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Monovex sets

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Abstract. A set A in a finite-dimensional Euclidean space is *monovex* if for any $x, y \in A$ there is a continuous path within A that connects x and y and is monotone (nonincreasing or nondecreasing) in each coordinate. We prove that every open monovex set and every closed monovex set are contractible, and we provide an example of a nonopen and nonclosed monovex set that is not contractible. Our proofs reveal additional properties of monovex sets.

1. Introduction. A set A in a finite-dimensional Euclidean space is *monovex* if for any $x, y \in A$ there is a continuous path within A that connects x and y and is monotone (nonincreasing or nondecreasing) in each coordinate. In particular, whether or not a set is monovex depends on the choice of basis for the space.

Monovex sets arise in the study of stochastic games [S] where an extension of the Kakutani's fixed-point theorem [K] for set-valued functions with a closed graph and nonempty monovex values is needed (1). By the Eilenberg–Montgomery fixed-point theorem [EM] any set-valued functions from a convex compact subset of $\mathbb{R}^n$ to itself with a closed graph and nonempty contractible values has a fixed point. Consequently, our goal is to study contractibility of monovex sets.

In this paper we prove that every open monovex set, as well as every closed monovex set, is contractible. We also provide an example of a nonopen and nonclosed monovex set that is not contractible.

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(1) Kakutani's fixed-point theorem states that any set-valued function F from a convex and compact subset of $\mathbb{R}^n$ to itself with a closed graph and nonempty convex values has a fixed point.

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2. Definition and main results

DEFINITION 2.1. A set $A \subseteq \mathbb{R}^n$ is *monovex* if for any $x, y \in A$ there is a continuous path $\gamma : [0,1] \to A$ that has the following properties:

(M1) $\gamma(0) = x$ and $\gamma(1) = y$.
(M2) $\gamma_i : [0,1] \to \mathbb{R}$ is a monotone function (nondecreasing or nonincreasing) for every $i \in \{1, \dots, n\}$.

A path γ that satisfies (M2) is called *monotone*.

The image of a monovex set under a diagonal affine transformation is monovex, yet a rotation of a monovex set need not be monovex. Every convex set is monovex. If A is a monovex set, then so is the projection of A onto any "coordinate subspace", that is, a subspace spanned by a collection of elements of the standard basis of $\mathbb{R}^n$. Every monovex subset of $\mathbb{R}$ is convex, yet there are monovex subsets of $\mathbb{R}^2$ that are not convex (see Figure 1).

Fig. 1. A monovex set (left) and a nonmonovex set (right) in the plane

As the following example shows, monovex sets may be complex objects. In particular, they need not be CW-complexes.

EXAMPLE 2.2. Let $A \subset [0,1]^2$ be the following set (see Figure 2):

$$A = \{(0,0)\} \cup \bigcup_{k=0}^{\infty} \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right]^2.$$

It is evident that A is monovex, yet it is not a CW-complex.

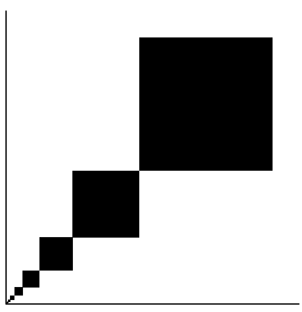

Fig. 2. The monovex set A of Example 2.2

The Minkowski sum of two convex sets is a convex set. This property is not shared by monovex sets. In fact, as the following example shows, the Minkowski sum of a monovex set and a convex set need not be monovex. In Lemma 3.4 below we will prove that the Minkowski sum of a monovex set in $\mathbb{R}^n$ and an n-dimensional box whose faces are parallel to the axes is monovex.

EXAMPLE 2.3. Let A be the union of the line segments $[(0,0,0),(0,1,1)]$ and $[(0,1,1),(1,1,2)]$, which is monovex. Let B be the line segment $[(0,0,0),(-1,-1,2)]$. The intersection of $A+B := \{a+b\colon a \in A,\ b \in B\}$ and the line $L := \{x \in \mathbb{R}^3\colon x_1 = x_2 = 0\}$ is the two points $(0,0,0)$ and $(0,0,4)$. Indeed, all points $b \in B$ satisfy $b_1 = b_2$, while the only points $a \in A$ that satisfy $a_1 = a_2$ are $(0,0,0)$ and $(1,1,2)$. Hence $a+b \in A+B$ is on L if and only if $a = b = (0,0,0)$, or $a = (1,1,2)$ and $b = (-1,-1,2)$.

Since $(A+B) \cap L$ contains two points, there is no monotone path that connects these points and lies in $A+B$, and therefore $A+B$ is not monovex.

The Minkowski sum of the sets in Example 2.3 is contractible. As the following example shows, the Minkowski sum of a monovex set and a convex set can be homotopy equivalent to the circle S^1.

EXAMPLE 2.4. Let A be the union of the line segments $[(0,0,0),(1,0,0)]$, $[(1,0,0),(1,1,0)]$, and $[(1,1,0),(1,1,1)]$, which is monovex. Let $B = \{(x,x,x)\colon x \in \mathbb{R}\}$, which is convex. Denote by C the triangle in $\mathbb{R}^3$ with vertices $(0,0,0)$, $(2/3,-1/3,-1/3)$, and $(1/3,1/3,-2/3)$. The Minkowski sum of A and B is $A+B = C+B$, which is homotopy equivalent to the circle.

