Counting upper triangular symplectic matrices

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For any field F, define

$$Sp_{2m}(F) = \{g \in GL_{2m}(F) \mid g^t w_{2m} g = w_{2m}\},$$

where

$$J_m = \left.\begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}\right\} m,$$

$$w_{2m} = \begin{pmatrix} & J_m \\ -J_m & \end{pmatrix}$$

For q an odd prime power, $c \in \mathbb{F}_q$, ℓ and k non-negative integers, and m a positive integer, define $\alpha^c_{2m,2\ell}(k)$ to be the number of upper triangular unipotent symplectic matrices $u \in Sp_{2m}(\mathbb{F}_q)$ such that $rank(u-1) = k$, u has $I_{2\ell}$ for its middle block:

$$u = \begin{pmatrix} u' & M & Z \\ & I_{2\ell} & M' \\ & & u'^{*} \end{pmatrix} \in Sp_{2m}(\mathbb{F}_q)$$

and

$$u_{1,2} + u_{2,3} + ... + u_{m-\ell,m-\ell+1} = c.$$

For convenience we call this last sum $Tr^*(u)$. It is not the usual trace, it is not preserved under conjugation of the whole group, and it is not standard notation. But it is kinda like trace.

Theorem 1. *$\alpha^c_{2m,2\ell}(k)$ is a polynomial in q, and satisfies:*

1. *If* $cc' \neq 0$, *then*
$$\alpha^{c}_{2m,2\ell}(k) = \alpha^{c'}_{2m,2\ell}(k)$$
2. *If* $2\ell \leq 2m$ *then*
$$\alpha^{0}_{2m,2\ell}(0) = 1,\ \alpha^{1}_{2m,2\ell}(0) = 0$$
3. *If* $2\ell < 2m$ *then*
$$deg\ \alpha^{0}_{2m,2\ell}(1) = m - \ell,\ \alpha^{1}_{2m,2\ell}(1) = 0$$
4. *If* $2 \leq 2k \leq 2(m-\ell)$ *then*
$$deg\ \alpha^{0}_{2m,2\ell}(2k) = deg\ \alpha^{1}_{2m,2\ell}(2k) = 2km - k^2 - 1$$
5. *If* $1 < 2k+1 < 2(m-\ell)$ *then*
$$deg\ \alpha^{0}_{2m,2\ell}(2k+1) = deg\ \alpha^{1}_{2m,2\ell}(2k+1) = 2km + (m-\ell) - k(k+1) - 1$$

Proof. (2) and the second equation of (3) are clear. We prove the rest by induction. If u is an upper triangular unipotent symplectic matrix with middle identity block, then it has the following form:

$$u = \begin{pmatrix} 1 & h & z \\ & u' & h' \\ & & 1 \end{pmatrix}$$

where h' is determined by h, u', and u' is an upper triangular unipotent symplectic matrix with the same middle identity block. Thus, to count the u's such that $rank(u-1) = k$, we count those u' such that $rank(u'-1) \in \{k, k-, k-2\}$, and the possible h, z that complement the rank and Tr^*.

First, say $rank(u'-1) = k$. For u to satisfy $rank(u-1) = k$, h must be in the row space of $u'-1$, and z is determined by h since it must be the same combination of h'-rows. But being such a combination implies that $h_1 = 0$, so $Tr^*(u) = Tr^*(u')$. The number of h's in the row space of $u'-1$, given that $rank(u'-1) = k$, is q^k. Thus the contribution of such u's to $\alpha^{c}_{2m,2\ell}(k)$ is
$$\alpha^{c}_{2(m-1),2\ell}(k)q^k.$$

Second, say $rank(u'-1) = k-1$. Again, h must be in the row space of $u'-1$, but this time z must not be the same combination of h'-rows. Hence the number of such h's is q^{k-1}, and for each h we have $q-1$ choices of z, so that $rank(u-1)=k$. As in the first case, $Tr^*(u)=Tr^*(u')$. Hence the contribution of such u's to $\alpha^c_{2m,2\ell}(k)$ is

$$\alpha^c_{2(m-1),2\ell}(k-1)q^{k-1}(q-1).$$

Third, $rank(u'-1)=k-2$. This time h must be outside of the row space of $u'-1$, because then $u-1$ gains rank both from h on the top, and from h' on the right. To satisfy $Tr^*(u)=c$, h must satisfy $h_1=c-Tr^*(u')$. If $Tr^*(u')\neq c$, then for any h with $h_1=c-Tr^*(u')\neq 0$, u will have rank k. Thus these (u',h,z)'s contribute

$$\sum_{\substack{c'\in\mathbb{F}_q\\ c'\neq c}} \alpha^{c'}_{2(m-1),2\ell}(k-2)q^{2(m-1)}.$$

