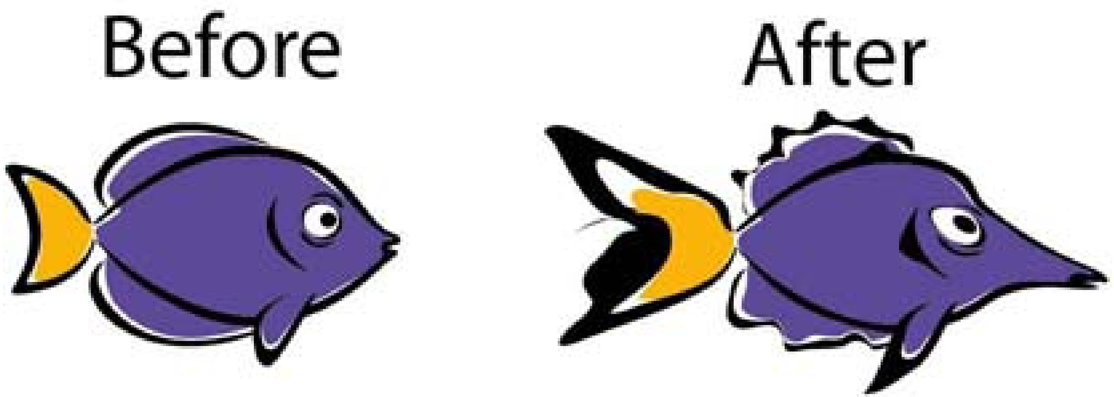
Transformations

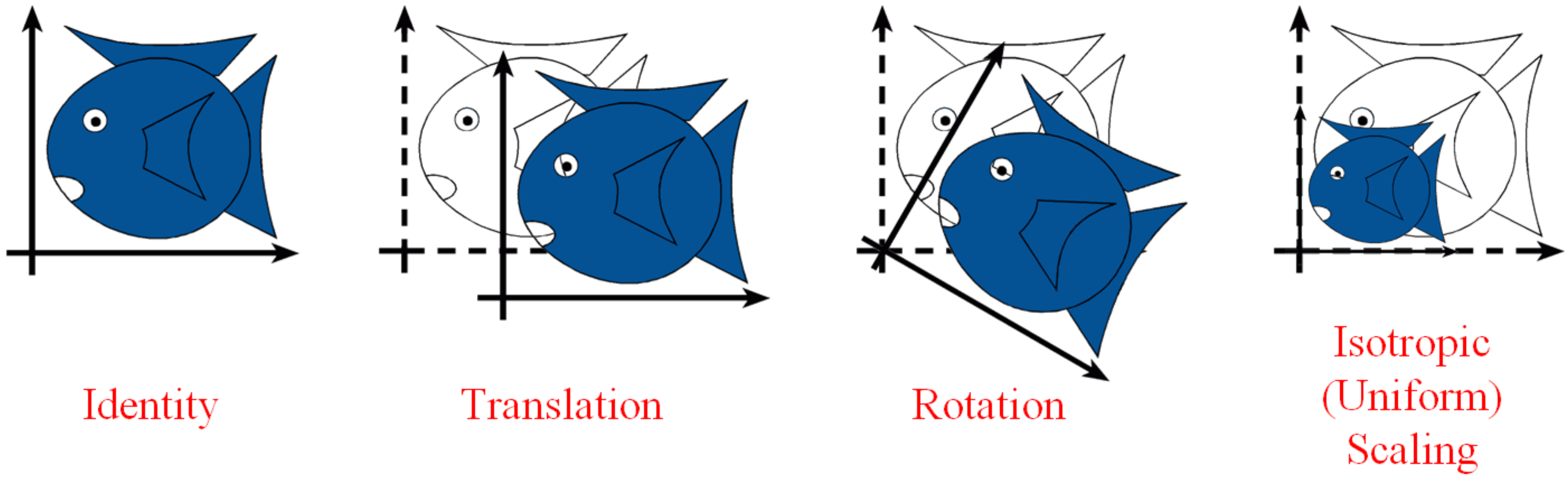


Many of the slides are taken from MIT EECS 6.837, Durand and Cutler

Transformations are used:

- Position objects in a scene (modeling)
- Change the shape of objects
- Create multiple copies of objects
- Projection for virtual cameras
- Animations

Simple Transformations

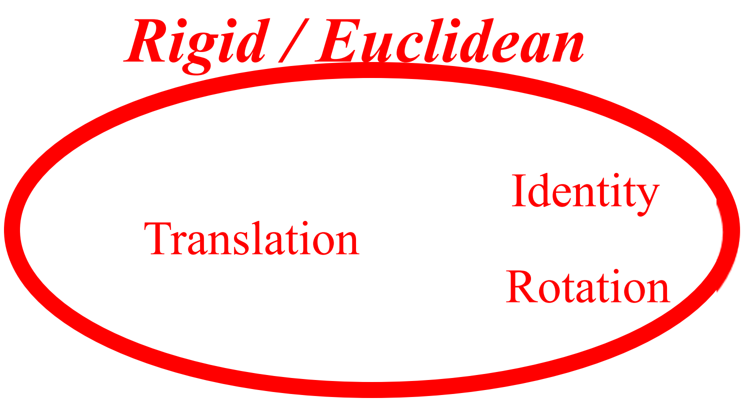


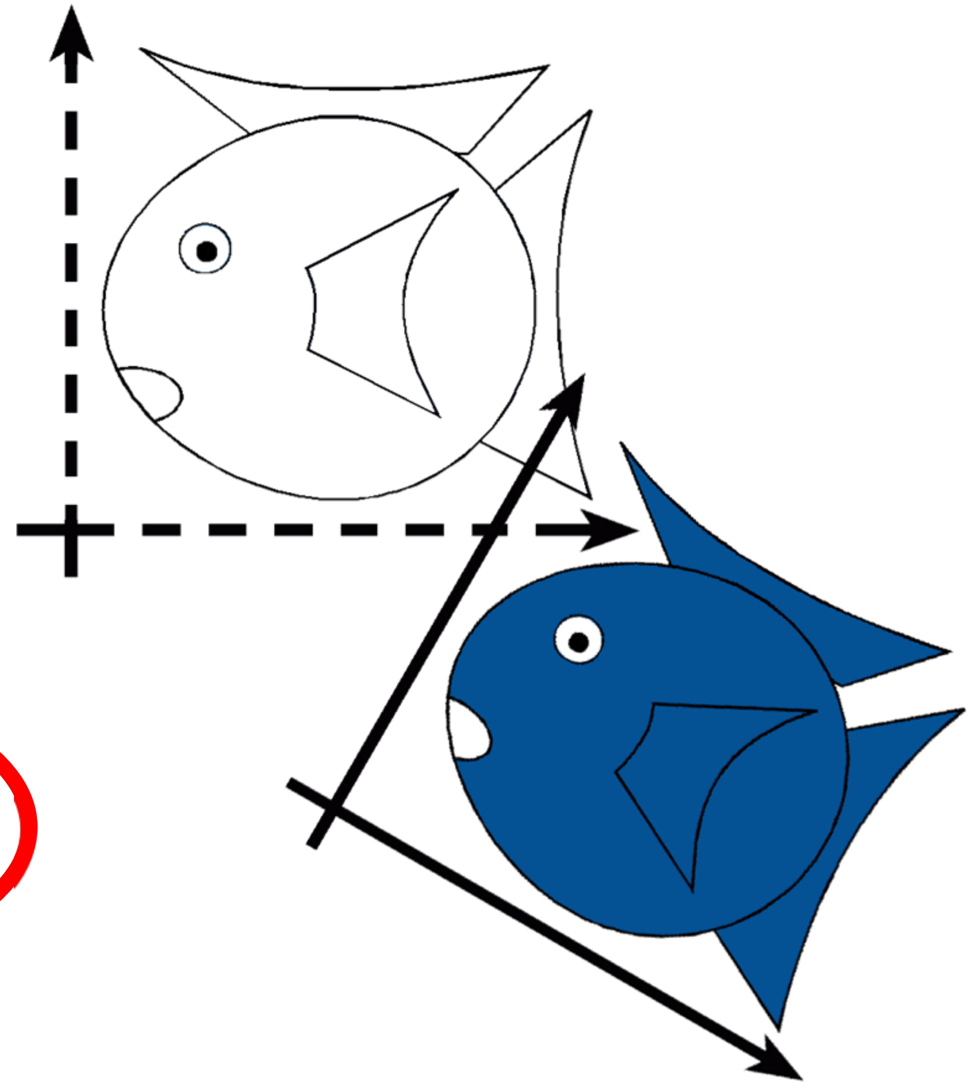
- Can be combined
- Are these operations invertible?

  Yes, except scale = 0

Rigid-Body / Euclidean Transforms

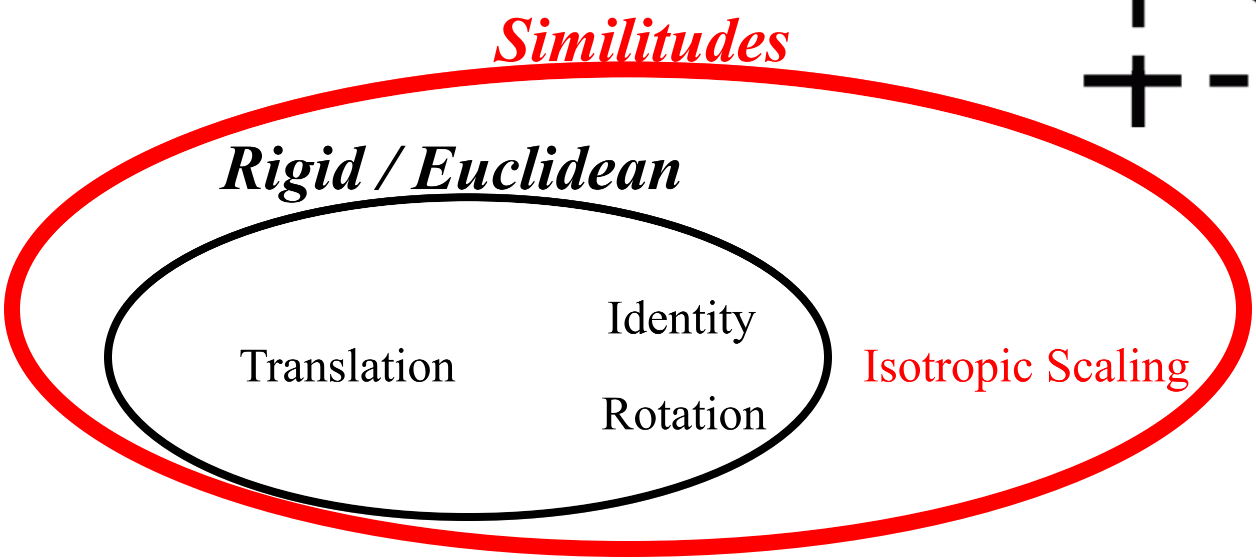
- Preserves distances
- Preserves angles

