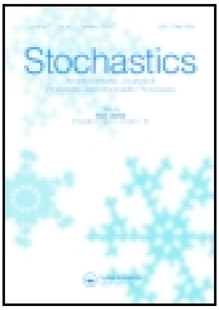

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Rida Laraki [a] & Eilon Solan [b]

[a] CNRS and Laboratoire d'Econométrie de l'Ecole Polytechnique, Palaiseau, 91128, France

[b] School of Mathematical Sciences, Tel Aviv University, Tel Aviv, 69978, Israel

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Equilibrium in two-player non-zero-sum Dynkin games in continuous time†

Rida Laraki[a1] and Eilon Solan[b]*

[a]*CNRS and Laboratoire d'Econométrie de l'Ecole Polytechnique, Palaiseau 91128, France;*
[b]*School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel*

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We prove that every two-player non-zero-sum Dynkin game in continuous time admits an ε-equilibrium in randomized stopping times. We provide a condition that ensures the existence of an ε-equilibrium in non-randomized stopping times.

Keywords: Dynkin games; stopping games; equilibrium; stochastic analysis; continuous time

2000 Mathematics Subject Classification: 91A05; 91A10; 91A55; 60G40

1. Introduction

Dynkin games [8] serve as a model of optimal stopping. These games were applied in various set-ups, including wars of attrition (see, e.g. [12,16,23]), pre-emption games (see, e.g. [11, Section 4.5.3]), duels (see, e.g. [2,6,17,29] and the survey by Radzik and Raghavan [26]) and pricing of options (see, e.g. [4,9,13,14,18]).

The existence of equilibria (in non-randomized strategies) in Dynkin games has been extensively studied when the payoffs satisfy certain conditions (see, e.g. [3,5,22] for the zero-sum case, and [24,25] for the non-zero-sum case).

Without conditions on the payoff processes, Dynkin games may fail to have equilibria, even in the one-player case (see Examples 4). Two ways to obtain a positive result are to look for ε-equilibria and to allow the players to use randomized stopping times. As Example 5 shows, 0-equilibria in randomized strategies may fail to exist in two-player zero-sum Dynkin games, as well as ε-equilibria in non-randomized strategies. The existence of an ε-equilibrium in randomized strategies in two-player zero-sum Dynkin games, in its general setting, has been settled only recently (see [27] for discrete time games, and [20] for continuous time games). The existence of an ε-equilibrium in randomized strategies in non-zero-sum games has been proven for two-player games in discrete time [28] and for games in continuous time under certain conditions (see, e.g. [21]).

In this paper we prove that every two-player non-zero-sum Dynkin game in continuous time admits an ε-equilibrium in randomized strategies for every $\varepsilon > 0$. We further show

*Corresponding author. Email: eilons@post.tau.ac.il
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how such an equilibrium can be constructed, and we provide a condition under which there exists an ε-equilibrium in non-randomized strategies. Rather than using the Snell envelope, as, e.g. in Hamadène and Zhang [15], our technique is to use results from zero-sum games.

We note that three-player Dynkin games in continuous time may fail to admit an ε-equilibrium in randomized strategies, even if the payoff processes are constant [21, Section 5.2]. Thus, our result completes the mapping of Dynkin games in continuous time that admit an ε-equilibrium in randomized stopping times.

This paper is organized as follows. The model and the main results appear in Section 2. In Section 3, we review known results regarding zero-sum games that are then used in Section 4 to prove the main theorem.

2. Model and results

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration in continuous time that satisfies 'the usual conditions'. That is, $\mathcal{F}$ is right continuous, and $\mathcal{F}_0$ contains all P-null sets: for every $B \in \mathcal{A}$ with $P(B) = 0$ and every $A \subseteq B$, one has $A \in \mathcal{F}_0$. All stopping times in the sequel are w.r.t. the filtration $\mathcal{F}$.

Denote $\mathcal{F}_\infty := \vee_{t\geq 0}\mathcal{F}_t$. Assume without loss of generality that $\mathcal{F}_\infty = \mathcal{A}$. Hence, $(\Omega, \mathcal{A}, P)$ is a complete probability space.

Let $(X_i, Y_i, Z_i)_{i=1,2}$ be uniformly bounded $\mathcal{F}$-adapted real-valued processes, and let $(\xi_i)_{i=1,2}$ be two bounded real-valued $\mathcal{F}_\infty$-measurable functions.[2] In the sequel we will assume that the processes $(X_i, Y_i, Z_i)_{i=1,2}$ are right continuous.

Definition 1. A *two-player non-zero-sum Dynkin game* over $(\Omega, \mathcal{A}, P, \mathcal{F})$ with payoffs $(X_i, Y_i, Z_i, \xi_i)_{i=1,2}$ is the game with player set $N = \{1,2\}$, the set of pure strategies of each player is the set of stopping times, and the payoff function of each player $i \in \{1,2\}$ is

$$\gamma_i(\lambda_1, \lambda_2) := \mathbf{E}\left[X_i(\lambda_1)1_{\{\lambda_1<\lambda_2\}} + Y_i(\lambda_2)1_{\{\lambda_2<\lambda_1\}} + Z_i(\lambda_1)1_{\{\lambda_1=\lambda_2<\infty\}} + \xi_i 1_{\{\lambda_1=\lambda_2=\infty\}}\right], \tag{1}$$

where λ_1 and λ_2 are the stopping times chosen by the two players, respectively.

In other words, the process X_i represents the payoff to player i if player 1 stops before player 2, the process Y_i represents the payoff to player i if player 2 stops before player 1, the process Z_i represents the payoff to player i if the two players stop simultaneously and the function ξ_i represents the payoff to player i if no player ever stops.

The game is *zero-sum* if $X_1 + X_2 = Y_1 + Y_2 = Z_1 + Z_2 = \xi_1 + \xi_2 = 0$.

In non-cooperative game theory, a randomized strategy is a probability distribution over pure strategies, with the interpretation that at the outset of the game the player randomly chooses a pure strategy according to the probability distribution given by the randomized strategy, and uses it along the game. In the set-up of Dynkin games in continuous time, a randomized strategy is a randomized stopping time, which is defined as follows.

Definition 2. A *randomized stopping time* for player i is a measurable function $\varphi_i:[0,1] \times \Omega \to [0, +\infty]$ such that the function $\varphi_i(r, \cdot):\Omega \to [0, +\infty]$ is a stopping time for every $r \in [0, 1]$ (see [1]).