As mentioned in the introduction, our goal is to study whether monovexity implies contractibility. It is a little technical but not difficult to show that every monovex subset of $\mathbb{R}^2$ is contractible. As the following example shows, not every three-dimensional monovex set is contractible.

EXAMPLE 2.5. Let $A \subset [-1,1]^3$ be the set of all points that have at least one negative coordinate and at least one nonnegative coordinate. The reader can verify that A is monovex. It is disjoint from the line $\{(x,x,x)\colon x \in \mathbb{R}\}$, and it contains the loop γ depicted in Figure 3 and is not contractible in $\mathbb{R}^3 \setminus \{(x,x,x)\colon x \in \mathbb{R}\}$. In particular, A is not contractible. In fact, one can show that it is homotopy equivalent to the circle S^1.

THEOREM 2.6. *Every open monovex subset of $\mathbb{R}^n$ is contractible.*

Proof. The proof is by induction on n. For $n = 1$, an open monovex set is an open interval, hence contractible. Assume now that A is an open monovex subset of $\mathbb{R}^n$ with $n > 1$. Let B be the projection of A onto its first $n-1$ coordinates, and let $F : B \rightrightarrows \mathbb{R}$ be the set-valued function whose

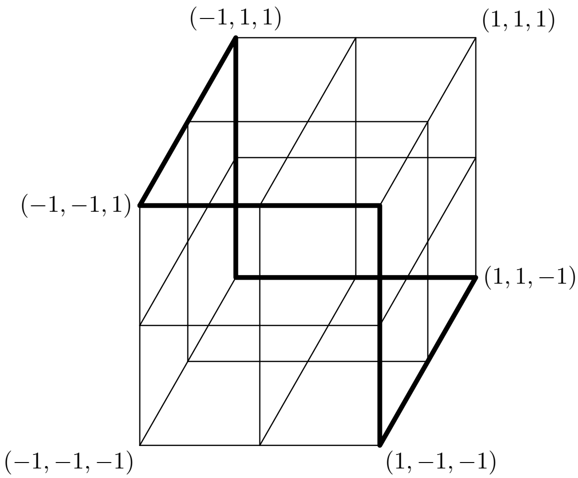

Fig. 3. The path γ in Example 2.5 (dark)

graph is A, that is,

$$F(x) := \{t \in \mathbb{R}\colon (x,t) \in A\}, \qquad \forall x \in B.$$

Note that $F(x)$ is an open interval for every $x \in B$. The set B is open and monovex, so by the induction hypothesis it is contractible. The set-valued function F satisfies the conditions of Michael's selection theorem [M, Theorem 3.1‴] ([2]), hence there is a continuous function $f : B \to \mathbb{R}$ such that $f(x) \in F(x)$ for every $x \in B$. This implies that A is contractible; indeed first contract A to graph(f), and then contract graph(f) to a point. ∎

The main result of the paper is the following.

Theorem 2.7. *Every closed monovex subset of $\mathbb{R}^n$ is contractible.*

We provide two proofs to Theorem 2.7, each using different properties of monovex sets, which may be of independent interest. The first proof, provided in Section 3.1, relies on the property that one can assign, in a continuous way, to every pair of points in a monovex set a (not necessarily monotone) path that connects these points and lies in the set. The second proof, provided in Section 3.2, relies on the stronger property that the complement of a monovex set can be continuously projected onto the set. In the first proof we will provide a direct argument that shows the existence of a continuous map from pairs of points in the monovex set to paths that connect the points and lie in the set.

([2]) Michael's selection theorem implies in particular that for every subset $X \subseteq \mathbb{R}^n$ and every set-valued function $F : X \rightrightarrows \mathbb{R}^m$ with an open graph and nonempty convex values there exists a continuous function $f : X \to \mathbb{R}^n$ such that $f(x) \in F(x)$ for every $x \in X$.

Several open problems regarding contractibility of monovex sets still remain. We prove that every closed monovex set is contractible. We do not know whether for every such set there is a Lipschitz continuous contraction. Another issue is whether our results hold for infinite-dimensional spaces.

3. Proof of Theorem 2.7. Throughout the paper we use the maximum metric in $\mathbb{R}^n$, that is, $d_\infty(x,y) := \max_{1\le i\le n}|x_i-y_i|$ for all $x,y\in\mathbb{R}^n$. The distance between a point $x\in\mathbb{R}^n$ and a set $A\in\mathbb{R}^n$ is $d_\infty(x,A) := \inf_{y\in A} d_\infty(x,y)$, and the distance between two sets $A,B\subset\mathbb{R}^n$ is the Hausdorff distance

$$d_\infty(A,B) := \max\Big\{\sup_{x\in A} d_\infty(x,B),\ \sup_{y\in B} d_\infty(y,A)\Big\}.$$

For every $x\in\mathbb{R}^n$ and $r>0$ we write $B(x,r) := \{y\in\mathbb{R}^n : d_\infty(x,y)<r\}$ and $\overline{B}(x,r) := \{y\in\mathbb{R}^n : d_\infty(x,y)\le r\}$. We denote by $\vec{0}$ the vector $(0,\dots,0)$ in $\mathbb{R}^n$.