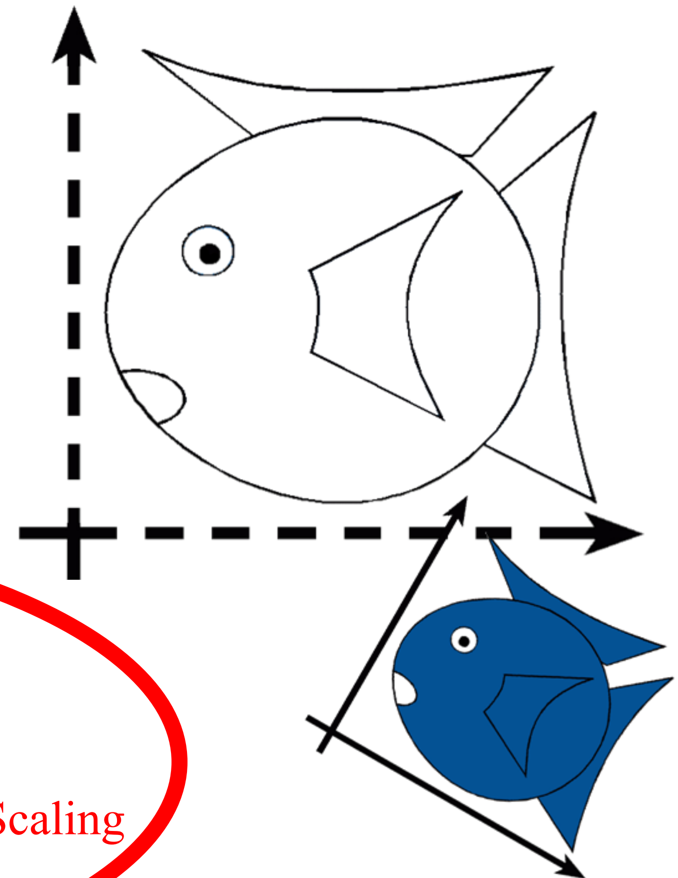




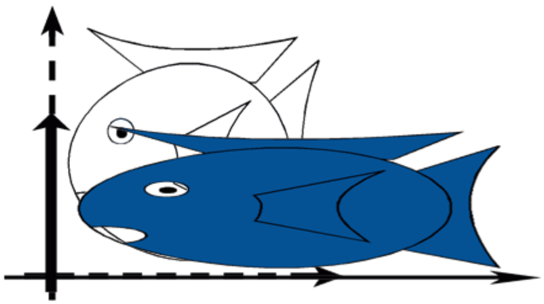
Similitudes / Similarity Transforms

- Preserves angles

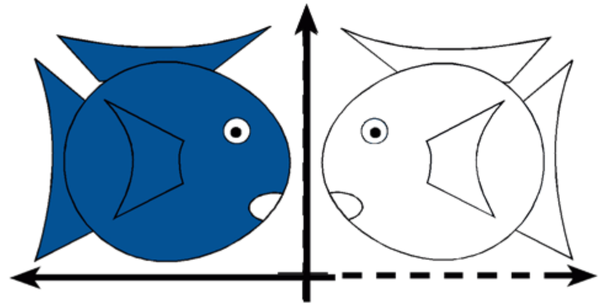




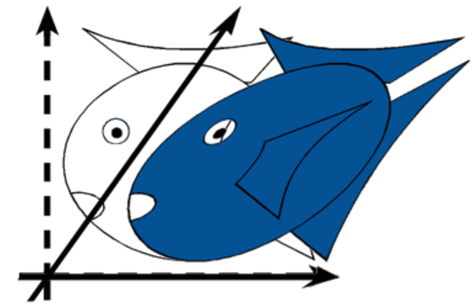
Linear Transformations



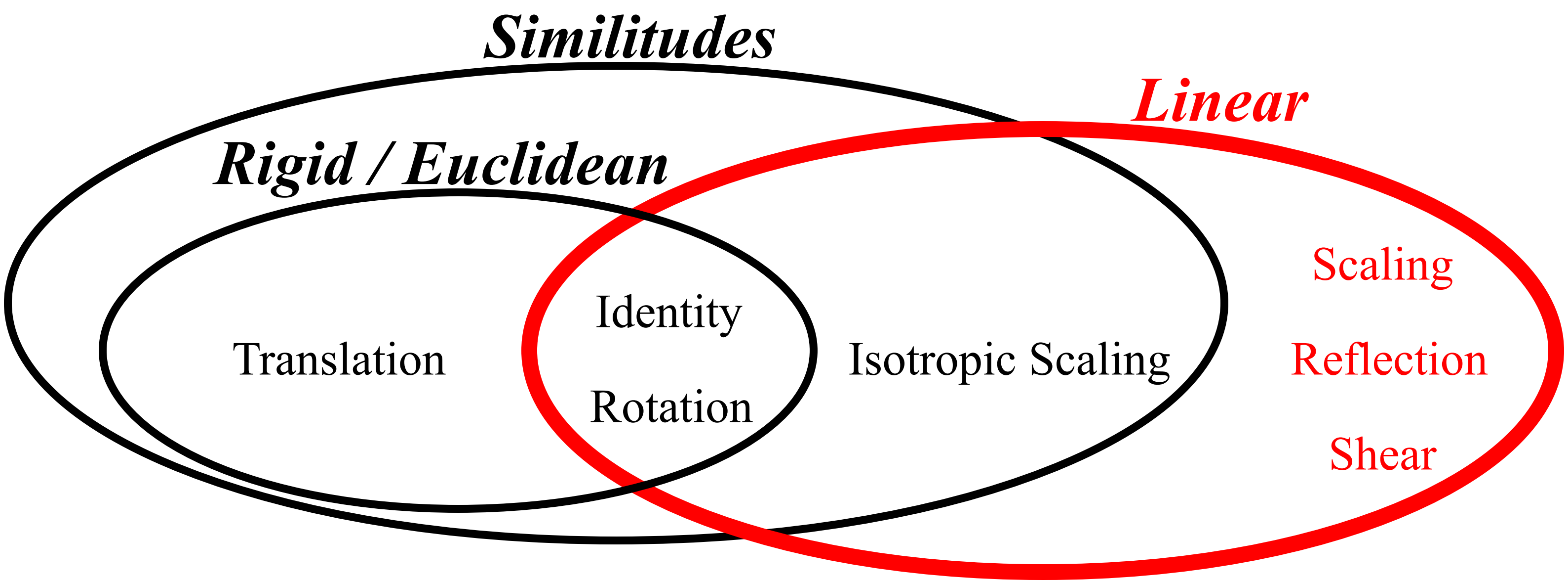
Scaling



Reflection

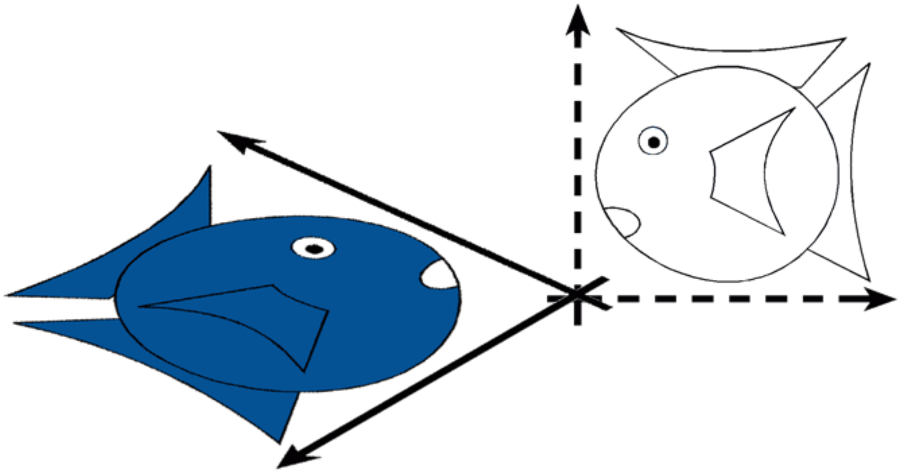


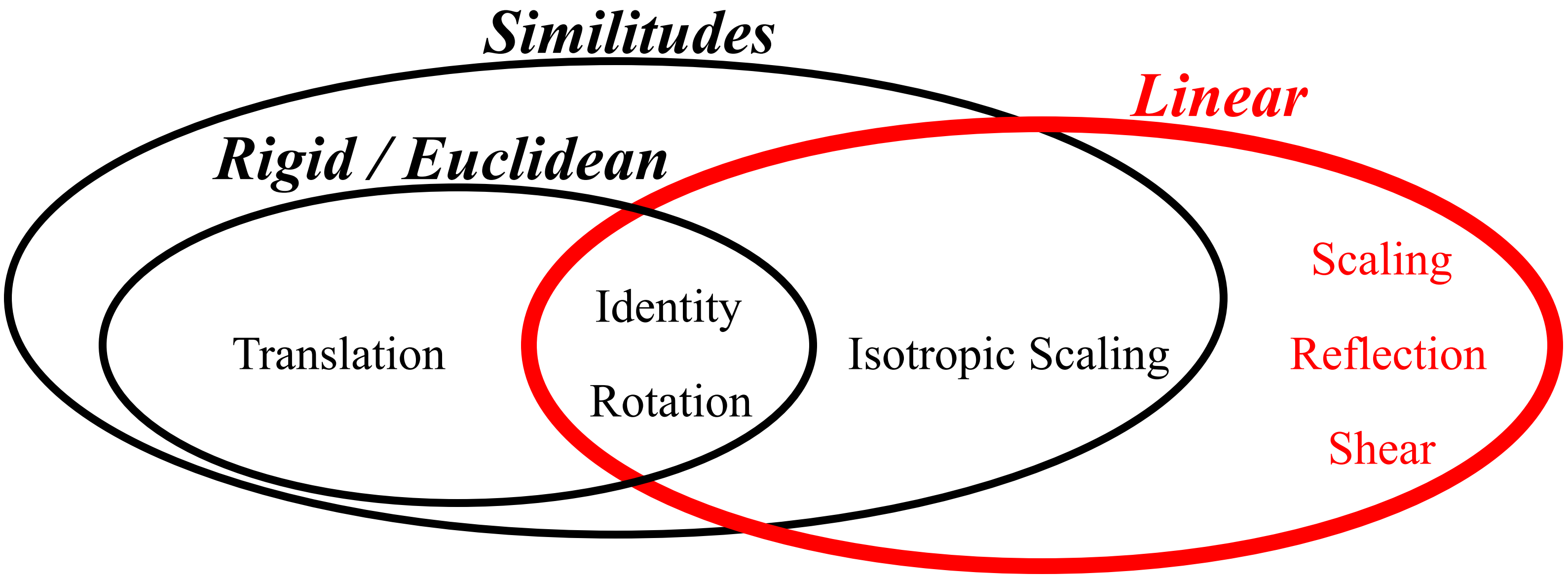
Shear



Linear Transformations

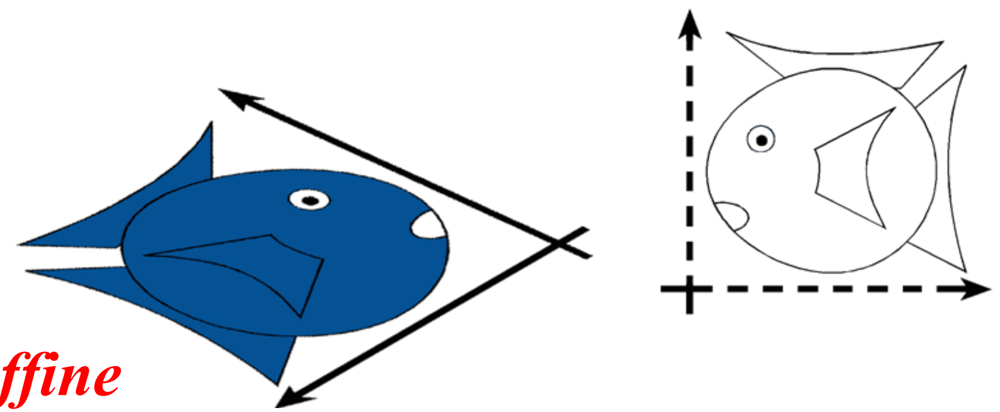
- $L(p + q) = L(p) + L(q)$
- $L(ap) = a\,L(p)$