Here, the interval $[0, 1]$ is endowed with the Borel σ-field. For strategically equivalent definitions of randomized stopping times, see Touzi and Vieille [30]. The interpretation of Definition 2 is that player i chooses r in $[0, 1]$ according to the uniform distribution, and then stops at the stopping time $\varphi_i(r, \cdot)$. Throughout the paper, the symbols λ, μ and τ stand for stopping times, and φ and ψ stand for randomized stopping times.

The *expected payoff* for player i that corresponds to a pair of randomized stopping times (φ_1, φ_2) is

$$\gamma_i(\varphi_1, \varphi_2) := \int_{[0,1]^2} \gamma_i(\varphi_1(r, \cdot), \varphi_2(s, \cdot))\mathrm{d}r\, \mathrm{d}s, \quad i = 1, 2.$$

In the sequel we will also consider the expected payoff at a given time t. We therefore define for every $t \geq 0$ and every pair of randomized stopping times φ_1, $\varphi_2 \geq t$:

$$\gamma_i(\varphi_1, \varphi_2 | \mathcal{F}_t) := \mathbf{E}\left[X_i(\varphi_1)1_{\{\varphi_1<\varphi_2\}} + Y_i(\varphi_2)1_{\{\varphi_2<\varphi_1\}} + Z_i(\varphi_1)1_{\{\varphi_1=\varphi_2<\infty\}} + \xi_i 1_{\{\varphi_1=\varphi_2=\infty\}} | \mathcal{F}_t\right]. \tag{2}$$

A pair of randomized stopping times $(\varphi_1^*, \varphi_2^*)$ is an ε-equilibrium if no player can profit more than ε by deviating from φ_i^*.

DEFINITION 3. Let $\varepsilon \geq 0$. A pair of randomized stopping times $(\varphi_1^*, \varphi_2^*)$ is an *ε-equilibrium* if for every two randomized stopping times φ_1, φ_2 the following inequalities hold:

$$\gamma_1\left(\varphi_1, \varphi_2^*\right) \leq \gamma_1\left(\varphi_1^*, \varphi_2^*\right) + \varepsilon, \tag{3}$$

and

$$\gamma_2\left(\varphi_1^*, \varphi_2\right) \leq \gamma_2\left(\varphi_1^*, \varphi_2^*\right) + \varepsilon. \tag{4}$$

Because of the linearity of the payoff function, Equations (3) and (4) hold for every randomized stopping times φ_1 and φ_2, respectively, as soon as they hold for non-randomized stopping times.

We now provide two examples that show that 0-equilibria may fail to exist.

Example 4. We here provide a one-player Dynkin game with trivial filtration, which fails to have a zeor-equilibrium. A one-player Dynkin game is given by a process X_1 and a bounded real-valued function ξ_1. The payoff function is given by $\gamma_1(\lambda_1) = \mathbf{E}[X_1(\lambda_1)1_{\{\lambda_1<\infty\}} + \xi_1 1_{\{\lambda_1=\infty\}}]$. For $\varepsilon \geq 0$, an ε-equilibrium (in non-randomized stopping times) is a stopping time λ_1^* that satisfies $\gamma_1(\lambda_1^*) \geq \sup_{\lambda_1} \gamma_1(\lambda_1) - \varepsilon$. Consider the one-player game with trivial filtration, where X_1 is a strictly increasing, non-negative and bounded function, and $\xi_1 = 0$. Since X_1 is strictly increasing and positive there are no 0-equilibria but there are ε-equilibria for every $\varepsilon > 0$: for every t such that $X_1(t) \geq \lim_{t\to\infty} X_1(t) - \varepsilon$, the stopping times $\lambda_1 = t$ (that stops at time t with probability 1) is an ε-equilibrium.

Example 5. We now provide an example of a two-player zero-sum game with trivial filtration that has neither an ε-equilibrium in non-randomized stopping times nor a 0-equilibrium in randomized stopping times. Consider the two-player zero-sum games were $X_1(t) = Y_1(t) = 1$ and $Z_1(t) = 0$ for every $t \geq 0$, and $\xi_1 = 0$. It follows that $X_2(t) = Y_2(t) = -1$ and $Z_2(t) = 0$ for every $t \geq 0$, and $\xi_2 = 0$.

Suppose by contradiction that the game has an ε-equilibrium $(\lambda_1^*, \lambda_2^*)$ in non-randomized stopping times. Since $\gamma_2 = -\gamma_1$, the ε-equilibrium condition (3) implies that

$$\gamma_2\left(\lambda_1^*, \lambda_2\right) \geq \gamma_2\left(\lambda_1^*, \lambda_2^*\right) - \varepsilon,$$

for every stopping time λ_2 of player 2. Since $\gamma_2(\lambda_1^*, \lambda_1^*) = 0$ we deduce that $\gamma_2(\lambda_1^*, \lambda_2^*) \leq \varepsilon$. Similarly, the ε-equilibrium condition (4) implies that

$$\gamma_1(\lambda_1, \lambda_2^*) \leq \gamma_1(\lambda_1^*, \lambda_2^*) + \varepsilon,$$

for every stopping time λ_1 of player 1. For every stopping time λ_2^* one has $\sup_{\lambda_1} \gamma_1(\lambda_1, \lambda_2^*) = 1$, and therefore $\gamma_1(\lambda_1^*, \lambda_2^*) \geq 1 - \varepsilon$. Provided that $\varepsilon < 1/2$, there is no pair of stopping times $(\lambda_1^*, \lambda_2^*)$ for which $\gamma_2(\lambda_1^*, \lambda_2^*) \leq \varepsilon$ and $\gamma_1(\lambda_1^*, \lambda_2^*) \geq 1 - \varepsilon$, so that an ε-equilibrium in non-randomized stopping times does not exist.

We now argue that a 0-equilibrium in randomized stopping times does not exist as well. Indeed, suppose by contradiction that $(\varphi_1^*, \varphi_2^*)$ is a 0-equilibrium in randomized stopping times. One has $\sup_{\varphi_1} \gamma_1(\varphi_1, \varphi_2^*) = 1$, which implies that $\gamma_1(\varphi_1^*, \varphi_2^*) = 1$, and therefore $\gamma_2(\varphi_1^*, \varphi_2^*) = -1$. However, for every randomized stopping time φ_1 there is a randomized stopping time φ_2 such that $\gamma_2(\varphi_1^*, \varphi_2) > -1$, implying that $(\varphi_1^*, \varphi_2^*)$ cannot be a 0-equilibrium.

Our goal in this paper is to prove the existence of an ε-equilibrium in randomized strategies in two-player non-zero-sum games, and to construct such an ε-equilibrium.