A (closed) *box* in $\mathbb{R}^n$ is a set of the form $\times_{i=1}^n[a_i,b_i]$, where $a_i\le b_i$ for each $i\in\{1,\dots,n\}$. A box is *l-dimensional* if the number of indices i such that $a_i<b_i$ is l. The set of vertices of a box R is denoted $\mathrm{vert}(R)$. The smallest box that contains a set A is called the *b-hull* of A and denoted $\text{b-hull}(A)$.

A *b-lattice* is a set of the form $\Gamma = \{(a_1k_1,\dots,a_nk_n) : k_1,\dots,k_n\in\mathbb{Z}\}$, where $a_1,\dots,a_n>0$. Denote by $P_l(\Gamma)$ the set of l-dimensional elementary boxes having vertices in Γ, that is, the collection of all sets $\times_{i=1}^n J_i$ such that for each i either $J_i=\{a_ik_i\}$ or $J_i=[a_ik_i, a_i(k_i+1)]$ for some $k_i\in\mathbb{Z}$, and moreover the second condition holds for exactly l values of i. Denote by $P(\Gamma) := P_n(\Gamma)$ the set of full-dimensional elementary boxes with vertices in Γ.

3.1. First proof. The following proposition states that any function f from an m-dimensional grid to a monovex set A can be extended to a continuous function (still denoted by f) from the m-dimensional space to A with the property that the image under f of any elementary l-dimensional box whose vertices are points in the grid is a subset of the b-hull of the image under f of the vertices of the box.

Proposition 3.1. *Let $A\subset\mathbb{R}^n$ be a closed monovex set and let $\Gamma\subset\mathbb{R}^m$ be a b-lattice. Let $X\subset\mathbb{R}^m$ be a (finite or infinite) union of boxes in $P(\Gamma)$, and let $S := X\cap\Gamma$ be the set of their vertices. Let $f: S\to A$. Then f can be extended to a continuous function $f: X\to A$ with the following property:*

(P) *For every $1\le l\le m$ and $R\in P_l(\Gamma)$ with $R\subset X$, the image $f(R)$ is a subset of the b-hull of $f(\mathrm{vert}(R))$.*

Proof. Assume without loss of generality that $\Gamma=\mathbb{Z}^m$. We will define the extension f iteratively on the sets $X\cap((2^{-k-1}\Gamma)\setminus(2^{-k}\Gamma))$, $k=0,1,\dots,$ and show that f can be further extended to a continuous function over X.

For every $r \in \mathbb{N}$ and $1 \le i \le n$, define $\varphi_{r,i} : A^r \to A$ as follows. Let $(q^{(1)}, \dots, q^{(r)}) \in A^r$, let $j_{\min}$ be an index at which $\min_{1\le j\le r} q_i^{(j)}$ is attained, and let $j_{\max}$ be an index at which $\max_{1\le j\le r} q_i^{(j)}$ is attained. Choose a continuous monotone curve $\gamma : [0,1] \to A$ connecting $q^{(j_{\min})}$ and $q^{(j_{\max})}$ (if $j_{\min} = j_{\max}$, the curve is constant). By continuity there exists $t_0 \in [0,1]$ such that $\gamma_i(t_0) = (q_i^{(j_{\min})} + q_i^{(j_{\max})})/2$. Set

$$\varphi_{r,i}(q^{(1)}, \dots, q^{(r)}) := \gamma(t_0).$$

We now extend f from $X \cap 2^{-k}\Gamma$ to $X \cap 2^{-k-1}\Gamma$, for $k = 0, 1, \dots$. Suppose that $f : X \cap 2^{-k}\Gamma \to A$ is given, and set $i := k+1 \pmod n$. Every $q \in X \cap ((2^{-k-1}\Gamma) \setminus (2^{-k}\Gamma))$ is the center of a unique l-dimensional box $R \in P_l(2^{-k}\Gamma)$ contained in X (where $1 \le l \le n$). Define

$$f(q) := \varphi_{2^l,i}(q^{(1)}, \dots, q^{(2^l)}),$$

where $q^{(1)}, \dots, q^{(2^l)}$ are the f-images of the vertices of R.

We have extended f to a function $f : X \cap \bigcup_{k=0}^{\infty} \frac{1}{2^k}\Gamma \to A$. Note that a property reminiscent of (P) is satisfied: for any integers $k \ge 0$ and $1 \le l \le m$, and for any box $R \in P_l(2^{-k}\Gamma)$ that is a subset of X, we have $f\big(R \cap \bigcup_{k=0}^{\infty} \frac{1}{2^k}\Gamma\big) \subseteq \text{b-hull}(f(\text{vert}(R)))$. We moreover claim that the function f is locally uniformly continuous ([3]), and in fact, locally $1/n$-Hölder. Indeed, for $k \in \mathbb{N}$ and a box $R \in P(2^{-k}\Gamma)$ denote $M_i(R) := \max\{|f_i(q^{(j)}) - f_i(q^{(m)})| : q^{(j)}, q^{(m)} \in \text{vert}(R)\}$. Let $N_i(R) := \max\{M_i(S)\colon S \in P(2^{-k-1}\Gamma),\ S \subset R\}$. If $i = k+1 \pmod n$ then $N_i(R) \le M_i(R)/2$, while if $i \ne k+1 \pmod n$ then $N_i(R) \le M_i(R)$. Since $i = k+1 \pmod n$ infinitely often with step n as k increases, f is indeed locally $1/n$-Hölder continuous.