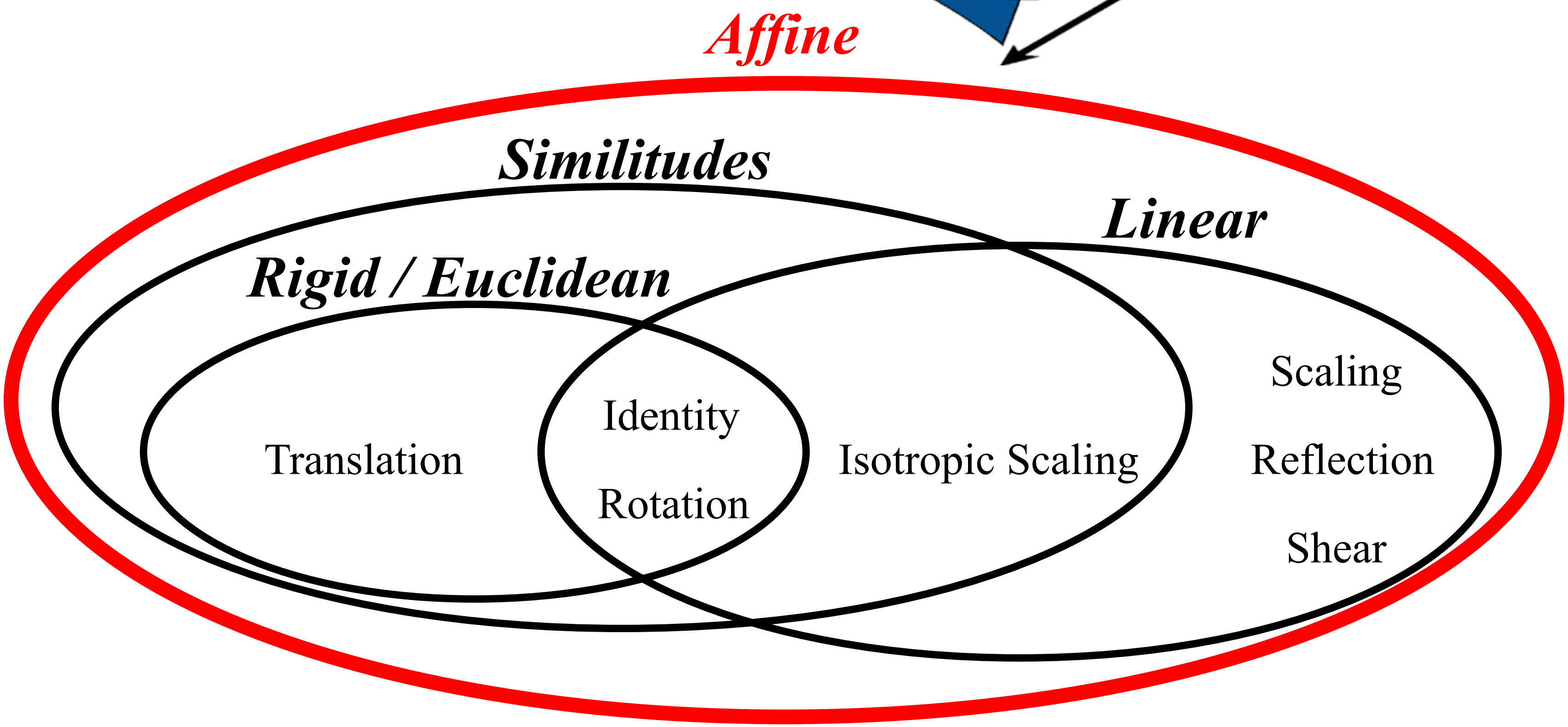




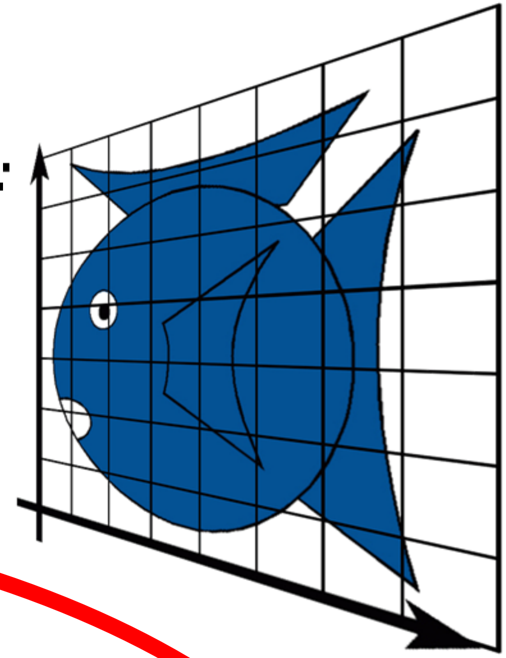
Affine Transformations

- preserves parallel lines

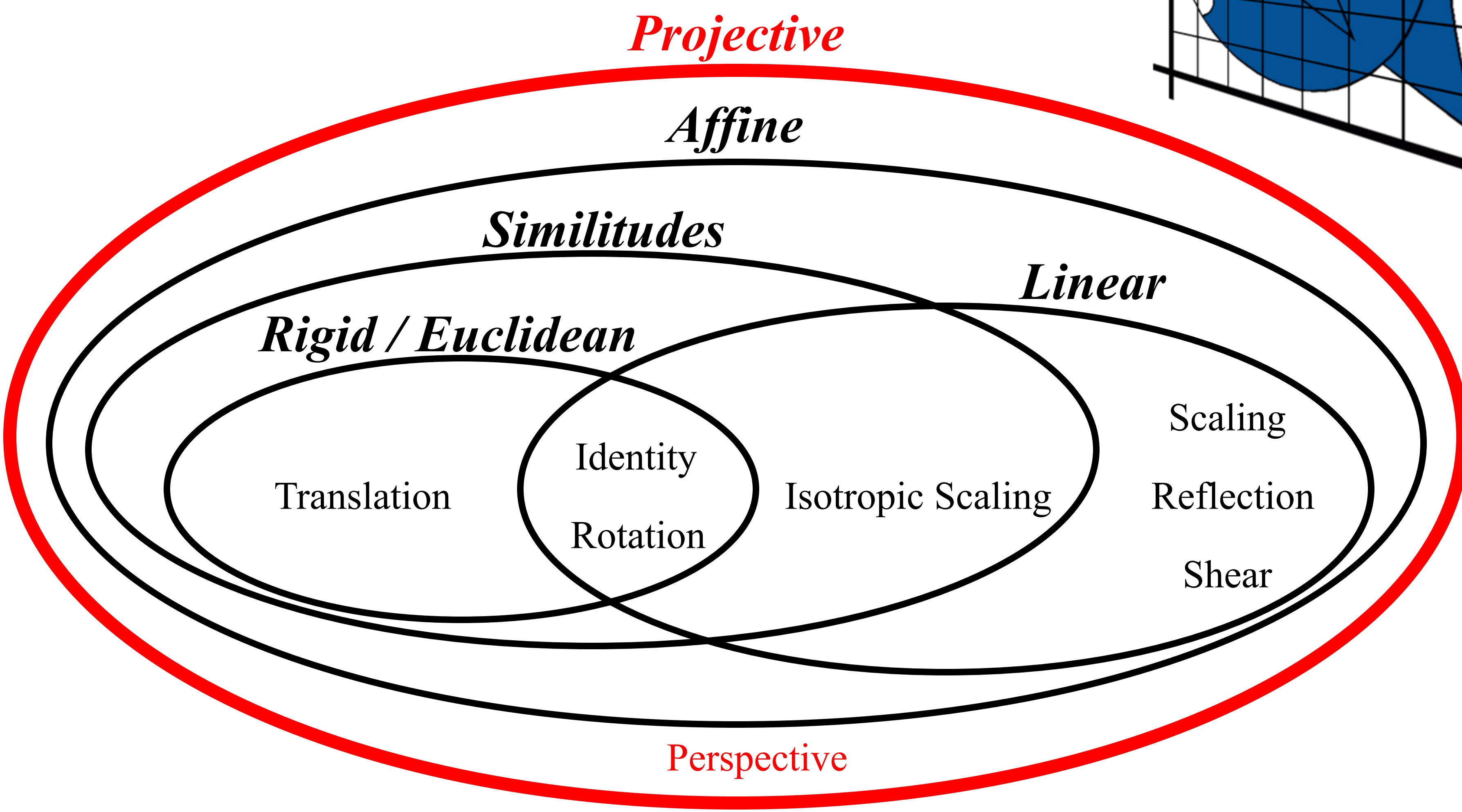




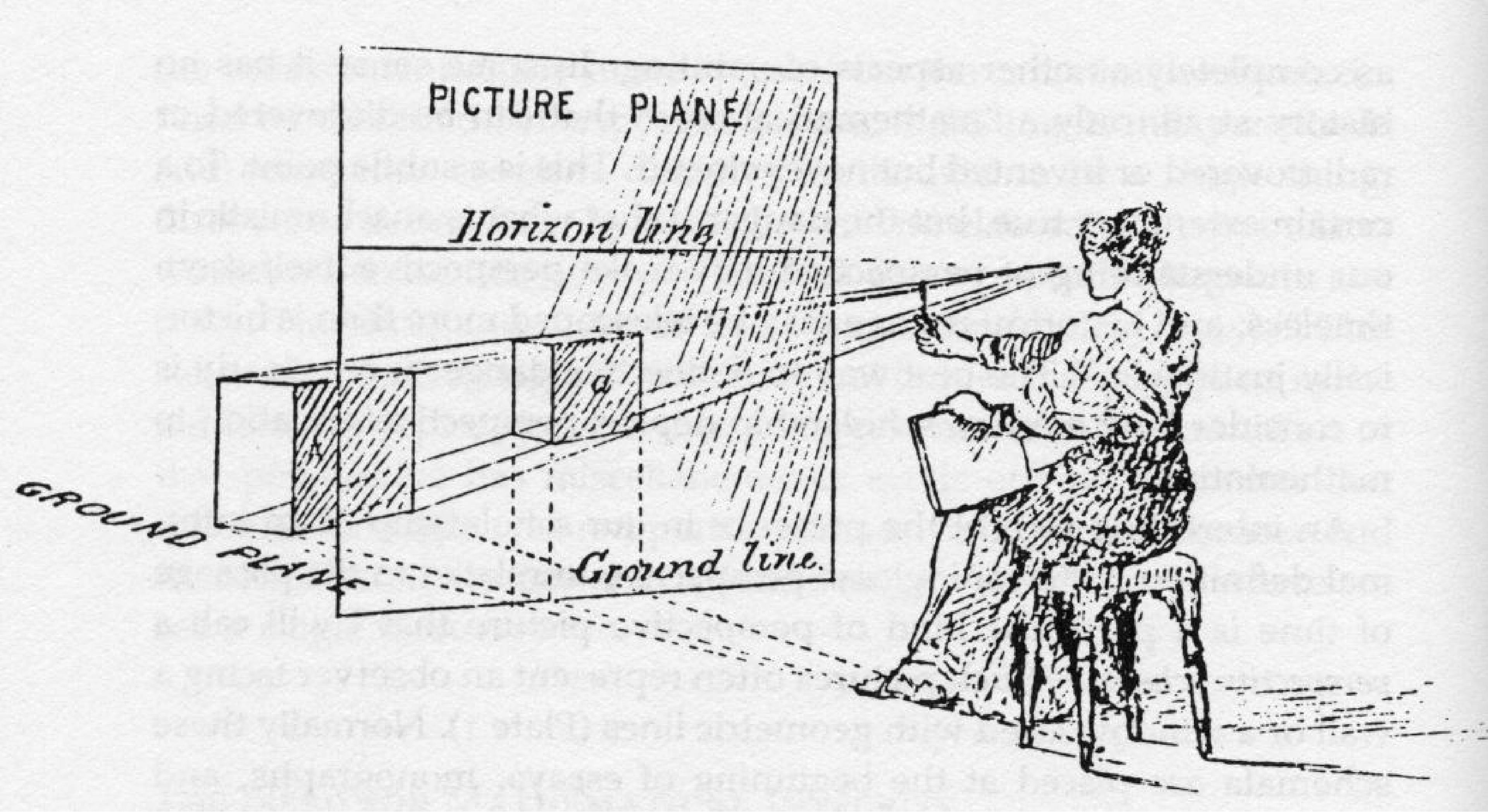
Projective Transformations



- preserves lines



Perspective Projection



Outline

- Assignment 0 Recap
- Intro to Transformations
- Classes of Transformations
- Representing Transformations
- Combining Transformations
- Change of Orthonormal Basis

