Viewing in 3D

(Chapt. 6 in FVD, Chapt. 12 in Hearn & Baker)
Common Coordinate Systems

• Object space
  – local to each object

• World space
  – common to all objects

• Eye space / Camera space
  – derived from view frustum

• Screen space
  – indexed according to hardware attributes
Specifying the Viewing Coordinates

- Viewing Coordinates system, \([u, v, w]\), describes 3D objects with respect to a viewer.

- A viewing plane (projection plane) is set up perpendicular to \(w\) and aligned with \((u,v)\).

- To set a view plane we have to specify a view-plane normal vector, \(N\), and a view-up vector, \(Up\), (both, in world coordinates):
Reminder: Vector Product

\[ \mathbf{U} \times \mathbf{V} = \hat{n} \| \mathbf{U} \| \| \mathbf{V} \| \sin \theta \]

\[ \mathbf{U} \times \mathbf{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \begin{bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{bmatrix} \]
• $P_0=(x_0, y_0, z_0)$ is a point where a camera is located.
• $P$ is a point to look-at.
• $N=(P_0-P)/|P_0-P|$ is the view-plane normal vector.
• $Up$ is the view up vector, whose projection onto the view-plane is directed up.
• How to form Viewing coordinate system:

\[ w = \frac{N}{|N|} ; \quad u = \frac{Up \times N}{|Up \times N|} ; \quad v = w \times u \]

• The transformation, \( M \), from world-coordinate into viewing-coordinates is:

\[
M = \begin{bmatrix}
u_x & u_y & u_z & 0 \\
v_x & v_y & v_z & 0 \\
w_x & w_y & w_z & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}1 & 0 & 0 & -x_0 \\
0 & 1 & 0 & -y_0 \\
0 & 0 & 1 & -z_0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
How to form Viewing coordinate system

First, normalize the look-at vector to form the $w$-axis

$$w = \frac{N}{\| N \|}$$
Create $U$ perpendicular to $U_p$ and $W$

\[ U = \frac{U_p \times W}{\left| U_p \times W \right|} \]
Create $V$ perpendicular to $U$ and $W$

$$V = W \times U$$
Projections

• Viewing 3D objects on a 2D display requires a mapping from 3D to 2D.

• A projection is formed by the intersection of certain lines (projectors) with the view plane.

• Projectors are lines from the center of projection through each point in the object.
• Center of projection at infinity results with a parallel projection.

• A finite center of projection results with a perspective projection.
Parallel Projection

A parallel projection preserves relative proportions of objects, but does not give realistic appearance (commonly used in engineering drawing).
Perspective Projection

A perspective projection produces realistic appearance, but does not preserve relative proportions.
Parallel Projection

• Projectors are all parallel.

• **Orthographic**: Projectors are perpendicular to the projection plane.

• **Oblique**: Projectors are not necessarily perpendicular to the projection plane.

Orthographic

\[\uparrow \quad \uparrow\]

Oblique

\[\parallel\parallel\]
Orthographic Projection

• Since the viewing plane is aligned with \((x_v, y_v)\), orthographic projection is performed by:

\[
\begin{bmatrix}
  x_p \\
  y_p \\
  0 \\
  1
\end{bmatrix}
= \begin{bmatrix}
  x_v \\
  y_v \\
  0 \\
  1
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 & 0 & 0 & x_v \\
  0 & 1 & 0 & 0 & y_v \\
  0 & 0 & 0 & 0 & z_v \\
  0 & 0 & 0 & 1 & 1
\end{bmatrix}
\]
• Lengths and angles of faces parallel to the viewing planes are preserved.
• **Problem**: 3D nature of projected objects is difficult to deduce.
Oblique Projection

- Projectors are *not* perpendicular to the viewing plane.
- Angles and lengths are preserved for faces parallel to the plane of projection.
- Somewhat preserves 3D nature of an object.
• **Cavalinear projection**:  
  – Preserves lengths of lines perpendicular to the viewing plane.  
  – 3D nature can be captured but shape seems distorted.

• **Cabinet projection**:  
  – lines perpendicular to the viewing plane project at 1/2 of their length.  
  – A more realistic view than the Cavalinear projection.
Perspective Projection

• In a perspective projection, the center of projection is at a finite distance from the viewing plane.
• Parallel lines that are not parallel to the viewing plane, converge to a vanishing point.

A vanishing point is the projection of a point at infinity.
Vanishing Points

• There are infinitely many general vanishing points.

• There can be up to three *axis vanishing points* (principal vanishing points).

• Perspective projections are categorized by the number of principal vanishing points, equal to the number of principal axes intersected by the viewing plane.

