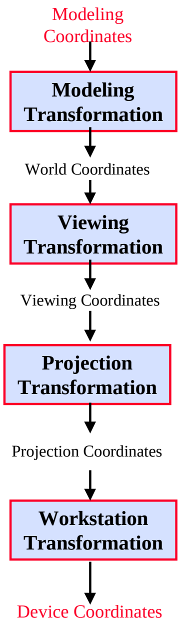
- Similarly, Y goes into V, and Z goes into W.
- The inverse transform, $T^{-1}R^{-1}$, provides the mapping from UVW back to XYZ. For the rotation matrix $R^{-1}=R^T$:

$$R^T U = \begin{bmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ w_x & w_y & w_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} u_x^2 + u_y^2 + u_z^2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = X$$

- **Comment**: Very useful if an arbitrary plane is to be mapped to the XY plane or vice versa.
- Possible to apply if an arbitrary vector is to be mapped to an axis (How?).

# Transformation Pipe-Line




# Modeling Coordinate Hierarchy