Suppose that the payoff processes are right continuous and that a player wants to stop at the stopping time λ, but he would like to mask the exact time at which he stops (e.g. so that the other player cannot stop at the very same moment as he does). To this end, he can stop at a randomly chosen time in a small interval $[\lambda, \lambda + \delta]$, and since the payoff processes are right continuous, he will not lose (or gain) much relative to stopping at time λ. This leads us to the following class of simple randomized stopping times that will be extensively used in the sequel.

DEFINITION 6. A randomized stopping time φ is *simple* if there exist a stopping time λ and a $\mathcal{F}_\lambda$-measurable non-negative function $\delta \geq 0$, such that for every $r \in [0, 1]$ one has $\varphi(r, \cdot) = \lambda + r\delta$. The stopping time λ is called the *basis* of φ, and the function δ is called the *delay* of φ.

In Definition 6, $\varphi(r, \cdot) \geq \lambda$ and $\varphi(r, \cdot)$ is $\mathcal{F}_\lambda$-measurable. By Dellacherie and Meyer [7, §IV-56] $\varphi(r, \cdot)$ is a stopping time for every $r \in [0, 1]$. Consequently, φ is indeed a randomized stopping time.

Definition 6 does not require that λ is finite:[3] on the set $\{\lambda = \infty\}$ we have $\varphi(r, \cdot) = \infty$ for every $r \in [0, 1]$. On the set $\{\delta = 0\}$, the randomized stopping time φ that is defined in Definition 6 stops at time λ with probability 1. On the set $\{\delta > 0\}$ the stopping time is 'nonatomic' yet finite, and in particular for every stopping time μ we have $\mathbf{P}(\{\delta > 0\} \cap \{\varphi = \mu\}) = 0$.

We now state our main results.

THEOREM 7. Every two-player non-zero-sum Dynkin game with right continuous and uniformly bounded payoff processes admits an ε-equilibrium in simple randomized stopping times for every $\varepsilon > 0$.

Moreover, the delay of the simple randomized stopping time that constitutes the ε-equilibrium can be arbitrarily small.

Theorem 7 was proved by Laraki and Solan [20] for two-player *zero-sum* games. Our proof heavily relies on the results of Laraki and Solan [20], and we use ε-equilibria in zero-sum games to construct an ε-equilibrium in the non-zero-sum game.

Under additional conditions on the payoff processes, the ε-equilibrium is given in non-randomized stopping times.

THEOREM 8. Under the assumptions of Theorem 7, if $Z_1(t) \in co\{X_1(t), Y_1(t)\}$ and $Z_2(t) \in co\{X_2(t), Y_2(t)\}$ for every $t \geq 0$, then the game admits an ε-equilibrium in non-randomized stopping times for every $\varepsilon > 0$.

Hamadène and Zhang [15] proved the existence of a 0-equilibrium in non-randomized stopping times under stronger conditions than those in Theorem 8, using the notion of Snell envelope of processes (see, e.g. [10] for more detail).

The rest of the paper is devoted to the proofs of Theorems 7 and 8. We will assume w.l.o.g. that the payoff processes are bounded between 0 and 1.

3. The zero-sum case

In this section we summarize several results on zero-sum games, taken from Laraki and Solan [20], that will be used in the sequel, and prove some new results on zero-sum games.

For every $t \geq 0$ denote

$$v_1(t) := \text{ess} - \sup_{\varphi_1 \geq t} \text{ess} - \inf_{\lambda_2 \geq t} \mathbf{E}[X_1(\varphi_1) 1_{\{\varphi_1 < \lambda_2\}} + Y_1(\lambda_2) 1_{\{\lambda_2 < \varphi_1\}} \\ + Z_1(\varphi_1) 1_{\{\varphi_1 = \lambda_2 < \infty\}} + \xi_1 1_{\{\varphi_1 = \lambda_2 = \infty\}} | \mathcal{F}_t], \tag{5}$$

where the supremum is over all randomized stopping times $\varphi_1 \geq t$, and the infimum is over all (non-randomized) stopping times $\lambda_2 \geq t$. This is the highest payoff that player 1 can guarantee in the zero-sum Dynkin game $\Gamma_1(t)$, where the payoffs are those of player 1, player 1 is the maximizer, player 2 is the minimizer and the game starts at time t. Similarly, the highest payoff that player 2 can guarantee in the zero-sum Dynkin game $\Gamma_2(t)$, where the payoffs are those of player 2, player 2 is the maximizer, player 1 is the minimizer and the game starts at time t is given by

$$v_2(t) := \text{ess} - \sup_{\varphi_2 \geq t} \text{ess} - \inf_{\lambda_1 \geq t} \mathbf{E}[X_2(\lambda_1) 1_{\{\lambda_1 < \varphi_2\}} + Y_2(\varphi_2) 1_{\{\varphi_2 < \lambda_1\}} \\ + Z_2(\lambda_1) 1_{\{\lambda_1 = \varphi_2 < \infty\}} + \xi_2 1_{\{\lambda_1 = \varphi_2 = \infty\}} | \mathcal{F}_t]. \tag{6}$$

The next lemma, which is proved in Laraki and Solan [20], states that $v_1(t)$ (respectively, $v_2(t)$) is in fact the value of the zero-sum games $\Gamma_1(t)$ (respectively, $\Gamma_2(t)$). This lemma is proved in Laraki and Solan [20] when $\mathcal{F}_t$ is the trivial σ-algebra. Its proof can be adapted to a general $\mathcal{F}_t$ (see the discussion in Appendix A).

LEMMA 9.

$$v_1(t) = \text{ess} - \inf_{\psi_2 \geq t} \text{ess} - \sup_{\lambda_1 \geq t} \mathbf{E}[X_1(\lambda_1) 1_{\{\lambda_1 < \psi_2\}} + Y_1(\psi_2) 1_{\{\psi_2 < \lambda_1\}}$$
$$+ Z_1(\lambda_1) 1_{\{\lambda_1 = \psi_2 < \infty\}} + \xi_1 1_{\{\lambda_1 = \psi_2 = \infty\}} | \mathcal{F}_t] \quad (7)$$

and

$$v_2(t) = \text{ess} - \inf_{\psi_1 \geq t} \text{ess} - \sup_{\lambda_2 \geq t} \mathbf{E}[X_2(\psi_1) 1_{\{\psi_1 < \lambda_2\}} + Y_2(\lambda_2) 1_{\{\lambda_2 < \psi_1\}}$$
$$+ Z_2(\psi_1) 1_{\{\psi_1 = \lambda_2 < \infty\}} + \xi_2 1_{\{\psi_1 = \lambda_2 = \infty\}} | \mathcal{F}_t], \quad (8)$$

where the infimum in (7) is over all randomized stopping times $\psi_2 \geq t$ for player 2, the supremum in (7) is over all (non-randomized) stopping times $\lambda_1 \geq t$ for player 1, the infimum in (8) is over all randomized stopping times $\psi_1 \geq t$ for player 1 and the supremum in (8) is over all (non-randomized) stopping times $\lambda_2 \geq t$ for player 2.