The set $X \cap \bigcup_{k=0}^{\infty} 2^{-k}\Gamma$, where f is now defined, is dense in X, hence f can be extended by continuity to a continuous function from X to A. The extended function f is locally $1/n$-Hölder as well, and it satisfies (P). ∎

We would like to prove that there is a continuous function $f : A \times A \times [0,1] \to A$ that satisfies $f(x,y,0) = x$ and $f(x,y,1) = y$ for every $x, y \in A$. In the next lemma we prove an approximate version of this result. We will use it in Proposition 3.3 below to prove a stronger version.

Lemma 3.2. *Let $A \subset \mathbb{R}^n$ be a closed monovex set. For every $\delta > 0$ there exists a continuous function $g_\delta : A \times A \times [0,1] \to A$ such that for every $x, y \in A$ we have:*

(1) $d_\infty(x, g_\delta(x,y,0)) \le \delta$ *and* $d_\infty(y, g_\delta(x,y,1)) \le \delta$.
(2) $d_\infty(g_\delta(x,y,t), \text{b-hull}(\{x,y\})) \le \delta$ *for every* $t \in [0,1]$.

([3]) A function is *locally uniformly continuous* if it is uniformly continuous on every bounded subset.

Proof. Fix $\delta > 0$. Consider the lattice $\Gamma := (\delta/2)\mathbb{Z}^n$, and denote by X the union of all boxes $R \in P(\Gamma)$ that satisfy $R \cap A \neq \emptyset$. Denote $\widetilde{X} := X \times X \times [0,1] \subseteq \mathbb{R}^{2n+1}$, $\widetilde{\Gamma} := \Gamma \times \Gamma \times \mathbb{Z}$, and $\widetilde{S} := \widetilde{X} \cap \widetilde{\Gamma}$.

Let $f : \widetilde{S} \to A$ be any function with the following property: for all $x, y \in X \cap \Gamma$ we have $d_\infty(x, f(x,y,0)) \le \delta/2$ and $d_\infty(y, f(x,y,1)) \le \delta/2$. Such a function exists since every $x \in X \cap \Gamma$ is a vertex of a box $R \in P(\Gamma)$ whose sidelength is $\delta/2$ with $R \cap A \neq \emptyset$.

By Proposition 3.1, the function f can be extended to a continuous function $f : \widetilde{X} \to A$ that satisfies (P). In particular, for any two boxes $Q, R \in P(\Gamma)$ lying in X, we have:

- $f(Q \times R \times \{0\})$ is contained in the b-hull of $f(\mathrm{vert}(Q) \times \mathrm{vert}(R) \times \{0\})$;
- $f(Q \times R \times \{1\})$ is contained in the b-hull of $f(\mathrm{vert}(Q) \times \mathrm{vert}(R) \times \{1\})$.

Moreover, for all $x \in Q$, $y \in R$, $q \in \mathrm{vert}(Q)$, and $r \in \mathrm{vert}(R)$ we have $d_\infty(x,q) \le \delta/2$, $d_\infty(y,r) \le \delta/2$, $d_\infty(q, f(q,r,0)) \le \delta/2$, and $d_\infty(r, f(q,r,1)) \le \delta/2$. By the triangle inequality it follows that $d_\infty(x, f(q,r,0)) \le \delta$ and $d_\infty(y, f(q,r,1)) \le \delta$. We conclude that given $x \in Q$ and $y \in R$, for every $q \in \mathrm{vert}(Q)$ and $r \in \mathrm{vert}(R)$ we have $d_\infty(x, f(q,r,0)) \le \delta$ and $d_\infty(y, f(q,r,1)) \le \delta$. Therefore, since $f(x,y,0) \in \text{b-hull}(f(\mathrm{vert}(Q) \times \mathrm{vert}(R) \times \{0\}))$ and $f(x,y,1) \in \text{b-hull}(f(\mathrm{vert}(Q) \times \mathrm{vert}(R) \times \{1\}))$, we have $d_\infty(x, f(x,y,0)) \le \delta$ and $d_\infty(y, f(x,y,1)) \le \delta$.

In addition, since f satisfies (P), the image $f(Q \times R \times [0,1])$ is contained in the b-hull of $f(\mathrm{vert}(Q) \times \mathrm{vert}(R) \times \{0,1\})$, and hence we also conclude that $d_\infty(f(x,y,t), \text{b-hull}(\{x,y\})) \le \delta$ for every $t \in [0,1]$.

To summarize, for all $x, y \in X$ and $t \in [0,1]$ we have:

- $d_\infty(x, f(x,y,0)) \le \delta$,
- $d_\infty(y, f(x,y,1)) \le \delta$, and
- $d_\infty(f(x,y,t), \text{b-hull}(\{x,y\})) \le \delta$.

To end the proof, define g_δ to be the restriction of f to $A \times A \times [0,1]$. ∎

Proposition 3.3. *There exists a continuous function*

$$\varphi : A \times A \times [0,1] \to A$$

such that $\varphi(x,y,0) = x$ *and* $\varphi(x,y,1) = y$ *for all* $x, y \in A$.