How are Transforms Represented?

$$x' = ax + by + c$$
$$y' = dx + ey + f$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

$$p' = M p + t$$

Homogeneous Coordinates

- Add an extra dimension
  - in 2D, we use 3 x 3 matrices
  - In 3D, we use 4 x 4 matrices
- Each point has an extra value, *w*

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$p' = M p$$

Homogeneous Coordinates

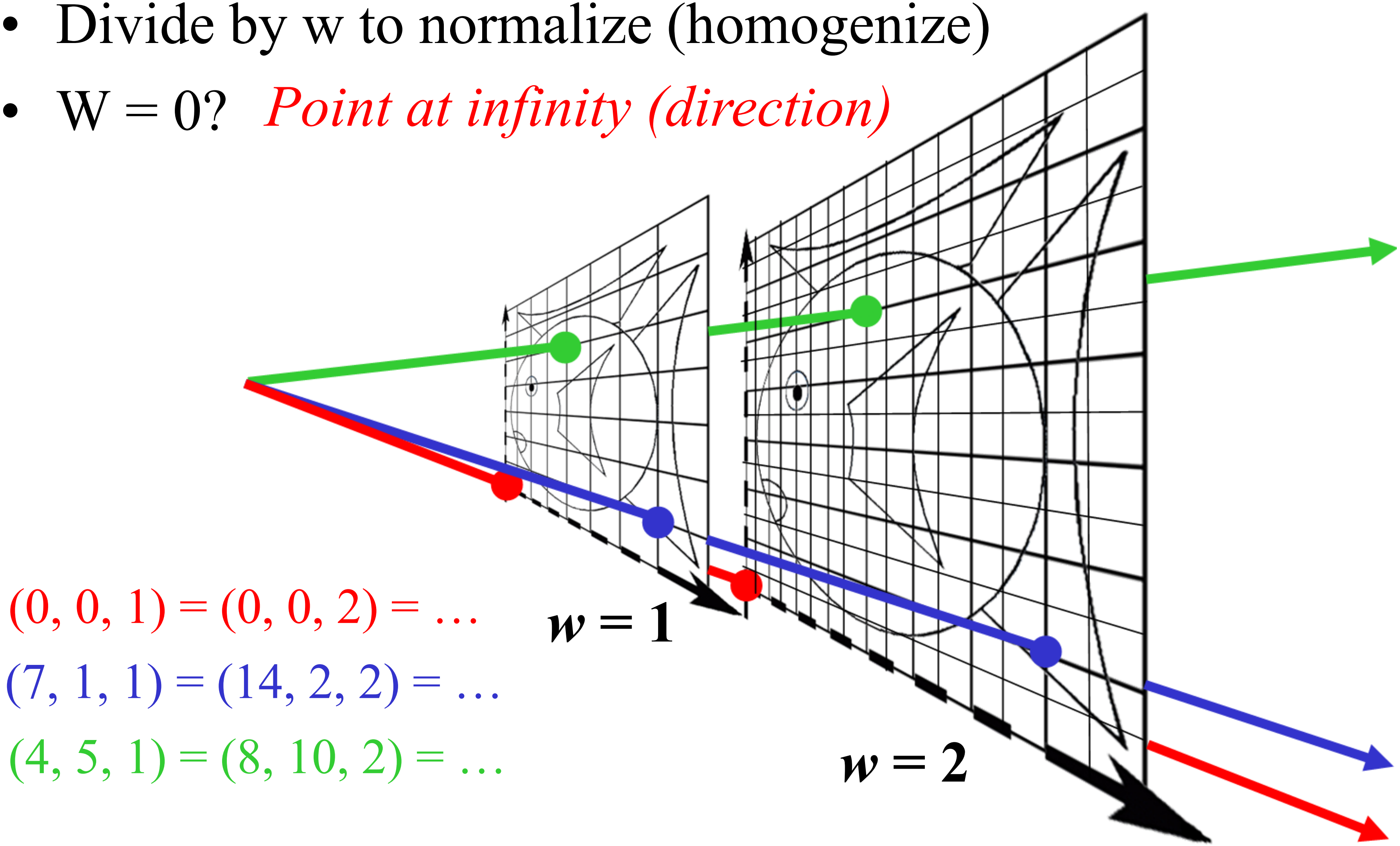
- Most of the time w = 1, and we can ignore it