A stopping time φ_1 (respectively, ψ_1) that achieves the supremum in (5) (respectively, infimum in (8)) up to ε is called an *ε-optimal* stopping time for player 1 in $\Gamma_1(t)$ (respectively, $\Gamma_2(t)$). Similarly, a stopping time φ_2 (respectively, ψ_2) that achieves the supremum in (6) (respectively, infimum in (7)) up to ε is called an *ε-optimal* stopping time for player 2 in $\Gamma_2(t)$ (respectively, $\Gamma_1(t)$).

The proof of Laraki and Solan [20, Proposition 7] can be adapted to show that the value process is right continuous (see Appendix A).

LEMMA 10. The process $(v_i(t))_{t \geq 0}$ is right continuous, for each $i \in \{1, 2\}$.

The following two lemmas provide crude bounds on the value process.

LEMMA 11. For every $t \geq 0$ and each $i = 1, 2$ one has

$$\min\{X_i(t), Y_i(t)\} \leq v_i(t) \leq \max\{X_i(t), Y_i(t)\} \quad \text{on } \Omega.$$

Proof. We start by proving the left-hand side inequality for $i = 2$. Let $\varepsilon > 0$ be arbitrary, and let $\delta > 0$ be sufficiently small such that

$$\mathbf{P}\left(\sup_{\rho \in [0,\delta]} |X_2(t) - X_2(t+\rho)| > \varepsilon \right) \leq \varepsilon, \quad (9)$$

$$\mathbf{P}\left(\sup_{\rho \in [0,\delta]} |Y_2(t) - Y_2(t+\rho)| > \varepsilon \right) \leq \varepsilon. \quad (10)$$

Such δ exists because the processes X_2 and Y_2 are right continuous.

Let φ_2 be the simple randomized stopping time $\varphi_2(r, \cdot) = t + r\delta$, and let $\lambda_1 \geq t$ be any non-randomized stopping time for player 1. The definition of φ_2 implies that the probability that $\lambda_1 = \varphi_2$ is 0: $\mathbf{P}(\lambda_1 = \varphi_2) = 0$. Moreover, $\varphi_2 < \infty$. Therefore

$$\gamma_2(\lambda_1, \varphi_2 | \mathcal{F}_t) = \mathbf{E}[X_2(\lambda_1) 1_{\{\lambda_1 < \varphi_2\}} + Y_2(\varphi_2) 1_{\{\varphi_2 < \lambda_1\}} | \mathcal{F}_t].$$

By (9) and (10), and since payoffs are bounded by 1, this implies that

$$\mathbf{P}(\gamma_2(\lambda_1, \varphi_2|\mathcal{F}_t) < \min\{X_2(t), Y_2(t)\} - \varepsilon) \leq 2\varepsilon.$$

Because λ_1 is arbitrary, Equation (6) implies that

$$\mathbf{P}(v_2(t) < \min\{X_2(t), Y_2(t)\} - \varepsilon) \leq 2\varepsilon.$$

The left-hand side inequality for $i = 2$ follows because ε is arbitrary.

The proof of the right-hand side inequality for $i = 2$ follows the same arguments by using the simple randomized stopping time $\varphi_1(r, \cdot) = t + r\delta$. Indeed, for every stopping time λ_2 for player 2 we then have

$$\gamma_2(\varphi_1, \lambda_2|\mathcal{F}_t) = \mathbf{E}[X_2(\varphi_1)1_{\{\varphi_1<\lambda_2\}} + Y_2(\lambda_2)1_{\{\varphi_1>\lambda_2\}}|\mathcal{F}_t].$$

The same argument as above, using (8), delivers the desired inequality. The proof for $i = 1$ is analogous. □

Lemma 12. For every $t \geq 0$, one has

$$v_1(t) \leq \max\{Y_1(t), Z_1(t)\} \quad \text{on } \Omega,$$
$$v_2(t) \leq \max\{X_2(t), Z_2(t)\} \quad \text{on } \Omega.$$

Proof. We prove the Lemma for $i = 1$. Let $\psi_2 = t$: player 2 stops at time t. By (7),

$$v_1(t) \leq \text{ess} - \sup_{\lambda_1 \geq t} \gamma_1(\lambda_1, \psi_2|\mathcal{F}_t).$$

Because for every (non-randomized) stopping time λ_1 for player 1, $\gamma_1(\lambda_1,\psi_2|\mathcal{F}_t)$ is either $Y_1(t)$ (if $\lambda_1 > t$) or $Z_1(t)$ (if $\lambda_1 = t$), the result follows. □

Following Lepeltier and Maingueneau [22], for every $\eta > 0$ let μ_1^η and μ_2^η be the stopping times defined as follows:

$$\mu_1^\eta := \inf\{s \geq 0 : X_1(s) \geq v_1(s) - \eta\} \tag{11}$$

and

$$\mu_2^\eta := \inf\{s \geq 0 : Y_2(s) \geq v_2(s) - \eta\}. \tag{12}$$

As the following example shows, the stopping times μ_1^η and μ_2^η may be infinite. Consider the following Dynkin game, where the payoffs are constants: $X_1 = 0$, $Y_1 = Z_1 = 2$ and $\xi_1 = 1$. Then $v_1(t) = 1$ for every t, and $\mu_1^\eta = \infty$, provided $\eta \in (0, 1)$.

Observe that $\mu_2^\eta \leq \mu_2^{\eta'}$ whenever $\eta > \eta'$. Moreover, because the processes X_1, Y_2, v_1 and v_2 are right continuous, we have

$$X_1(\mu_1^\eta) \geq v_1(\mu_1^\eta) - \eta, \tag{13}$$

and

$$Y_2(\mu_2^\eta) \geq v_2(\mu_2^\eta) - \eta. \tag{14}$$

For every $t < \mu_1^\eta$, by the definition of μ_1^η and Lemma 11, we have

$$X_1(t) < v_1(t) - \eta < v_1(t) \le \max\{X_1(t), Y_1(t)\},$$

and therefore

$$Y_1(t) > X_1(t), \quad \forall\, t < \mu_1^\eta. \tag{15}$$

The analogous inequality for player 2 holds as well.