• Most commonly used: one-point and two-points perspective.
One point (z axis) perspective projection

Two points perspective projection

x axis vanishing point.

z axis vanishing point.
3-point Perspective

M.C. Escher's "Relativity" where 3 worlds co-exist thanks to 3-point perspective.
3-point Perspective
Perspective Projection

https://vimeo.com/48425421
A perspective projection produces realistic appearance, but does not preserve relative proportions.
Using similar triangles it follows:

\[
\frac{x_p}{d} = \frac{x}{z + d} ; \quad \frac{y_p}{d} = \frac{y}{z + d} ;
\]

\[
\Rightarrow x_p = \frac{d \cdot x}{z + d} ; \quad y_p = \frac{d \cdot y}{z + d} ; \quad z_p = 0
\]
Thus, a perspective projection matrix is defined:

$$M_{per} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{d} & 1 \\
\end{bmatrix}$$

$$M_{per}P = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{d} & 1 \\
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z \\
1 \\
\end{bmatrix} = \begin{bmatrix}
x \\
y \\
0 \\
\frac{z + d}{d} \\
\end{bmatrix}$$

$$x_p = \frac{d \cdot x}{z + d}; \quad y_p = \frac{d \cdot y}{z + d}; \quad z_p = 0$$
Observations

• $M_{\text{per}}$ is singular ($|M_{\text{per}}|=0$), thus $M_{\text{per}}$ is a many to one mapping.

• Points on the viewing plane ($z=0$) do not change.

• The vanishing point of parallel lines directed to $(U_x, U_y, U_z)$ is at $[dU_x/U_z, dU_y/U_z]$.

• When $d \to \infty$, $M_{\text{per}} \to M_{\text{ort}}$. 
Zoom-in

Center of Projection

Projection plane
Zoom-in

Center of Projection

Projection plane
What is the difference between moving the center of projection and moving the projection plane?

**Original**

![Original diagram showing the center of projection and the projection plane.]

**Moving the Center of Projection**

![Diagram showing the effect of moving the center of projection.]

**Moving the Projection Plane**

![Diagram showing the effect of moving the projection plane.]

Original

Moving the Center of Projection

Moving the Projection Plane
Summary

Planar geometric projections

Parallel
- Oblique
  - Cavalinear
  - Cabinet
- Cabinet
- Other

Orthographic
- Top
- Front
- Side
- Other

Perspective
- One point
- Two point
- Three point
Another view in Perspective
Leonardo da Vinci 1498–1495
Another view in Perspective
Two-point perspective

Three-point perspective
Three-point perspective

Four-point perspective
Five-point perspective?

Six-point perspective?
Fisheye views of the Hagia Sophia (Istanbul) (also known as Aya Sofya)
Fisheye view
Vertical lines
House of Stairs

M.C. Escher's House of Stairs
Shepard’s Table Illusion:
Both tabletops are the same size and shape
https://www.youtube.com/watch?v=PhRbsaSMMVU&t=29s
View Window

- After objects were projected onto the viewing plane, an image is taken from a View Window.
- A view window can be placed anywhere on the view plane.
- In general the view window is aligned with the viewing coordinates and is defined by its extreme points: \((x_{w\text{min}}, y_{w\text{min}})\) and \((x_{w\text{max}}, y_{w\text{max}})\).
View Volume

- Given the specification of the view window, we can set up a View Volume.
- Only objects inside the view volume might appear in the display, the rest are clipped.
• In order to limit the infinite view volume we define two additional planes: *Near Plane* and *Far Plane*.

• Only objects in the bounded view volume can appear.

• The near and far planes are parallel to the view plane and specified by $z_{\text{near}}$ and $z_{\text{far}}$.

• A limited view volume is defined:
  – For orthographic: a rectangular parallelepiped.
  – For oblique: an oblique parallelepiped.
  – For perspective: a frustum.
Canonical View Volumes

• In order to determine the objects that are seen in the view window we have to clip objects against six planes forming the view volume.

• Clipping against arbitrary 3D plane requires considerable computations.

• For fast clipping we transform the general view volume to a **canonical view volume** against which clipping is easy to apply.

```
Canonical view
Transformation

Clipping

Projection
Transformation

Viewing Coordinates
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Trivial Accept/Reject

Red polygons are rejected
Black are accepted
And
Blue are tagged for clipping
Classify the vertices of the polygons against each side $S_i$ of the frustum in turn:

If all the vertices are on the outside of some $S_i$, then cull the polygon, otherwise,

Polygons with all its vertices inside all $S_j$ are accepted.

All others are tagged.
Affine Transformation:

$$Ax + By + C = 0$$

$x = y$
OpenGL Transformation Pipe-Line

Homogeneous coordinates in World System

ModelView Matrix

→ Viewing Coordinates

Projection Matrix

→ Clip Coordinates

Clipping

Viewport Transformation

→ Window Coordinates