We note that Proposition 3.3 implies Theorem 2.7. Indeed, choose any $x_0 \in A$. The function $G : A \times [0,1] \to A$ defined by $G(x,t) := \varphi(x, x_0, t)$ for all $x \in A$ and $t \in [0,1]$ is a homotopy between A and $\{x_0\}$.

Proof of Proposition 3.3. Let $(\delta_k)_{k=1}^\infty$ be a sequence of positive reals such that $\sum_{k=1}^\infty \delta_k < \infty$. We define the function φ in several steps by a Cantor set construction. Define $C_0 := \{0,1\}$, $C_1 := \{[1/3, 2/3]\}$, and for every $k \ge 2$

let C_k be the collection of all closed intervals $[s, s']$ where

$$s = \frac{1}{3^k} + \sum_{j=1}^{k-1} \frac{\alpha_j}{3^j} \quad \text{and} \quad s' = \frac{2}{3^k} + \sum_{j=1}^{k-1} \frac{\alpha_j}{3^j},$$

for some $\alpha_j \in \{0, 2\}$, $j = 1, \ldots, k-1$ (see Figure 4).

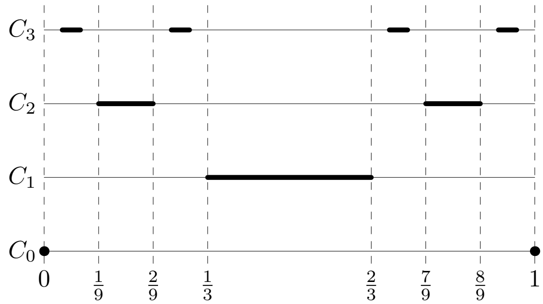

Fig. 4. The sets C_k for $k = 0, 1, 2, 3$

For $k = 0, 1, \ldots,$ in step k we define φ on $A \times A \times \bigcup_{[s,s'] \in C_k} [s, s']$. For $k = 0$ set

$$\varphi(x, y, 0) := x, \quad \varphi(x, y, 1) := y, \quad \forall x, y \in A.$$

For $k \geq 1$, consider an interval $[s, s'] \in C_k$ and set $t := s - 1/3^k$ and $t' := s' + 1/3^k$. Each of the points t and t' is an endpoint of an interval in C_j for some $j < k$. Hence $\varphi(\cdot, \cdot, t)$ and $\varphi(\cdot, \cdot, t')$ were already defined. Set

$$\varphi(x, y, (1-\lambda)s + \lambda s') := g_{\delta_k}(\varphi(x, y, t), \varphi(x, y, t'), \lambda), \quad \forall x, y \in A, \forall \lambda \in [0, 1],$$

where g_{δ_k} satisfies the statement of Lemma 3.2. The above procedure defines φ on $A \times A \times \bigcup_{k=0}^{\infty} \bigcup_{[s,s'] \in C_k} [s, s']$. The latter union is dense in $[0, 1]$, hence φ is defined in a dense subset of $A \times A \times [0, 1]$. Since $\sum_{k=1}^{\infty} \delta_k < \infty$, the function φ is in fact locally uniformly continuous, hence it can be extended to a continuous function $\varphi : A \times A \times [0, 1] \to A$, as desired. ∎

3.2. Second proof. We first argue that if A is a monovex set and R is an open box whose faces are parallel to the axes, then $A + R$ is monovex. We note that the proof is valid also when the box R is closed.

LEMMA 3.4. *If $A \subset \mathbb{R}^n$ is monovex and $R \subset \mathbb{R}^n$ is an open box whose faces are parallel to the axes, then $A + R$ is monovex.*

Proof. Let $x, y \in A + R$. Then $x = x' + a'$ and $y = y' + b'$, where $x', y' \in A$ and $a', b' \in R$. Assume without loss of generality that $x'_i \leq y'_i$ for every $i \in \{1, \ldots, n\}$, and let $\gamma' : [0, 1] \to A$ be a continuous monotone path that connects x' to y'. Let $J := \{i \colon 1 \leq i \leq n,\ x'_i < y'_i\}$. There are a diagonal matrix $D \in \mathcal{M}_{n,n}(\mathbb{R})$ and a vector $v \in \mathbb{R}^n$ such that $(Dx' + v)_i = a'_i$ and

$(Dy' + v)_i = b'_i$ for every $i \in J$. Define for every $i \in \{1, \ldots, n\}$ a continuous function $\delta_i : [0, 1] \to \mathbb{R}$ as follows:

- If $i \in J$ then $\delta_i(t) := (D\gamma'(t) + v)_i$.
- If $i \notin J$ then δ_i is any continuous monotone function with $\delta_i(0) = a'_i$ and $\delta_i(1) = b'_i$.

Since δ_i is monotone for every i and since $\delta(0) = a'$ and $\delta(1) = b'$, we have $\delta(t) \in R$ for every $t \in [0, 1]$.

The path $\gamma := \gamma' + \delta$ has the following properties, which imply that it is a continuous monotone path in $A + R$ from x to y:

- $\gamma(0) = x' + a' = x$ and $\gamma(1) = y' + b' = y$.
- $\gamma(t) \in A + R$ for every $t \in [0, 1]$.
- For every $i \in J$ we have $\gamma_i = ((I + D)\gamma' + v)_i$. Since $I + D$ is a diagonal matrix, the function γ_i is monotone.
- For every $i \notin J$ we have $\gamma_i = x'_i + \delta_i$, and therefore in this case γ_i is monotone as well.

