Evolutionary Stability for Large Populations and Backward Induction

by

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Evolutionary Games

- Population Game

- The dynamics:
  - Selection: Toward “better” actions
  - Mutation: Random and relatively rare
Stability

- Stability with respect to mutation
  - Evolutionary stability for fixed populations
  - Evolutionary stability for large populations: Stability for all population sizes

Which outcomes are evolutionarily stable (for large populations)?
Hart [2002]:

- Finite extensive-form games with perfect information (backward induction equilibria)
- Population games
- Evolutionary dynamics (selection and mutation)

1. For fixed populations, the backward induction equilibrium outcome is stable.

2. Only the backward induction equilibrium may be stable for large populations.

Main Result. The backward induction equilibrium is stable for large populations.
Two pure Nash equilibria:
- \( R = (R^1, R^2) \): Backward Induction (BI) equilibrium (subgame perfect)
- \( L = (L^1, L^2) \)

Evolutionary dynamics:
- Starting at \( R \) - no moves by selection
- Starting at \( L \) - after probability > 1/2 of \( R^2 \) is eventually reached (by mutation), selection towards \( R^1 \) and \( R^2 \)

\( \Rightarrow \) The BI equilibrium \( R \) is the (only) “evolutionarily stable“ equilibrium.
Example 2

- Two pure Nash equilibria:
  - \( R = (R^1, R^2, R^3) \): BI equilibrium
  - \( L = (L^1, L^2, L^3) \)

- Evolutionary dynamics:
  - Starting at \( R \) - no selection in node 2 and 3
  - The payoffs depend on the strategies in other nodes, therefore, there may be selection towards \( L^1 \) and then towards \( L^2 \)
  - \( L \) is obtained
  \[ \Rightarrow \] \( L \) is stable.
The Game

- Finite $N$-person game

- Extensive-form game (tree) with perfect information

- Generic: unique backward induction equilibrium - $b = (b^i)_{i \in N}$

- For each player $i \in N$:
  * $A^i$ - pure strategies of $i$
  * $X^i = \Delta(A^i)$ - mixed strategies of $i$

- $X = \prod_{i \in N} X^i$ - the space of mixed strategies
The Population Game

• At each node \( i \) there is a population \( M(i) \) of individuals playing the game in the role of \( i \) (for simplicity we will assume that all populations are of equal size - \( m \))

• The populations are assumed to be distinct

• Each individual \( q \in M(i) \) has an associated pure action \( \omega_q^i \in A^i \)

• Notations:

\[
\omega^i = (\omega_q^i)_{q \in M(i)} \quad \omega = (\omega_i^i)_{i \in \mathbb{N}}
\]

\[
\Omega_m = \left\{ \omega = (\omega_q^i) \mid i \in \mathbb{N}, \ q \in M(i) \right\}
\]

\[
x : \Omega_m \longrightarrow X
\]

\[
x_{a_i}^i(\omega^i) := \frac{\left\{ q \in M(i) : \omega_q^i = a^i \right\}}{m}
\]
The Dynamics

- Let $m \in \mathbb{N}$ be the populations size ($|M(i)| = m$ for all $i$)

- Let $0 < \mu < 1$ be the mutation rate

The dynamic process is a discrete time stationary Markov chain on $\Omega_m$
At each period $t = 1, 2, \ldots$, do the following (performed independently over $i$):

- Choose an individual $q(i)$ in population $i$ at random; all other individuals in $i$ do not change their actions.

- Change the action of $q(i)$ in one of the following ways:
  - **Mutation** (with probability $\mu$): an action in $A^i$ is chosen at random (i.e., all $a^i$ in $A^i$ with probability $1/|A^i|$).
  - **Selection** (with probability $1 - \mu$): an action in the set of strictly better actions (relative to $q(i)$’s current action) is chosen at random. If there are no better actions (for example, if node $i$ is not reached), then selection makes no change.
Remarks:

- In general, there is some freedom in choosing the probabilities and the sizes of the populations.

- The probability to change to some strategy by selection may depend on whether that strategy already present in the population or not.

- For every $m$ and $\mu$ we have an irreducible and aperiodic Markov chain on $\Omega_m$ with a unique invariant distribution $\pi_{m,\mu}$ over $\Omega_m$.