LEMMA 13. Let $\varepsilon, \eta > 0$, let τ be a stopping time, and let $A \in \mathcal{F}_\tau$ satisfy $\mathbf{P}(A \setminus \{\mu_1^\eta = \infty\}) < \varepsilon$. Then

$$\mathbf{E}[v_1(\tau)1_A] \le \mathbf{E}[\xi_1 1_{A\cap\{\mu_1^\eta=\infty\}}] + 3\varepsilon + \frac{6\varepsilon}{\eta}. \tag{16}$$

Proof. Let $\psi_2 = \infty$: player 2 never stops. By (7),

$$v_1(\tau) \le \text{ess} - \sup_{\lambda_1 \ge \tau} \gamma_1(\lambda_1, \psi_2 | \mathcal{F}_\tau). \tag{17}$$

Let $\lambda_1 \ge \tau$ be a stopping time for player 1 that achieves the supremum in (17) up to ε. Let λ_1' be the following stopping time:

- On $A \cap \{\lambda_1 < \infty\}$, λ_1' is an $\eta/2$-optimal stopping time for player 1 in $\Gamma_1(\lambda_1)$.
- On $A \cap \{\lambda_1 = \infty\}$, $\lambda_1' = \infty$.

It follows that

$$\begin{aligned}
\mathbf{E}[v_1(\tau)1_A] &\le \mathbf{E}[\gamma_1(\lambda_1, \psi_2 | \mathcal{F}_\tau)1_A] + \varepsilon \\
&= \mathbf{E}[X_1(\lambda_1)1_{A\cap\{\lambda_1<\infty\}} + \xi_1 1_{A\cap\{\lambda_1=\infty\}}] + \varepsilon \\
&< \mathbf{E}[(v_1(\lambda_1) - \eta)1_{A\cap\{\lambda_1<\mu_1^\eta=\infty\}} + X_1(\lambda_1)1_{A\cap\{\lambda_1<\infty\}\cap\{\mu_1^\eta<\infty\}} + \xi_1 1_{A\cap\{\lambda_1=\infty\}}] + \varepsilon \\
&\le \mathbf{E}[(v_1(\lambda_1) - \eta)1_{A\cap\{\lambda_1<\infty\}} + \xi_1 1_{A\cap\{\lambda_1=\infty\}}] + 3\varepsilon \\
&\le \mathbf{E}[\gamma_1(\lambda_1', \psi_2 | \mathcal{F}_\tau)1_A] - \frac{\eta}{2}\mathbf{E}[1_{A\cap\{\lambda_1<\infty\}}] + 3\varepsilon \\
&\le \mathbf{E}[\gamma_1(\lambda_1, \psi_2 | \mathcal{F}_\tau)1_A] - \frac{\eta}{2}\mathbf{E}[1_{A\cap\{\lambda_1<\infty\}}] + 4\varepsilon,
\end{aligned}$$

where the second inequality holds by the definition of μ_1^η, the third inequality holds since $\mathbf{P}(A \setminus \{\mu_1^\eta = \infty\}) < \varepsilon$ and since payoffs are bounded by 1, and the last inequality holds because λ_1 is ε-optimal.

This sequence of inequalities implies that

$$\mathbf{P}(A \cap \{\lambda_1 < \infty\}) \le \frac{6\varepsilon}{\eta},$$

and therefore

$$\mathbf{E}[v_1(\tau)1_A] \le \mathbf{E}[\xi_1 1_{A\cap\{\mu_1^\eta=\infty\}}] + 3\varepsilon + \frac{6\varepsilon}{\eta},$$

as desired. □

By Lepeltier and Maingueneau [22], for each $i = 1, 2$ the process v_i is a submartingale up to time μ_i^η.

LEMMA 14. For every $\eta > 0$ the process $(v_1(t))_{t=0}^{\mu_1^\eta}$ is a submartingale: for every pair of finite stopping times $\lambda < \lambda' \leq \mu_1^\eta$, one has $v_1(\lambda) \leq \mathbf{E}[v_1(\lambda')|\mathcal{F}_\lambda]$ on Ω.

Lemma 14 implies that before time $\sup_{\eta>0}\mu_1^\eta$, player 1 is better off waiting and not stopping. An analogue statement holds for player 2.

Lemmas 13 and 14 deliver the following result.

LEMMA 15. Let $\eta > 0$. For every stopping time λ_1 that satisfies $\lambda_1 \leq \mu_1^\eta$, one has

$$v_1(\lambda_1) \leq \mathbf{E}\Big[v_1\big(\mu_1^\eta\big)1_{\{\mu_1^\eta<\infty\}} + \xi_1 1_{\{\mu_1^\eta=\infty\}}|\mathcal{F}_{\lambda_1}\Big].$$

Proof. Let $\varepsilon > 0$ be arbitrary. By Lemma 14, for every $t \geq 0$ one has

$$v_1(\lambda_1) \leq \mathbf{E}[v_1(\min\{\mu_1^\eta, t\})].$$

Let t_0 be sufficiently large such that $\mathbf{P}(t_0 \leq \mu_1^\eta < \infty) < \varepsilon$. By Lemma 13 with $\tau = t_0$ and $A = \{t_0 \leq \mu_1^\eta\}$,

$$\mathbf{E}[v_1(t_0)1_{\{t_0\leq\mu_1^\eta\}}] \leq \mathbf{E}\Big[\xi_1 1_{\{t_0\leq\mu_1^\eta=\infty\}}\Big] + 3\varepsilon + \frac{6\varepsilon}{\eta}.$$

Therefore,

$$\begin{aligned} v_1(\lambda_1) &\leq \mathbf{E}[v_1(\min\{\mu_1^\eta, t_0\})] \\ &= \mathbf{E}\Big[v_1(\mu_1^\eta)1_{\{\mu_1^\eta<t_0\}} + v_1(t_0)1_{\{t_0\leq\mu_1^\eta\}}\Big] \\ &\leq \mathbf{E}\Big[v_1(\mu_1^\eta)1_{\{\mu_1^\eta<\infty\}} + \xi_1 1_{\{\mu_1^\eta=\infty\}}\Big] + 5\varepsilon + \frac{6\varepsilon}{\eta}. \end{aligned}$$

The result follows since ε is arbitrary. □

The proof of Laraki and Solan [20, Section 3.3] delivers the following result, which states that each player i has a simple randomized ε-optimal stopping time that is based on μ_i^η, provided η is sufficiently small.