Since x and y are arbitrary, the result follows. ∎

We will use the following extension of Michael's selection theorem to monovex-valued functions.

LEMMA 3.5. *Let $X \subseteq \mathbb{R}^n$ and let $F : X \rightrightarrows \mathbb{R}^m$ with open graph and nonempty monovex values. Then F has a continuous selection: there is a continuous function $f : X \to \mathbb{R}^m$ with $f(x) \in F(x)$ for every $x \in X$.*

Proof. We use induction on m. If $m = 1$ then the values of F are convex, hence one can apply Michael's selection theorem.

Assume now that $m > 1$. Let $F_1 : X \rightrightarrows \mathbb{R}$ be the projection of F to its first coordinate:

$$F_1(x) = \{y_1 \in \mathbb{R} \colon (y_1, y_2, \ldots, y_m) \in F(x) \text{ for some } (y_2, \ldots, y_m) \in \mathbb{R}^{m-1}\}.$$

Define $F_2 : \mathrm{graph}(F_1) \rightrightarrows \mathbb{R}^{m-1}$ by

$$F_2(x, y_1) := \{(y_2, \ldots, y_m) \in \mathbb{R}^{m-1} \colon (y_1, y_2, \ldots, y_m) \in F(x)\}.$$

Since F_1 and F_2 have open graphs and monovex values, by the induction hypothesis they have respective continuous selections f_1 and f_2. The function $g : X \to \mathbb{R}^m$ defined by $g(x) := (f_1(x), f_2(x, f_1(x)))$ is a continuous selection of F. ∎

DEFINITION 3.6. Let $U \subseteq \mathbb{R}^n$ be an open set, let $\varepsilon : U \to (0, 1]$ be a continuous function, and let $F : U \rightrightarrows \mathbb{R}^m$ be a set-valued function. The *ε-neighborhood* of F is the set

$$\mathcal{N}_\varepsilon(F) := \bigcup_{(x,y) \in \mathrm{graph}(F)} B((x, y), \varepsilon(x)) \subseteq \mathbb{R}^{n+m}.$$

The following result states that every set-valued function with a relatively closed graph and compact monovex values can be approximated by a set-valued function with an open graph and monovex values.

LEMMA 3.7. *Let $U \subseteq \mathbb{R}^n$ be an open set, let $\varepsilon : U \to (0,1]$ be a continuous function, and let $F : U \rightrightarrows \mathbb{R}^m$ with relatively closed graph and compact monovex values. There exists a set-valued function $G : U \rightrightarrows \mathbb{R}^m$ with open graph and monovex values satisfying* $\operatorname{graph}(F) \subseteq \operatorname{graph}(G) \subseteq \mathcal{N}_\varepsilon(F)$.

Proof. The proof is in several steps.

STEP 1: *Definitions.*

Define $\eta : U \to \{2^{-k} : k \in \mathbb{N}\}$ by

$$\eta(x) := \max\{2^{-k} : k \in \mathbb{N},\, 2^{-k} \le \varepsilon(x)/10\}.$$

This function is upper-semicontinuous: for every sequence $(x_k)_{k\in\mathbb{N}} \subset U$ that converges to a limit $x \in U$ we have $\limsup_{k\to\infty} \eta(x_k) \le \eta(x)$. Given $\delta > 0$, let $\mathcal{G}_\delta := P(\delta\mathbb{Z}^m)$ be the collection of elementary m-dimensional boxes in the lattice $\delta\mathbb{Z}^m$. Define

$$F_1(x) := \bigcup\{R \in \mathcal{G}_{\eta(x)} : R \cap F(x) \neq \emptyset\}.$$

The set $F_1(x)$ contains $F(x)$, it is a union of closed boxes, hence closed, and it approximates $F(x)$: $d_\infty(F_1(x), F(x)) \le \eta(x) \le \varepsilon(x)/10$ for every $x \in U$.

STEP 2: *The set $F_1(x)$ is monovex for every $x \in U$.*

Let $x \in U$ and $y, z \in F_1(x)$. By the definition of F_1, there are $y', z' \in F(x)$ and $R, S \in \mathcal{G}_{\eta(x)}$ such that $y, y' \in R$ and $z, z' \in S$. Since $F(x)$ is monovex, there is a continuous monotone path γ' that connects y' to z' within $F(x)$. We can assume that $y'_i \le z'_i$ for every $i = 1, \dots, n$.

We now define a path γ:

(B1) If there is $a_i \in \mathbb{Z}$ such that $a_i\eta(x) \le z_i \le (a_i + 1)\eta(x)$ and $a_i\eta(x) \le y_i \le (a_i + 1)\eta(x)$, set $\gamma_i(t) := (1 - t)y_i + tz_i$.