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

- If we multiply a homogeneous coordinate by an *affine matrix*, w is unchanged

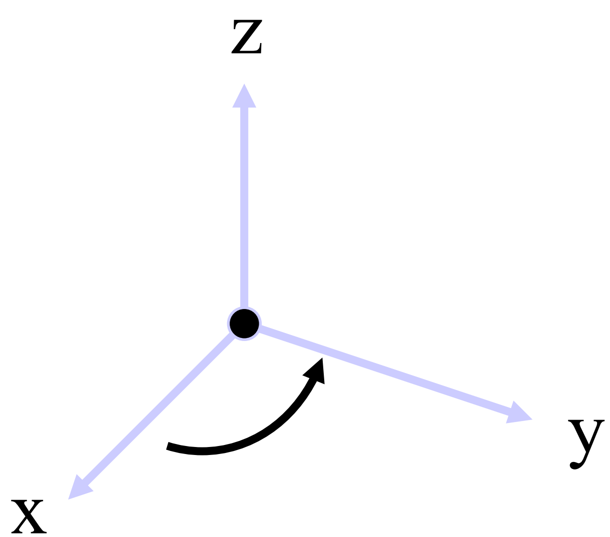
Homogeneous Visualization

- Divide by w to normalize (homogenize)
- W = 0? *Point at infinity (direction)*

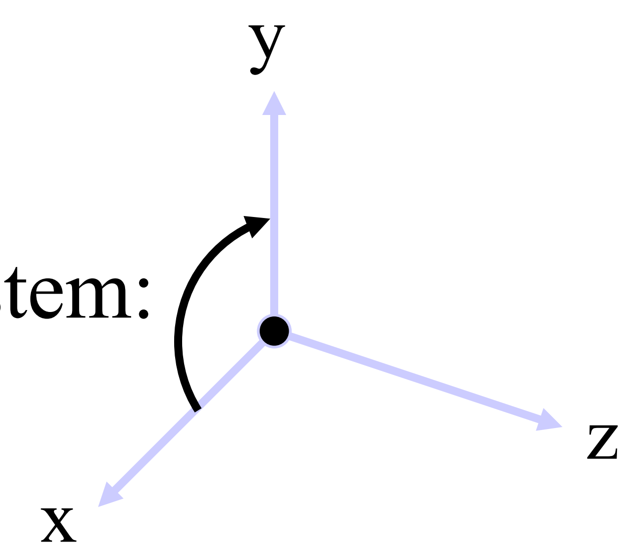


3D Coordinate Systems

- Right-handed coordinate system:



- Left-handed coordinate system:



3D Transformations

- In homogeneous coordinates, 3D transformations are represented by 4x4 matrices.
- A point transformation is performed:

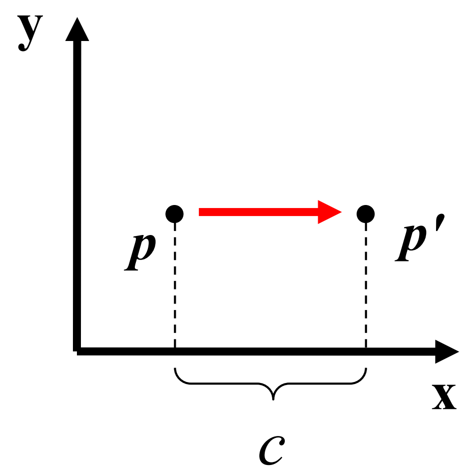
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Translate (t_x, t_y, t_z)

- Why bother with the extra dimension?

  Because now translations can be encoded in the matrix!

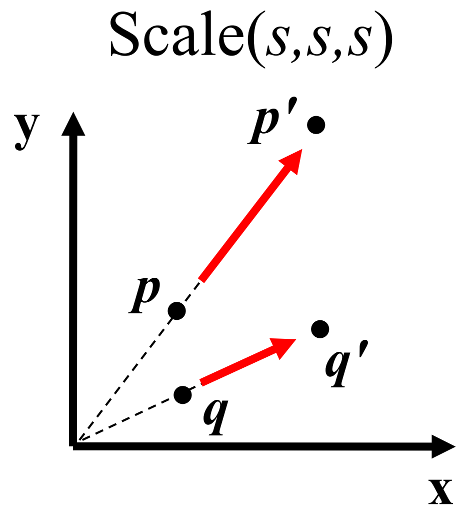
Translate($c,0,0$)



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Scale (s_x, s_y, s_z)

- Isotropic (uniform) scaling: $s_x = s_y = s_z$

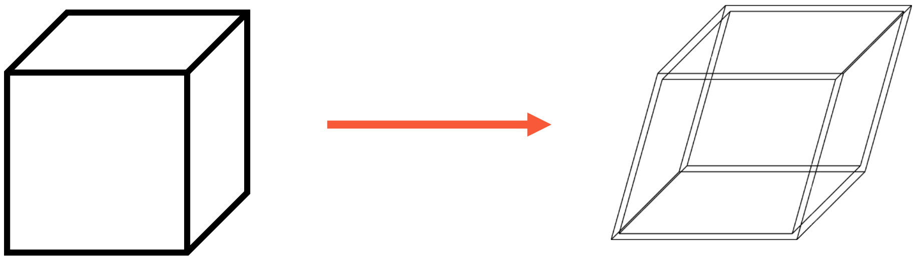


$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

3D Shearing

$$\begin{bmatrix} 1 & a & b & 0 \\ c & 1 & d & 0 \\ e & f & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + ay + bz \\ cx + y + dz \\ ex + fy + z \\ 1 \end{bmatrix}$$

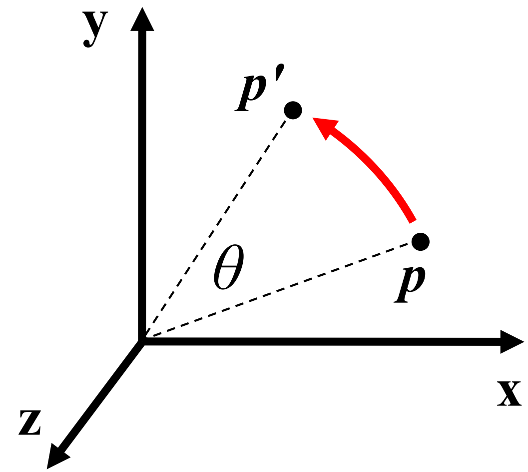
- The change in each coordinate is a linear combination of all three.
- Transforms a cube into a general parallelepiped.



Rotation

- About z axis

ZRotate(θ)



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Rotation

- About x axis:

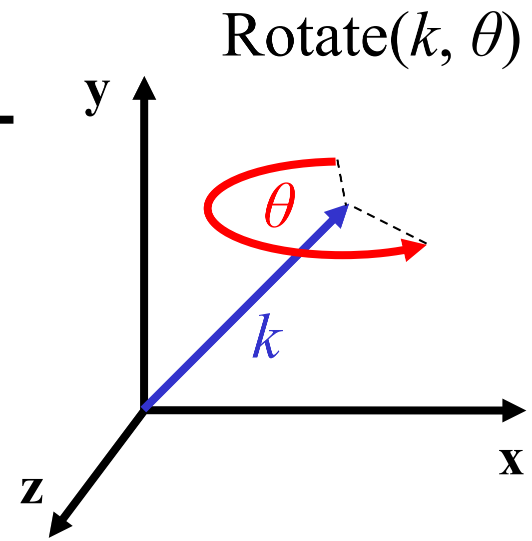
$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

- About y axis:

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Rotation

- About (k_x, k_y, k_z), a unit vector on an arbitrary axis (Rodrigues Formula)

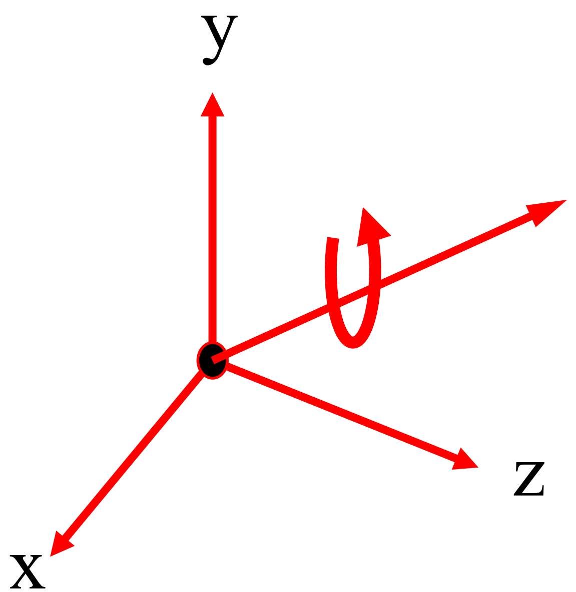


$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} k_xk_x(1-c)+c & k_zk_x(1-c)-k_zs & k_xk_z(1-c)+k_ys & 0 \\ k_yk_x(1-c)+k_zs & k_zk_x(1-c)+c & k_yk_z(1-c)-k_xs & 0 \\ k_zk_x(1-c)-k_ys & k_zk_x(1-c)-k_xs & k_zk_z(1-c)+c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

where $c = \cos\theta$ & $s = \sin\theta$

3D Rotation

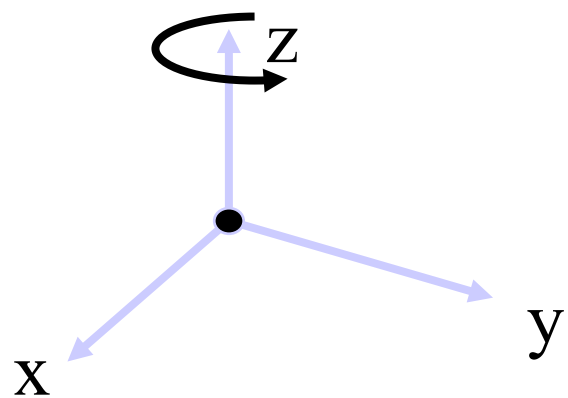
- To generate a rotation in 3D we have to specify:
  - axis of rotation (2 d.o.f)
  - amount of rotation (1 d.o.f)
- Note, the axis passes through the origin.