Finally, we must consider those u' with $Tr^*(u')=c$, so $h_1=0$. For u to have rank $k=rank(u')+2$, h must be outside the row space of u', which has size q^{k-2}, and z is arbitrary. Hence the contribution of such (u',h,z)'s is

$$\alpha^c_{2(m-1),2\ell}(k-2)(q^{2m-3}-q^{k-2})q.$$

Putting these cases together, we get the following formula:

$$\begin{aligned}\alpha^c_{2m,2\ell}(k) &= \alpha^c_{2(m-1),2\ell}(k)q^k + \alpha^c_{2(m-1),2\ell}(k-1)q^{k-1}(q-1)+\\ &+\sum_{\substack{c'\in\mathbb{F}_q\\ c'\neq c}} \alpha^{c'}_{2(m-1),2\ell}(k-2)q^{2(m-1)} + \alpha^c_{2(m-1),2\ell}(k-2)(q^{2m-3}-q^{k-2})q.\end{aligned}$$

At this point we define a matrix that we will use now, and also later. For $\varepsilon\in\mathbb{F}_q^\times$, let

$$\mathcal{E}_{2m,2\ell}(\varepsilon)=\begin{pmatrix} \varepsilon^{m-l} &&&&&& \\ & \ddots &&&&& \\ && \varepsilon &&&& \\ &&& I_{2\ell} &&& \\ &&&& \varepsilon^{-1} && \\ &&&&& \ddots & \\ &&&&&& \varepsilon^{-(m-\ell)} \end{pmatrix} \in Sp_{2m}(\mathbb{F}_q) \qquad (2)$$

A short computation shows that conjugation by $\mathcal{E}_{2m,2\ell}(\varepsilon)$ preserves upper triangular matrices, and multiplies the first $m-\ell$ entries above the diagonal by ε. This shows that if $cc' \neq 0$, then

$$\alpha^c_{2m,2\ell}(k) = \alpha^{c'}_{2m,2\ell}(k).$$

This simplifies the above formula to:

$$\begin{aligned}\alpha^c_{2m,2\ell}(k) = \alpha^c_{2(m-1),2\ell}(k)q^k + \alpha^c_{2(m-1),2\ell}(k-1)q^{k-1}(q-1)+ \\ + (\alpha^0_{2(m-1),2\ell}(k-2) + (q-1)\alpha^1_{2(m-1),2\ell}(k-2))q^{2(m-1)} \\ - \alpha^c_{2(m-1),2\ell}(k-2)q^{k-1}.\end{aligned}$$

Putting $c=0$, $k=1$, we get $\alpha^0_{2m,2\ell}(1) = \alpha^0_{2(m-1),2\ell}(1)q + (q-1)$. Using induction and that $\alpha^0_{2\ell,2\ell}(1) = 0$, we get the first part of (2).

To prove (4) and (5) we must start with the base cases, since they are different from the results of (2) and (3). Let us start with $k=2, m=\ell+1$. The above recursion formula gives

$$\alpha^c_{2(\ell+1),2\ell}(2) = \alpha^0_{2\ell,2\ell}(0)q^{2\ell} - \alpha^c_{2\ell,2\ell}(0)q = q^{2\ell} - \alpha^c_{2\ell,2\ell}(0)q$$

which has degree $2\ell = 2(\ell+1) - 1^2 - 1$ whether $c=0$ or not. This agrees with (3). Now let $k=3, m=\ell+2$. Using the recursion we get

$$\alpha^c_{2(\ell+2),2\ell}(3) = \alpha^c_{2(\ell+1),2\ell}(2)q^2(q-1) + \alpha^0_{2(\ell+1),2\ell}(1)q^{2(\ell+1)} - \alpha^c_{2(\ell+1),2\ell}(1)q^2$$

which has degree $2\ell+3 = 2(\ell+2) + (2) - 1\cdot 2 - 1$, and agrees with (4).

We can now apply induction on m. Assume (4) and (5) are true for all $m' < m$. We use the recursion formula for the even case:

$$\begin{aligned}deg(\alpha^c_{2m,2\ell}(2k)) = max\big\{ & deg(\alpha^c_{2(m-1),2\ell}(2k)) + k, \\ & deg(\alpha^c_{2(m-1),2\ell}(2k-1)) + k, \\ & deg(\alpha^0_{2(m-1),2\ell}(2k-2) + 2(m-1)), \\ & deg(\alpha^1_{2(m-1),2\ell}(2k-2) + 2m-1), \\ & deg(\alpha^c_{2(m-1),2\ell}(2k-2) + k)\big\}\end{aligned}$$

Applying the induction hypothesis:

$$
\begin{aligned}
= max\big\{&2k(m-1) - k^2 - 1 + k,\\
&2(k-1)(m-1) + (m-l) - (k-1)k - 1 + k,\\
&2(k-1)(m-1) - (k-1)^2 - 1 + 2(m-1),\\
&2(k-1)(m-1) - (k-1)^2 - 1 + 2m - 1,\\
&2(k-1)(m-1) - (k-1)^2 - 1 + k\big\}
\end{aligned}
$$

Which can be rewritten as:

$$
\begin{aligned}
= max\{&2km - k^2 - 1 - (k),\\
&2km - k^2 - 1 - (m + l - 3),\\
&2km - k^2 - 1 - (2),\\
&2km - k^2 - 1,\\
&2km - k^2 - 1 - (2m - k)\}\\
= 2&km - k^2 - 1
\end{aligned}
$$

The odd case is done similarly. □