(B2) Otherwise there is $a_i \in \mathbb{Z}$ such that $y_i \le a_i\eta(x) \le z_i$. We let $\gamma_i(t)$ be the projection of $\gamma'_i(t)$ to the line segment $[y_i, z_i]$:

$$\gamma_i(t) := \min\bigl\{\max\{\gamma'_i(t), y_i\}, z_i\bigr\}.$$

The reader can verify that γ is contained in $F_1(x)$. However, $\gamma(0)$ need not be y and $\gamma(1)$ need not be z. Indeed, for every i for which (B2) holds we have $\gamma_i(0) = \max\{y_i, y'_i\}$ and $\gamma_i(1) = \min\{z_i, z'_i\}$. Define then $\widetilde{y}, \widetilde{z} \in F_1(x)$ by

$$\widetilde{y}_i := \begin{cases} y_i & \text{if (B1) holds,} \\ \max\{y_i, y'_i\} & \text{if (B2) holds,} \end{cases} \tag{3.1}$$

$$(3.2) \qquad \widetilde{z}_i := \begin{cases} z_i & \text{if (B1) holds,} \\ \min\{z_i, z_i'\} & \text{if (B2) holds.} \end{cases}$$

A monotone path in $F_1(x)$ that connects y and z is the concatenation of (a) a monotone path that connects y and $\widetilde{y}$, (b) the path γ, and (c) a monotone path that connects $\widetilde{z}$ to z.

STEP 3: *For every $x \in U$ there is $\delta_x \in (0, \varepsilon(x)/10)$ such that $F_1(y) \subseteq F_1(x)$ and $\eta(y) \leq \eta(x)$ for every $y \in B(x, \delta_x)$.*

Since η is upper-semicontinuous and its image is discrete, for every $x \in U$ there is $\delta_x > 0$ such that $\eta(y) \leq \eta(x)$ for every $y \in B(x, \delta_x)$. We now prove the analogous property for F_1. If the property does not hold (4), then for every $k \in \mathbb{N}$ there exists $y_k \in B(x, 1/k)$ such that $F_1(y_k) \not\subseteq F_1(x)$. That is, there is $z_k \in F_1(y_k) \setminus F_1(x)$. Since $z_k \in F_1(y_k)$, the point z_k belongs to some box R_k of the lattice $\mathcal{G}_{\eta(y_k)}$, and in particular there is a point $w_k \in R_k \cap F(y_k)$. Since (i) F has a relatively closed graph and compact values, (ii) the image of η is discrete, and (iii) η is locally bounded from below, it follows that the number of boxes R_k that satisfy these properties is finite, hence by taking a subsequence we can assume that (a) $R_k = R$ for every $k \in \mathbb{N}$ and (b) $(w_k)_{k \in \mathbb{N}}$ converges to some $w \in \mathbb{R}^n$. In particular, $w \in R$. Since the graph of F is relatively closed, $w \in F(x) \cap R$. In particular, $F(x) \cap R \neq \emptyset$, and hence $R \subseteq F_1(x)$, which implies that $z_k \in F_1(x)$ for every $k \in \mathbb{N}$, a contradiction.

STEP 4: *Definition of G.*

For every $x \in U$ define

$$Q(x) := \{y \in U : x \in B(y, \delta_y/2)\}.$$

Thus, $y \in Q(x)$ if x and y are close when the distance is measured by δ_y. Note that $x \in Q(x)$ for every $x \in U$, and therefore Q has nonempty values. Define

$$G(x) := \bigcup_{y \in Q(x)} \big(F_1(y) + B(\vec{0}, \eta(y))\big), \qquad \forall x \in U.$$

We will prove that the set-valued function G satisfies the desired conditions.

Note that $G(x)$ is a union of open sets, and hence it is open. In addition, since $x \in Q(x)$, we have $G(x) \supseteq F_1(x) \supseteq F(x)$, hence $\operatorname{graph}(G) \supseteq \operatorname{graph}(F)$.

STEP 5: *If $y_1, y_2 \in Q(x)$ then either* (a) $F_1(y_1) \subseteq F_1(y_2)$ *and* $\eta(y_1) \leq \eta(y_2)$, *or* (b) $F_1(y_1) \supseteq F_1(y_2)$ *and* $\eta(y_1) \geq \eta(y_2)$.

(4) The index k always refers to an element of a sequence. Thus, y_k is the kth element of a sequence, and not the kth coordinate of y.

Let $y_1, y_2 \in Q(x)$; we can assume that $\delta_{y_1} \geq \delta_{y_2}$. Since $x \in B(y_1, \delta_{y_1}/2) \cap B(y_2, \delta_{y_2}/2)$, we deduce that these balls intersect. In particular,

$$d_\infty(y_1, y_2) < \delta_{y_1}/2 + \delta_{y_2}/2 \leq \delta_{y_1},$$

which implies that $y_2 \in B(y_1, \delta_{y_1})$. By Step 3 this implies that $F_1(y_1) \supseteq F_1(y_2)$ and $\eta(y_1) \geq \eta(y_2)$.

STEP 6: *The set $G(x)$ is monovex for every $x \in U$.*

Let $z_1, z_2 \in G(x)$. Then there are $y_1, y_2 \in Q(x)$ such that $z_1 \in F_1(y_1) + B(\vec{0}, \eta(y_1))$ and $z_2 \in F_1(y_2) + B(\vec{0}, \eta(y_2))$. By Step 5 we can assume that $z_2 \in F_1(y_1) + B(\vec{0}, \eta(y_1))$. By Step 2 and Lemma 3.4 the set $F_1(y_1) + B(\vec{0}, \eta(y_1))$ is monovex.