A counter-clockwise rotation about the z-axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

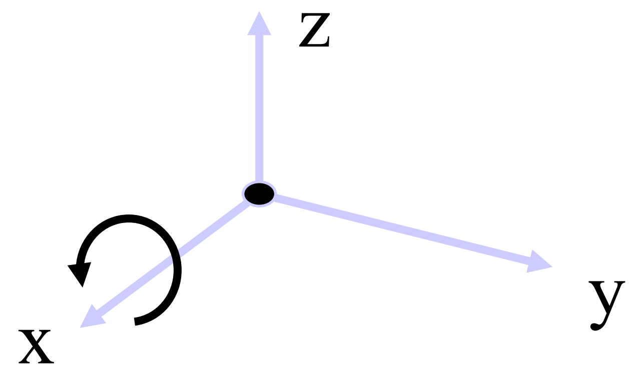
$$p' = R_z(\theta)\, p$$



A counter-clockwise rotation about the x-axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

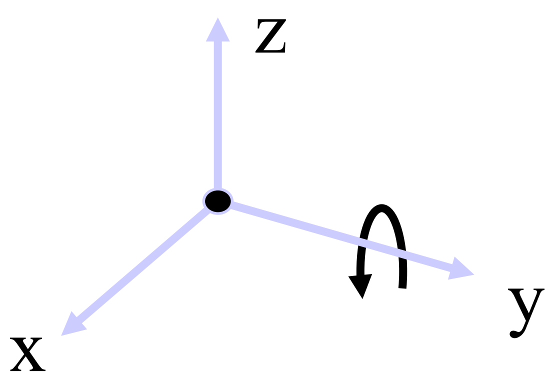
$$p' = R_x(\theta)\, p$$



A counter-clockwise rotation about the *y*-axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$p' = R_y(\theta)\, p$$



About Rotations

Inverse Rotation

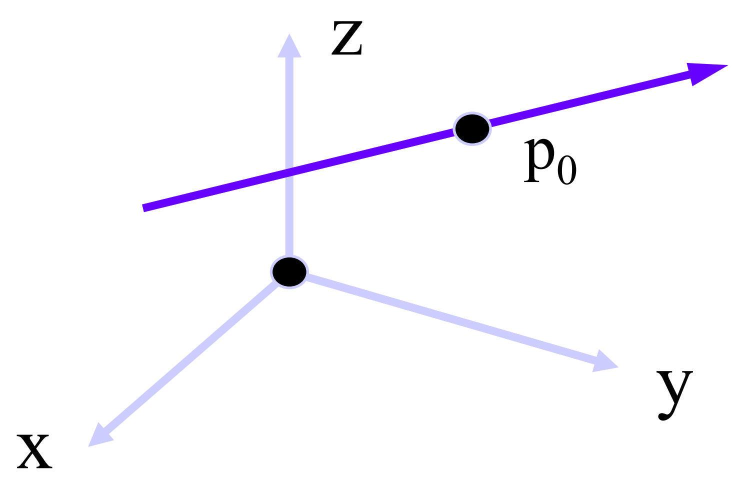
$$p = R^{-1}(\theta)\,p' = R(-\theta)\,p'$$

Composite Rotations

- R_x, R_y, and $R_{z,}$ can perform *any* rotation about an axis passing through the origin.

Rotation About an Arbitrary Axis

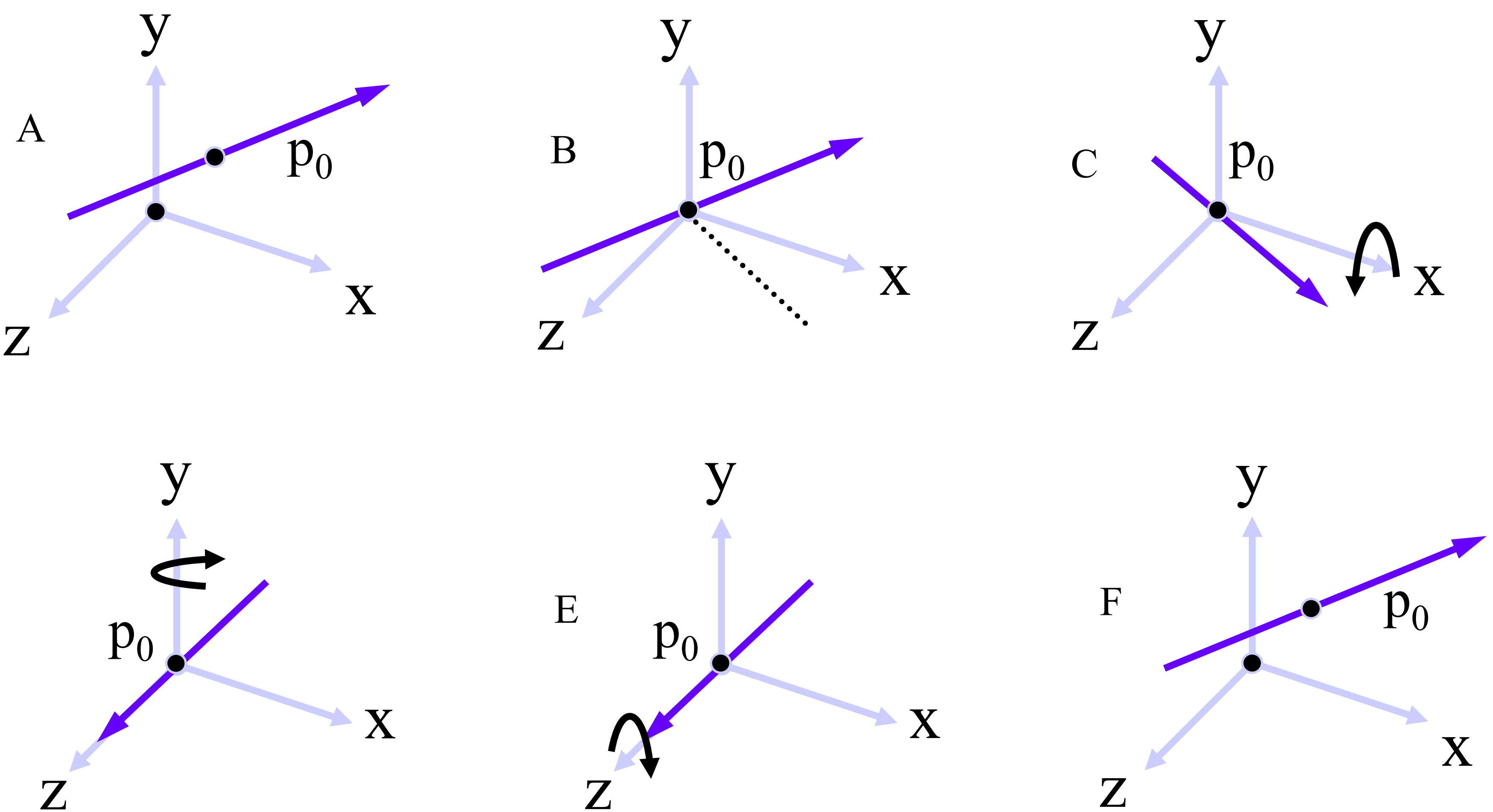
- Axis of rotation can be located at any point: 6 d.o.f.
- **The idea**: make the axis coincident with one of the coordinate axes (z axis), rotate, and then transform back.
- Assume that the axis passes through the point p_0.



Rotation About an Arbitrary Axis

- Steps:
  - Translate P_0 to the origin.
  - Make the axis coincident with the *z*-axis (for example):
    - Rotate about the *x*-axis into the *xz* plane.
    - Rotate about the *y*-axis onto the *z*-axis.
    - Rotate as needed about the *z*-axis.
    - Apply inverse rotations about *y* and *x*.
    - Apply inverse translation.

Rotation About an Arbitrary Axis

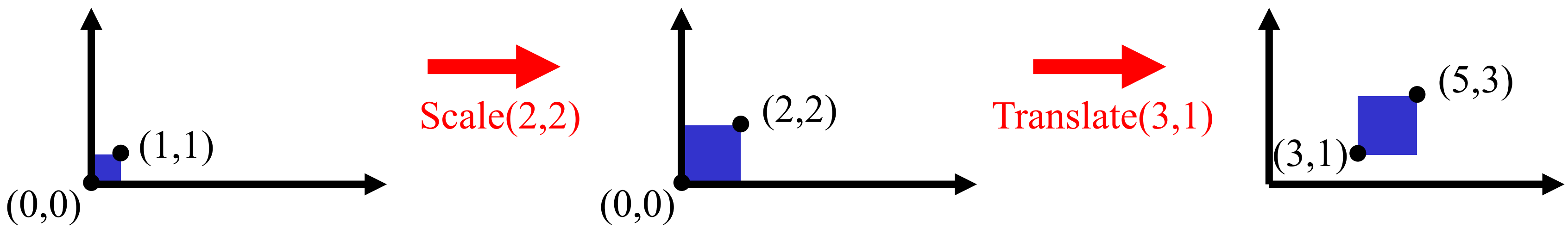


Outline

- Assignment 0 Recap
- Intro to Transformations
- Classes of Transformations
- Representing Transformations
- Combining Transformations
- Change of Orthonormal Basis

How are transforms combined?

Scale then Translate



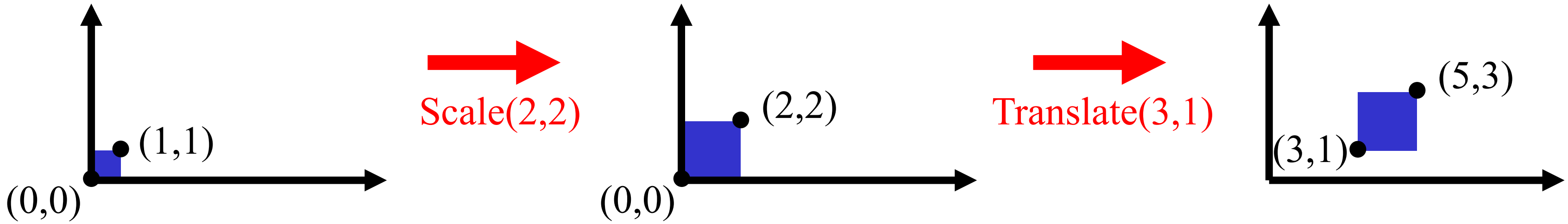
Use matrix multiplication: $p' = T(S\,p) = TS\,p$

$$TS = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

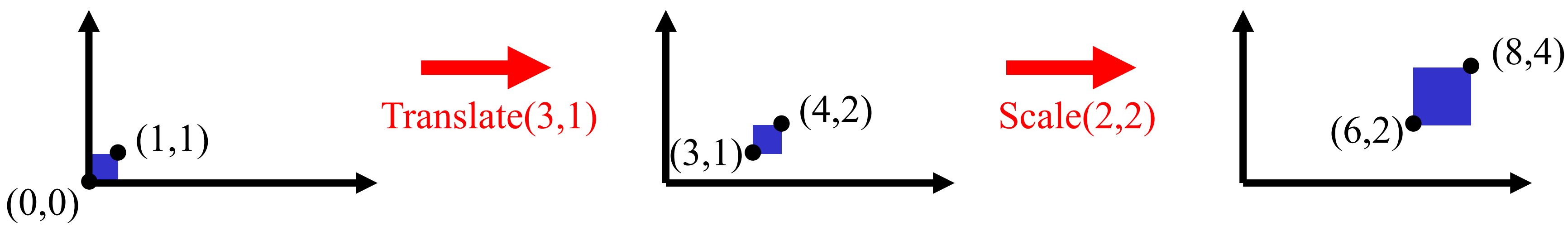
Caution: matrix multiplication is NOT commutative!