LEMMA 16. For every $i = 1, 2$, every $\varepsilon, \eta > 0$ and every positive $\mathcal{F}_{\mu_i^\eta}$-measurable function δ_i, there exists a simple randomized stopping time φ_i^η with basis μ_i^η and delay at most δ_i that satisfies

$$\gamma_i(\varphi_i^\eta, \lambda_{3-i}|\mathcal{F}_{\mu_i^\eta}) \geq v_i(\mu_i^\eta) - \varepsilon - \eta \quad \text{on } \Omega, \tag{18}$$

for every stopping time $\lambda_{3-i} \geq \mu_i^\eta$.

By Equation (15), before time μ_1^η one has $X_1 < Y_1$. When $X_1(t) \leq Z_1(t) \leq Y_1(t)$ for every t, a non-randomized ε-optimal stopping time exists [22]. Laraki and Solan [19, Section 4.1] use this observation to conclude the following.

LEMMA 17. If $Z_i(t) \in \mathrm{co}\{X_i(t), Y_i(t)\}$ for every $t \geq 0$ and each $i = 1, 2$, then the simple randomized stopping time φ_i^η in Lemma 16 can be taken to be non-randomized (i.e. the delay of both players is 0).

4. The non-zero-sum case

In this section we prove Theorems 7 and 8. Fix $\varepsilon > 0$ once and for all.

Let δ_0 (respectively, δ_1 and δ_2) be a positive $\mathcal{F}_\tau$-measurable function that satisfies the following inequalities for each $i \in \{1, 2\}$ and for the stopping time $\tau = 0$ (respectively, $\tau = \mu_1^\eta$ and $\tau = \mu_2^\eta$). Such δ_0 (respectively, δ_1 and δ_2) exists because the processes $(X_i, Y_i, v_i)_{i=1,2}$ are right continuous.

$$\mathbf{P}\left(\sup_{\rho\in[0,\delta_0]} |X_i(\tau) - X_i(\tau+\rho)| > \varepsilon\right) \leq \varepsilon, \tag{19}$$

$$\mathbf{P}\left(\sup_{\rho\in[0,\delta_0]} |Y_i(\tau) - Y_i(\tau+\rho)| > \varepsilon\right) \leq \varepsilon, \tag{20}$$

$$\mathbf{P}\left(\sup_{\rho\in[0,\delta_0]} |v_i(\tau) - v_i(\tau+\rho)| > \varepsilon\right) \leq \varepsilon. \tag{21}$$

We divide the set Ω into six $\mathcal{F}_0$-measurable subsets. For each of these subsets we then define a pair of randomized stopping times $(\varphi_1^*, \varphi_2^*)$, and we prove that, when restricted to each set, this pair is a $k\varepsilon$-equilibrium, for some $0 \leq k \leq 13$. It will then follow that $(\varphi_1^*, \varphi_2^*)$, when viewed as a randomized stopping time on Ω, is a 78ε-equilibrium. The partition is similar to that in Laraki et al. [21], and only the treatment on the last subset is different.

Denote by $\psi_i(t, \varepsilon)$ an ε-optimal stopping time of player i in the game $\Gamma_{3-i}(t)$; thus, the randomized stopping time $\psi_i(t, \varepsilon)$ is a punishment strategy against player $3 - i$, as it ensures that his payoff will not exceed $v_{3-i}(t) + \varepsilon$.

Part 1: The set $A_1 := \{X_1(0) \geq v_1(0)\} \cap \{X_2(0) \geq Z_2(0)\}$.

We prove that when restricted to the set A_1, the pair $(\varphi_1^*, \varphi_2^*)$ that is defined as follows is a 4ε-equilibrium:

- $\varphi_1^* = 0$: player 1 stops at time 0.
- $\varphi_2^* = \psi_2(\delta_0, \varepsilon)$: If player 1 does not stop before time δ_0, player 2 punishes him in the game $\Gamma_1(\delta_0)$ that starts at time δ_0.

If no player deviates, the game is stopped by player 1, and the payoff is

$$\gamma(\varphi_1^*, \varphi_2^* | \mathcal{F}_0) = (X_1(0), X_2(0)) \quad \text{on } A_1.$$

We argue that player 2 cannot profit by deviating. Indeed, let λ_2 be any non-randomized stopping time of player 2. Then on A_1

$$\gamma_2(\varphi_1^*, \lambda_2 | \mathcal{F}_0) = Z_2(0)1_{A_1\cap\{\lambda_2=0\}} + X_2(0)1_{A_1\cap\{\lambda_2>0\}} \leq X_2(0) = \gamma_2(\varphi_1^*, \varphi_2^* | \mathcal{F}_0),$$

and the claim follows.

We now argue that on A_1, player 1 cannot profit more than 4ε by deviating from φ_1^*. Let λ_1 be any non-randomized stopping time of player 1. Then by the definition of φ_2^*, on A_1

$$\gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_0) \leq \mathbf{E}[X_1(\lambda_1) 1_{\{\lambda_1 < \delta_0\}} + (v_1(\delta_0) + \varepsilon) 1_{\{\delta_0 \leq \lambda_1\}} | \mathcal{F}_0].$$

By (19), (21) and since $X_1(0) \geq v_1(0)$ on A_1, it follows that on A_1

$$\mathbf{P}(\gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_0) > \mathbf{E}[X_1(0) 1_{\{\lambda_1 < \delta_0\}} + (X_1(0) + \varepsilon) 1_{\{\delta_0 \leq \lambda_1\}} | \mathcal{F}_0] + \varepsilon) \leq 2\varepsilon.$$

Since $\gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_0) = X_1(0)$ on A_1, it follows that

$$\mathbf{P}(A_1 \cap \{\gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_0) > \gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_0) + 2\varepsilon\}) \leq 2\varepsilon, \tag{22}$$

and the desired results follow.

Part 2: The set $A_2 := \{Z_2(0) > X_2(0)\} \cap \{Z_1(0) \geq Y_1(0)\}$.

We prove that when restricted to the set A_2, the pair $(\varphi_1^*, \varphi_2^*)$ that is defined as follows is a 0-equilibrium:

- $\varphi_1^* = 0$: player 1 stops at time 0.
- $\varphi_2^* = 0$: player 2 stops at time 0.