STEP 7: *The graph of G is an open subset of the $\frac{3}{10}\varepsilon$-neighborhood of F.*

By the definition of G,

$$\operatorname{graph}(G) = \Big(\bigcup_{y \in U} B(y, \delta_y/2) \times (F_1(y) + B(\vec{0}, \eta(y)))\Big) \cap (U \times \mathbb{R}^m).$$

It follows that graph(G) is open. Moreover, it is a subset of the $(\delta_y/2 + 2\eta(y))$-neighborhood of F. The claim follows since $\delta_y/2 + 2\eta(y) \leq \frac{3}{10}\varepsilon$. ∎

The following result implies Theorem 2.7.

PROPOSITION 3.8. *Every closed monovex set $A \subseteq \mathbb{R}^n$ is a retract of $\mathbb{R}^n$: there is a continuous function $h : \mathbb{R}^n \to A$ which is the identity on A.*

We now show that Proposition 3.8 implies Theorem 2.7. Indeed, fix $x_0 \in \mathbb{R}^n$, and let h be the retraction of Proposition 3.8. The function $h^* : A \times [0,1] \to A$ defined by

$$h^*(x,t) := h((1-t)x + tx_0)$$

is a homotopy between A and $\{h(x_0)\}$, and so A is contractible, as claimed.

Proof of Proposition 3.8. We argue in several steps.

STEP 1: *Definitions.*

The real-valued function $x \mapsto d_\infty(x, A)$ is continuous and positive for $x \in \mathbb{R}^n \setminus A$. Define $F : \mathbb{R}^n \setminus A \rightrightarrows A$ by

$$F(x) := A \cap \overline{B}(x, d_\infty(x, A)).$$

The set $F(x)$ contains all points in A that are closest to x. Since A is a closed monovex set, and since the intersection of a monovex set and a ball in the maximum norm is monovex, F has a relatively closed graph and compact monovex values.

Set $\varepsilon(x) := d_\infty(x, A)/10$. By Lemma 3.7 there exists $G : \mathbb{R}^n \setminus A \rightrightarrows \mathbb{R}^n$ with open graph and monovex values such that graph$(G) \subseteq \mathcal{N}_\varepsilon(F)$.

Step 2: *For every $x \in \mathbb{R}^n \setminus A$ we have $G(x) \subseteq B\big(x, \frac{4}{3}d_\infty(x,A)\big)$.*

Let $(x,z) \in \operatorname{graph}(G)$. Since $\operatorname{graph}(G) \subseteq \mathcal{N}_\varepsilon(F)$, there is $(x',z') \in \operatorname{graph}(F)$ such that $d_\infty(x',x) < \varepsilon(x')$ and $d_\infty(z',z) < \varepsilon(x')$. Since $z' \in F(x') = A \cap \overline{B}(x', d_\infty(x',A))$, it follows that $d_\infty(z',x') = d_\infty(x',A)$. By the triangle inequality,

$$\begin{aligned} d_\infty(x,A) &\geq d_\infty(x',A) - d_\infty(x,x') > d_\infty(x',A) - \varepsilon(x') \\ &= d_\infty(x',A) - d_\infty(x',A)/10 = \tfrac{9}{10} d_\infty(x',A). \end{aligned} \tag{3.3}$$

By the triangle inequality once again and (3.3) we obtain

$$\begin{aligned} d_\infty(z,x) &\leq d_\infty(z,z') + d_\infty(z',x') + d_\infty(x',x) \\ &< 2\varepsilon(x') + d_\infty(x',A) = \tfrac{12}{10} d_\infty(x',A) \leq \tfrac{12}{9} d_\infty(x,A), \end{aligned}$$

as claimed.

Step 3: *Definition of a function g.*

By Michael's selection theorem (Lemma 3.5), there is a continuous selection g of G. Step 2 implies that

$$d_\infty(g(x),x) \leq \tfrac{4}{3} d_\infty(x,A), \qquad \forall x \in \mathbb{R}^n \setminus A. \tag{3.4}$$

As a consequence, for every sequence $(x_k)_{k\in\mathbb{N}}$ that converges to a limit $x \in A$ we have $\lim_{k\to\infty} d_\infty(g(x_k),x_k) = 0$. In particular, g can be extended to a continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$ that is the identity on A.

Since $(x,g(x)) \in \operatorname{graph}(G)$ and G lies in an ε-neighborhood of F, there is $(x',y') \in \operatorname{graph}(F)$ such that $d_\infty(x,x') < \varepsilon(x')$ and $d_\infty(g(x),y') < \varepsilon(x')$. Since in particular $y' \in A$, we deduce that

$$d_\infty(g(x),A) < \varepsilon(x') = d_\infty(x',A)/10 \leq d_\infty(x,A)/9,$$

where the last inequality follows from (3.3).

Step 4: *Definition of h.*

Let $g^1 := g$ and $g^k := g \circ g^{k-1}$ for $k > 1$. For every $k \in \mathbb{N}$ the function g^k is the identity on A and satisfies $d_\infty(g^k(x),A) \leq d_\infty(x,A)/9^k$ for every $x \in \mathbb{R}^n \setminus A$. Together with (3.4) we deduce that the functions $(g^k)_{k\in\mathbb{N}}$ converge locally uniformly to some continuous function h, which is the desired retraction. ∎

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