Non-commutative Composition

Scale then Translate: $p' = T(Sp) = TSp$



Translate then Scale: $p' = S(Tp) = STp$



Non-commutative Composition

Scale then Translate: p' = T (S p) = TS p

$$TS = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

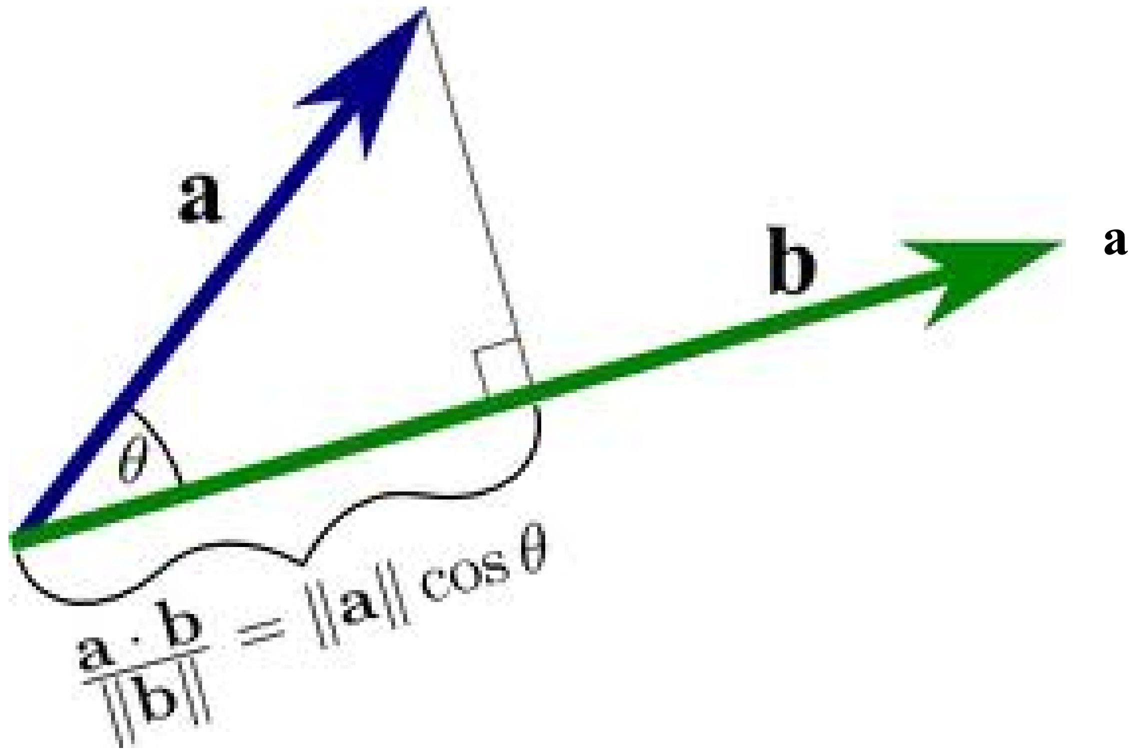
Translate then Scale: p' = S (T p) = ST p

$$ST = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Outline

- Assignment 0 Recap
- Intro to Transformations
- Classes of Transformations
- Representing Transformations
- Combining Transformations
- Change of Orthonormal Basis

Review of Dot Product



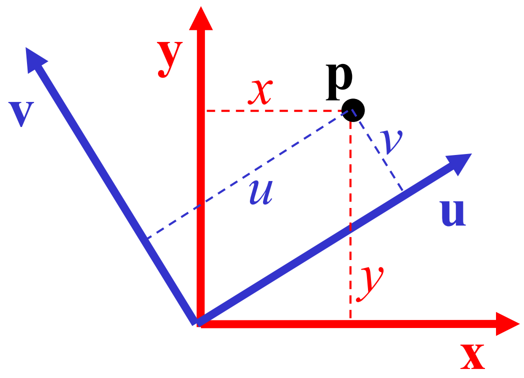
Change of Orthonormal Basis

- Given:

  coordinate frames

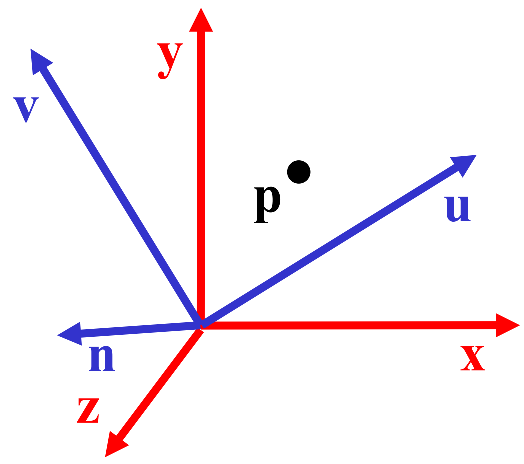
  xyz and **uvn**

  point $\mathbf{p} = (x,y,z)$

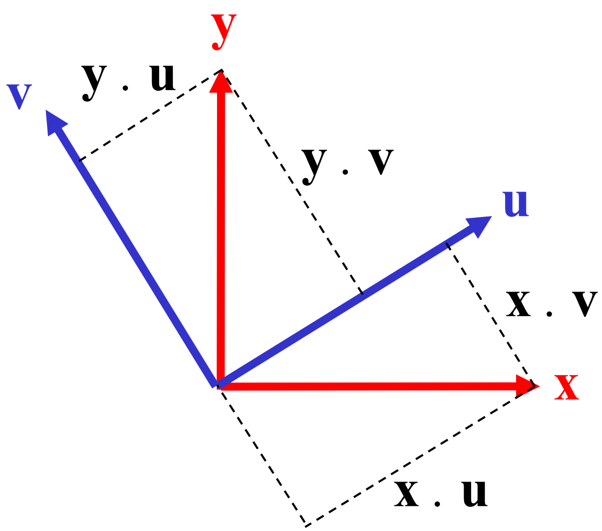


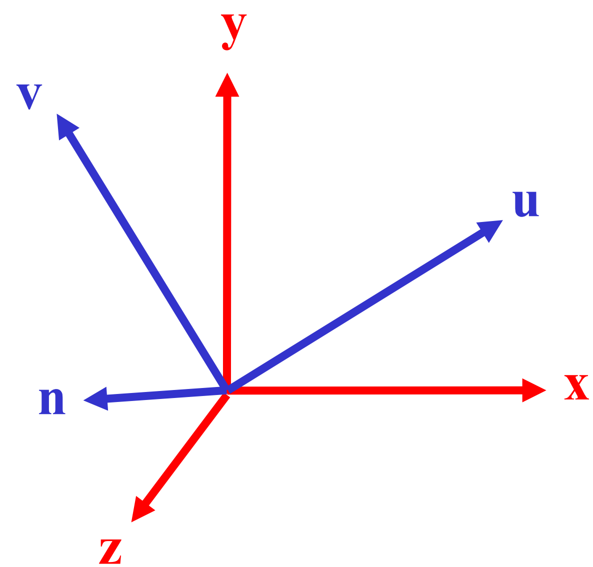
- Find:

  $\mathbf{p} = (u,v,n)$



Change of Orthonormal Basis





$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{x} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{x} \cdot \mathbf{n})\,\mathbf{n}$$

$$\mathbf{y} = (\mathbf{y} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{y} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{y} \cdot \mathbf{n})\,\mathbf{n}$$

$$\mathbf{z} = (\mathbf{z} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{z} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{z} \cdot \mathbf{n})\,\mathbf{n}$$

Change of Orthonormal Basis

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{x} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{x} \cdot \mathbf{n})\,\mathbf{n}$$
$$\mathbf{y} = (\mathbf{y} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{y} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{y} \cdot \mathbf{n})\,\mathbf{n}$$
$$\mathbf{z} = (\mathbf{z} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{z} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{z} \cdot \mathbf{n})\,\mathbf{n}$$

Substitute into equation for *p*:

$$\mathbf{p} = (x,y,z) = x\,\mathbf{x} + y\,\mathbf{y} + z\,\mathbf{z}$$

$$\begin{aligned}\mathbf{p} = \; & x\,[\,(\mathbf{x} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{x} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{x} \cdot \mathbf{n})\,\mathbf{n}\,] + \\ & y\,[\,(\mathbf{y} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{y} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{y} \cdot \mathbf{n})\,\mathbf{n}\,] + \\ & z\,[\,(\mathbf{z} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{z} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{z} \cdot \mathbf{n})\,\mathbf{n}\,]\end{aligned}$$

Change of Orthonormal Basis

$$
\begin{aligned}
\mathbf{p} = \; & x\,[\,(\mathbf{x}\,.\,\mathbf{u})\,\mathbf{u} + (\mathbf{x}\,.\,\mathbf{v})\,\mathbf{v} + (\mathbf{x}\,.\,\mathbf{n})\,\mathbf{n}\,] + \\
& y\,[\,(\mathbf{y}\,.\,\mathbf{u})\,\mathbf{u} + (\mathbf{y}\,.\,\mathbf{v})\,\mathbf{v} + (\mathbf{y}\,.\,\mathbf{n})\,\mathbf{n}\,] + \\
& z\,[\,(\mathbf{z}\,.\,\mathbf{u})\,\mathbf{u} + (\mathbf{z}\,.\,\mathbf{v})\,\mathbf{v} + (\mathbf{z}\,.\,\mathbf{n})\,\mathbf{n}\,]
\end{aligned}
$$

Rewrite:

$$
\begin{aligned}
\mathbf{p} = \; & [\,x\,(\mathbf{x}\,.\,\mathbf{u}) + y\,(\mathbf{y}\,.\,\mathbf{u}) + z\,(\mathbf{z}\,.\,\mathbf{u})\,]\,\mathbf{u} + \\
& [\,x\,(\mathbf{x}\,.\,\mathbf{v}) + y\,(\mathbf{y}\,.\,\mathbf{v}) + z\,(\mathbf{z}\,.\,\mathbf{v})\,]\,\mathbf{v} + \\
& [\,x\,(\mathbf{x}\,.\,\mathbf{n}) + y\,(\mathbf{y}\,.\,\mathbf{n}) + z\,(\mathbf{z}\,.\,\mathbf{n})\,]\,\mathbf{n}
\end{aligned}
$$