If no player deviates, both players stop at time 0, and the payoff is

$$\gamma(\varphi_1^*, \varphi_2^* | \mathcal{F}_0) = (Z_1(0), Z_2(0)) \quad \text{on} \ A_2.$$

To see that player 1 cannot profit by deviating, fix an arbitrary non-randomized stopping time λ_1 for player 1. On A_2 one has

$$\gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_0) = Z_1(0) 1_{\{\lambda_1 = 0\}} + Y_1(0) 1_{\{\lambda_1 > 0\}} \leq Z_1(0) = \gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_0), \tag{23}$$

as desired. A symmetric argument shows that player 2 cannot profit by deviating either.

Part 3: The set $A_3 := \{Y_1(0) > Z_1(0)\} \cap \{Y_2(0) \geq v_2(0)\}$.

The case of the set A_3 is analogous to Part 1: when restricted to A_3, the pair of randomized stopping times in which player 2 stops at time 0, and player 1 plays an ε-optimal stopping time $\psi_1(\delta_0, \varepsilon)$ in the game $\Gamma_2(\delta_0)$, is a 4ε-equilibrium.

Part 4: The set $A_4 := \{X_1(0) \geq v_1(0)\} \cap \{X_2(0) > Y_2(0)\}$.

We prove that when restricted to the set A_4, the pair $(\varphi_1^*, \varphi_2^*)$ that is defined as follows is a 6ε-equilibrium:

- $\varphi_1^*(r, \cdot) = r\delta_0$: player 1 stops at a random time between time 0 and time δ_0.
- $\varphi_2^* = \psi_2(\delta_0, \varepsilon)$: If player 1 does not stop before time δ_0, player 2 punishes him in the game $\Gamma_1(\delta_0)$ that starts at time δ_0.

If no player deviates, the game is stopped by player 1 before time δ_0, and by (19) the payoff is within 2ε of $(X_1(0), X_2(0))$:

$$\mathbf{P}(A_4 \cap \{|\gamma_i(\varphi_1^*, \varphi_2^*) - X_i(0)| > \varepsilon\}) \leq \varepsilon. \tag{24}$$

The same argument[4] as in Part 1 shows that

$$\mathbf{P}(A_4 \cap \{\gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_0) > \gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_0) + 3\varepsilon\}) \leq 3\varepsilon. \tag{25}$$

It follows that player 1 cannot profit more than 6ε by deviating from φ_1^*.

We now argue that player 2 cannot profit more than 5ε by deviating from φ_2^*. Fix a non-randomized stopping time λ_2 for player 2. On A_4 we have $\varphi_1^* \leq \delta_0$, and $\mathbf{P}(A_4 \cap \{\varphi_1^* = \lambda_2\}) = 0$, and therefore

$$\gamma_2(\varphi_1^*, \lambda_2) = \mathbf{E}[X_2(\varphi_1^*)1_{\{\varphi_1^* < \lambda_2\}} + Y_2(\lambda_2)1_{\{\lambda_2 < \varphi_1^*\}} | \mathcal{F}_0] \quad \text{on } A_4.$$

By (19) and (20),

$$\mathbf{P}(\gamma_2(\varphi_1^*, \lambda_2) > \mathbf{E}[(X_2(0) + \varepsilon)1_{\{\varphi_1^* < \lambda_2\}} + (Y_2(0) + \varepsilon)1_{\{\lambda_2 < \varphi_1^*\}} | \mathcal{F}_0]) \leq 2\varepsilon.$$

Because $X_2(0) > Y_2(0)$ on A_4 we have

$$\mathbf{P}(\gamma_2(\varphi_1^*, \lambda_2) > X_2(0) + \varepsilon) \leq 2\varepsilon.$$

Together with (24) we deduce that

$$\mathbf{P}(\gamma_2(\varphi_1^*, \lambda_2) > \gamma_2(\varphi_1^*, \varphi_2^*) + 2\varepsilon) \leq 3\varepsilon,$$

and the claim follows.

Part 5: The set $A_5 := \{X_1(0) \geq v_1(0)\} \backslash (A_1 \cup A_2 \cup A_3 \cup A_4)$.

We claim that $\mathbf{P}(A_5) = 0$. Since $X_1(0) \geq v_1(0)$ on A_5, and since $A_5 \cap A_1 = \emptyset$, it follows that $X_2(0) < Z_2(0)$ on A_5. Since $A_5 \cap A_2 = \emptyset$, it follows that $Z_1(0) < Y_1(0)$ on A_5. Since $A_5 \cap A_3 = \emptyset$, it follows that $Y_2(0) < v_2(0)$ on A_5. Since $A_5 \cap A_4 = \emptyset$, it follows that $Y_2(0) \geq X_2(0)$ on A_5. Lemma 11 then implies that

$$Y_2(0) < v_2(0) \leq \max\{X_2(0), Y_2(0)\} = Y_2(0) \quad \text{on } A_5,$$

which in turn implies that $\mathbf{P}(A_5) = 0$, as claimed.

The union $A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$ includes the set $\{X_1(0) \geq v_1(0)\}$. Thus, when restricted to this set, the game has a 7ε-equilibrium. By symmetric arguments, a 6ε-equilibrium exists on the set $\{Y_2(0) \geq v_2(0)\}$. We now construct a 13ε-equilibrium on the remaining set, $\{X_1(0) < v_1(0)\} \cap \{Y_2(0) < v_2(0)\}$.

Part 6: The set $A_6 := \{X_1(0) < v_1(0)\} \cap \{Y_2(0) < v_2(0)\}$.

Fix $\eta > 0$, and for each $i \in \{1, 2\}$ let φ_i^η be a simple randomized stopping time with basis μ_i^η and delay at most δ_i that satisfies Equation (18) for every stopping time $\lambda_{3-i} \geq \mu_i^\eta$ (see Lemma 16). Let $\psi_1(\mu_2^\eta + \delta_2, \varepsilon)$ (respectively, $\psi_2(\mu_1^\eta + \delta_1, \varepsilon)$) be a simple randomized ε-optimal stopping time for player 1 in the game $\Gamma_2(\mu_2^\eta + \delta_2)$ (respectively, in the game $\Gamma_1(\mu_1^\eta + \delta_1)$); that is a stopping time that achieves the infimum in (8) up to ε, for $t = \mu_2^\eta + \delta_2$ (the infimum in (7) up to ε, respectively for $t = \mu_1^\eta + \delta_1$).

Set $\mu^\eta = \min\{\mu_1^\eta, \mu_2^\eta\}$. We further divide A_6 into six $\mathcal{F}_{\mu^\eta}$-measurable subsets; the definition of $(\varphi_1^*, \varphi_2^*)$ is different in each subset, and is given in the second and third columns of Table 1. Under $(\varphi_1^*, \varphi_2^*)$ the game will be stopped at time μ^η or during a short interval after time μ^η, if $\mu^\eta < \infty$, and will not be stopped if $\mu^\eta = \infty$.