- The map $x : \Omega_m \rightarrow X$ induces a probability distribution over $X$. 
Definitions

Definition 1. A strategy profile $x \in X$ is $m$-evolutionarily stable if

$$\liminf_{\mu \to 0} \pi_{m,\mu}[x] > 0.$$ 

For every $\epsilon > 0$ and every strategy profile $x \in X$, let $x_\epsilon$ be the $\epsilon$-neighborhood of $x$, i.e.,

$$x_\epsilon := \{ y \in X : \|x - y\| < \epsilon \}$$

Definition 2. A strategy profile $x \in X$ is evolutionarily stable for large populations (ESLP) if for every $\epsilon > 0$

$$\liminf_{\mu \to 0} \pi_{m,\mu}[x_\epsilon] > 0.$$
Results

Theorem. (Hart[2002])
1. $b$ is $m$-evolutionarily stable for all $m$.
2. For all $x \neq b$, $x$ is not ESLP.

Main Theorem. $b$ is ESLP. Moreover, for every $\epsilon > 0$,

$$\lim_{\mu \to 0} \lim_{m \to \infty} \pi_{m,\mu}[b_{\epsilon}] = 1.$$ 

Thus, if the mutation rate is low and the populations are large, then in the long run the dynamic system is most of the time in states where almost every individual plays his backward induction action.
Sets of Equilibria

We use the following notations for different sets of strategy profiles in $X$:

- $ESLP$: the set of all ESLP strategies
- $ES_m$: the set of all $m$-evolutionarily stable strategies
- $BI$: the set of backward induction equilibria (i.e., $\{b\}$)
- $EQ$: the set of Nash equilibria

Proposition. $ESLP = BI \subseteq ES_m \subseteq EQ$, and each one of the inclusions may be strict.
A Sketch of the Proof

• The goal: Estimate the invariant distribution \( \pi_{m,\mu} \), and therefore the limit as \( m \to \infty \) and \( \mu \to 0 \)

• The means: using exit and entrance times to different sets of states - the easier it is to enter some set of states, the higher its invariant probability
**Proposition.** Let $\Omega_1 \subseteq \Omega$ such that:

1. The expected exit time from $\Omega_1 \geq T_{out}$.
2. The expected entrance time to $\Omega_1 \leq T_{in}$.

Then $\pi[\Omega_1] \geq \frac{T_{out}}{T_{out} + T_{in}}$.

We can get into the set $\Omega_1$ in no more then $T_{in}$ periods and then we stay in the set $\Omega_1$ at least $T_{out}$ periods, therefore, on average, the Markov chain will look like-

Thus, the system will be in $\Omega_1$ at least a proportion of $\frac{T_{out}}{T_{out} + T_{in}}$ of the time.
The dynamic system satisfies two conditions:

1. The expected exit time from $b_\epsilon$ is at least $C_1 m / \mu$

2. The expected entrance time to $b_\epsilon$ is at most $C_2 (m + 1 / \mu)$

Therefore,

$$\pi_{m, \mu}[b_\epsilon] \geq \frac{C_1 m / \mu}{C_1 m / \mu + C_2 (m + 1 / \mu)} \xrightarrow{\mu \to 0 \atop m \to \infty} 1.$$
The expected exit time from $b_\epsilon$:

- Everyone plays $b^i \Rightarrow b^i$ is a best reply

- The game is generic: Almost everyone plays $b^i \Rightarrow b^i$ is a best reply

Therefore,

- In $b_\epsilon$, $b^i$ is the (unique) best reply

- A change from $b^i$ to another action can only be done by mutation
The expected entrance time to $b_\varepsilon$

Moving in small (but stable) steps:

- A direction on the states towards $b_\varepsilon$

- The probability to go forward $>\$ The probability to go backward
1. Node 3 is reached $\sim 1/\mu$ periods

2. Most individuals in node 3 play $R^3 \sim m$ periods

3. Node 2 is reached $\sim 1/\mu$ periods

4. Most individuals in node 2 play $R^2 \sim m$ periods

5. ...

$\Rightarrow$ after $\sim (m + 1/\mu)$ periods, in all the nodes most individuals play the backward induction action
Conclusion

Selection can be done only when the node is reached

⇒ The backward induction action may have advantage only when the node is reached

The larger the populations the higher the probability for every node to be reached

Therefore,

• For large populations, selection chooses BI