Change of Orthonormal Basis

$$\mathbf{p} = [\, x\,(\mathbf{x}\,.\,\mathbf{u}) + y\,(\mathbf{y}\,.\,\mathbf{u}) + z\,(\mathbf{z}\,.\,\mathbf{u})\,]\,\mathbf{u} + [\, x\,(\mathbf{x}\,.\,\mathbf{v}) + y\,(\mathbf{y}\,.\,\mathbf{v}) + z\,(\mathbf{z}\,.\,\mathbf{v})\,]\,\mathbf{v} + [\, x\,(\mathbf{x}\,.\,\mathbf{n}) + y\,(\mathbf{y}\,.\,\mathbf{n}) + z\,(\mathbf{z}\,.\,\mathbf{n})\,]\,\mathbf{n}$$

$$\mathbf{p} = (u,v,n) = u\,\mathbf{u} + v\,\mathbf{v} + n\,\mathbf{n}$$

Expressed in **uvn** basis:

$$u = x\,(\mathbf{x}\,.\,\mathbf{u}) + y\,(\mathbf{y}\,.\,\mathbf{u}) + z\,(\mathbf{z}\,.\,\mathbf{u})$$

$$v = x\,(\mathbf{x}\,.\,\mathbf{v}) + y\,(\mathbf{y}\,.\,\mathbf{v}) + z\,(\mathbf{z}\,.\,\mathbf{v})$$

$$n = x\,(\mathbf{x}\,.\,\mathbf{n}) + y\,(\mathbf{y}\,.\,\mathbf{n}) + z\,(\mathbf{z}\,.\,\mathbf{n})$$

Change of Orthonormal Basis

$$u = x\,(\mathbf{x} \,.\, \mathbf{u}) + y\,(\mathbf{y} \,.\, \mathbf{u}) + z\,(\mathbf{z} \,.\, \mathbf{u})$$
$$v = x\,(\mathbf{x} \,.\, \mathbf{v}) + y\,(\mathbf{y} \,.\, \mathbf{v}) + z\,(\mathbf{z} \,.\, \mathbf{v})$$
$$n = x\,(\mathbf{x} \,.\, \mathbf{n}) + y\,(\mathbf{y} \,.\, \mathbf{n}) + z\,(\mathbf{z} \,.\, \mathbf{n})$$

In matrix form:

$$\begin{bmatrix} u \\ v \\ n \end{bmatrix} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ n_x & n_y & n_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where:

$u_x = \mathbf{x} \,.\, \mathbf{u}$

$u_y = \mathbf{y} \,.\, \mathbf{u}$

etc.

Change of Orthonormal Basis

$$\begin{pmatrix} u \\ v \\ n \end{pmatrix} = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ n_x & n_y & n_z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

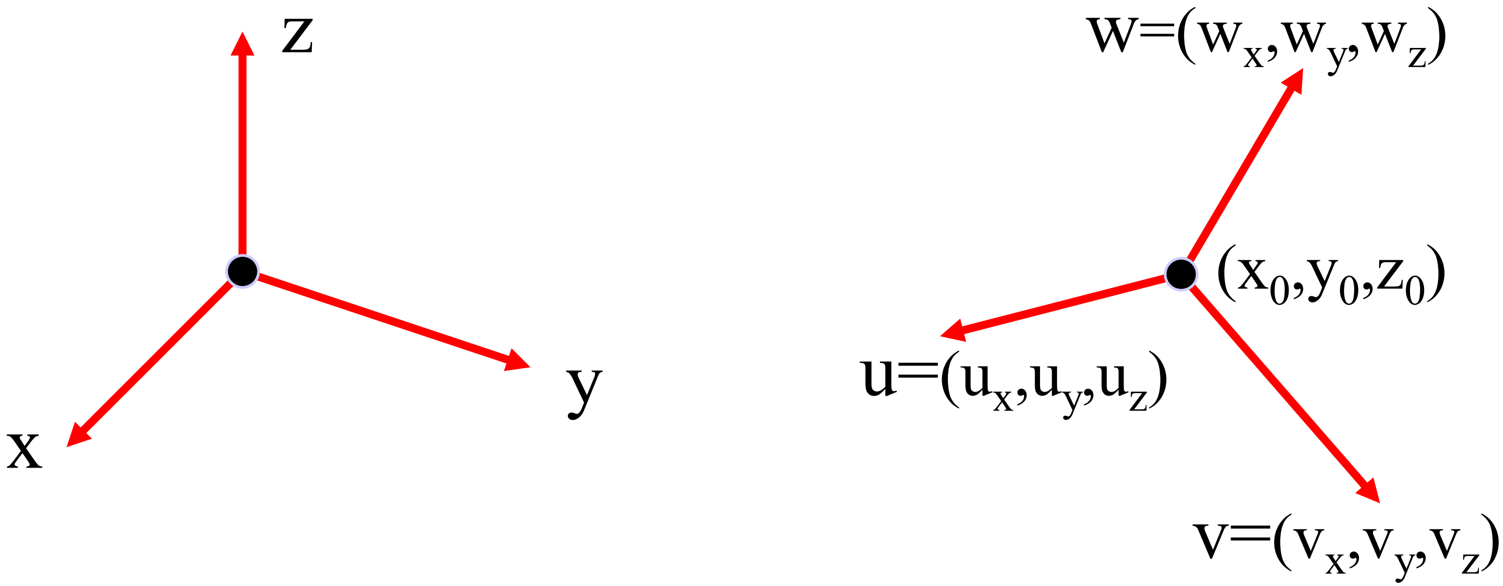
What's $\mathbf{M}^{-1}$, the inverse?

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_u & x_v & x_n \\ y_u & y_v & y_n \\ z_u & z_v & z_n \end{pmatrix} \begin{pmatrix} u \\ v \\ n \end{pmatrix}$$

$$u_x = \mathbf{x} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{x} = x_u$$

$$\mathbf{M}^{-1} = \mathbf{M}^{\mathrm{T}}$$

Changing Coordinate Systems



M is rotation matrix whose columns are U,V, and W:

$$MX = \begin{bmatrix} u_x & v_x & w_x & 0 \\ u_y & v_y & w_y & 0 \\ u_z & v_z & w_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} u_x \\ u_y \\ u_z \\ 1 \end{bmatrix} = U$$

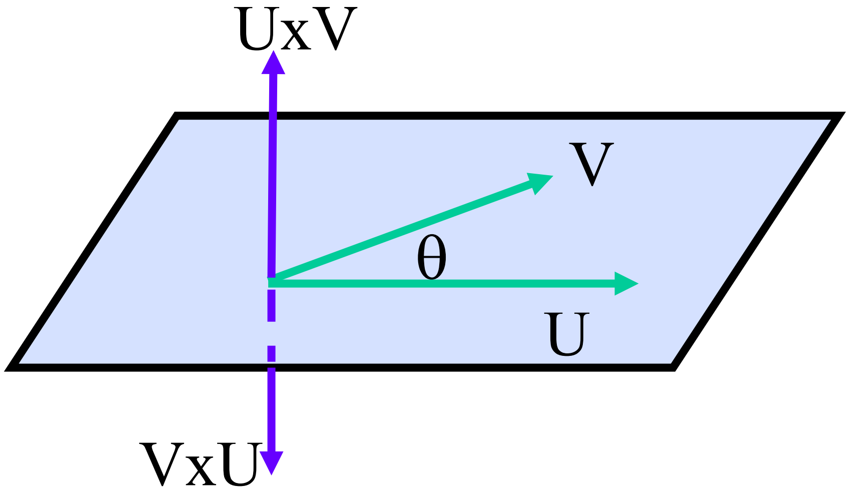
And the inverse…

For the rotation matrix: $R^T = R^{-1}$

$$M^T U = \begin{bmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ w_x & w_y & w_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} u_x^2 + u_y^2 + u_z^2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = X$$

Reminder: Vector Product



$$U \times V = \hat{n}|U||V|\sin\theta$$

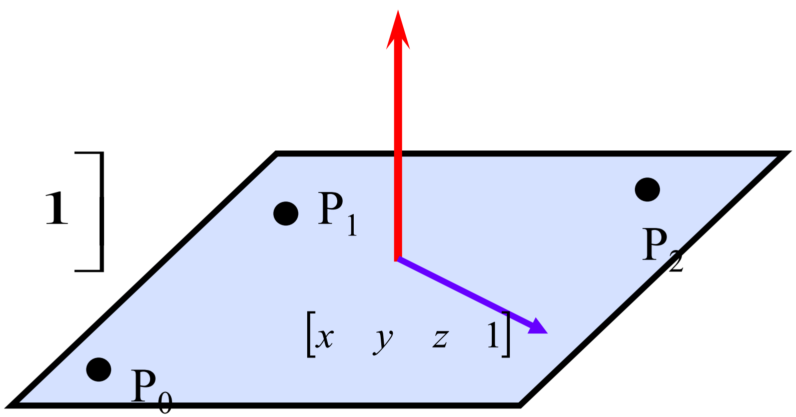
$$U \times V = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \begin{bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{bmatrix}$$

Transforming Planes

- Plane representation:
  - By three non-collinear points
  - By implicit equation:

$$Ax + By + Cz + D = \begin{bmatrix} A & B & C & D \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{A}{D} & \frac{B}{D} & \frac{C}{D} & 1 \end{bmatrix}$$



Transforming Planes

- One way to transform a plane is by transforming any three non-collinear points on the plane.
- Another way is to transform the plane equation: Given a transformation T that transforms $[x,y,z,1]$ to $[x',y',z',1]$ find [A',B',C',D'], such that:

$$\begin{bmatrix} A' & B' & C' & D' \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = 0 \qquad \begin{bmatrix} A & B & C & D \end{bmatrix} T^{-1} T \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

$$A' = AT^{-1} \Rightarrow (A')^T = (AT^{-1})^T \qquad \begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix} = \left(T^{-1}\right)^T \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$