Table 1. The randomized stopping times $(\varphi_1^*, \varphi_2^*)$ on A_6 with the payoff to player 1.

Subset	φ_1^*	φ_2^*	$\gamma_1(\varphi_1^*, \varphi_2^*)$
$A_{61} := A_6 \cap \{\mu_1^\eta < \mu_2^\eta\}$	φ_1^η	$\psi_2(\mu^\eta + \delta_1)$	$\geq X_1(\mu^\eta) - 2\varepsilon$
$A_{62} := A_6 \cap \{\mu_2^\eta < \mu_1^\eta\}$	$\psi_1(\mu^\eta + \delta_2)$	φ_2^η	$\geq Y_1(\mu^\eta) - 2\varepsilon$
$A_{63} := A_6 \cap \{\mu_1^\eta = \mu_2^\eta = \infty\}$	∞	∞	$= \xi_1$
$A_{64} := A_6 \cap \{\mu_1^\eta = \mu_2^\eta < \infty\} \cap \{Z_1(\mu_1^\eta) < Y_1(\mu_1^\eta)\}$	$\psi_1(\mu_1^\eta + \delta_2, \varepsilon)$	μ^η	$= Y_1(\mu^\eta)$
$A_{65} := A_6 \cap \{\mu_1^\eta = \mu_2^\eta < \infty\} \cap \{Z_2(\mu_1^\eta) < X_2(\mu_1^\eta)\}$	μ^η	$\psi_2(\mu^\eta + \delta_1,\varepsilon)$	$= X_1(\mu^\eta)$
$A_{66} := A_6 \cap \{\mu_1^\eta = \mu_2^\eta < \infty\} \cap \{Y_1(\mu_1^\eta) \leq Z_1(\mu_1^\eta)\} \cap \{X_2(\mu^\eta) \leq Z_2(\mu^\eta)\}$	μ^η	μ^η	$= Z_1(\mu^\eta)$

We argue that when restricted to A_6, the pair $(\varphi_1^*, \varphi_2^*)$ is a 13ε-equilibrium. Note that the roles of the two players in the definition of $(\varphi_1^*, \varphi_2^*)$ are symmetric: $\varphi_1^* = \varphi_2^*$ on A_{63} and A_{66}, and the role of player 1 (respectively, player 2) in A_{61} and A_{64} is similar to the role of player 2 (respectively, player 1) in A_{62} and A_{65}. To prove that $(\varphi_1^*, \varphi_2^*)$ is a 13ε-equilibrium, it is therefore sufficient to prove that the probability that player 1 can profit more than 3ε by deviating from φ_1^* is at most 10ε.

We start by bounding the payoff $\gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_{\mu^\eta})$ (the bound that we derive appears on the right-most column in Table 1), and by showing that

$$\gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_{\mu^\eta}) \geq v_1(\mu^\eta) - 3\varepsilon - \eta \quad \text{on } A_6 \setminus A_{63}. \tag{26}$$

We prove this in turn on each of the sets $A_{61}, \ldots, A_{66}$:

- On A_{61} we have $\mu^\eta = \mu_1^\eta$, and the game is stopped by player 1 between times μ^η and $\mu^\eta + \delta_1$, so that by (19) we have

$$\mathbf{P}(A_{61} \cap \{\gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_{\mu^\eta}) < X_1(\mu^\eta) - \varepsilon\}) \leq \varepsilon. \tag{27}$$

  By (13) we have $X_1(\mu^\eta) \geq v_1(\mu^\eta) - \eta$, and therefore (26) holds on A_{61}.
- On A_{62} we have $\mu^\eta = \mu_2^\eta$, and the game is stopped by player 2 between times μ^η and $\mu^\eta + \delta_2$, so that by (20) we have

$$\mathbf{P}(A_{62} \cap \{\gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_{\mu^\eta}) < Y_1(\mu^\eta) - \varepsilon\}) \leq \varepsilon. \tag{28}$$

  By (15) we have $X_1(\mu^\eta) < Y_1(\mu^\eta)$ on A_{62}, so that by Lemma 11 we have $Y_1(\mu^\eta) \geq v_1(\mu^\eta)$. It follows that (26) holds on A_{62}.
- On A_{63} no player ever stops, and therefore $\gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_{\mu^\eta}) = \xi_1$.
- On A_{64} player 2 stops at time μ^η, and therefore $\gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_{\mu^\eta}) = Y_1(\mu^\eta)$. By Lemma 12, on A_{64} we have

$$v_1(\mu^\eta) \leq \max\{Y_1(\mu^\eta), Z_1(\mu^\eta)\} = Y_1(\mu^\eta),$$

  and therefore (26) holds on A_{64}.
- On A_{65} player 1 stops at time μ^η, and therefore $\gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_{\mu^\eta}) = X_1(\mu^\eta)$. By (13) we have $X_1(\mu^\eta) \geq v_1(\mu^\eta) - \eta$, and therefore (26) holds on A_{65}.

- On A_{66} both players stop at time μ^η, and therefore $\gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_{\mu^\eta}) = Z_1(\mu^\eta)$. By Lemma 12 on this set we have

$$v_1(\mu^\eta) \leq \max\{Y_1(\mu^\eta), Z_1(\mu^\eta)\} = Z_1(\mu^\eta),$$

and therefore (26) holds on A_{66}.

Fix a stopping time λ_1 for player 1. To complete the proof of Theorem 7 we prove that

$$\mathbf{P}(A_6 \cap \{\gamma_1(\lambda_1, \varphi_2^*) > \gamma_1(\varphi_1^*, \varphi_2^*) + 3\varepsilon\}) \leq 10\varepsilon.$$

- On the set $A_6 \cap \{\lambda_1 < \mu^\eta\}$ we have by the definition of μ_1^η, since $\mu^\eta \leq \mu_1^\eta$, by Lemma 15, and by (26),

$$\begin{aligned}
\gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_{\lambda_1}) &= X_1(\lambda_1) \\
&< v_1(\lambda_1) - \eta \\
&\leq \mathbf{E}[v_1(\mu^\eta) 1_{A_6 \cap \{\lambda_1 < \mu^\eta < \infty\}} + \xi_1 1_{A_6 \cap \{\lambda_1 < \mu^\eta = \infty\}} | \mathcal{F}_{\lambda_1}] - \eta \\
&\leq \gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_{\lambda_1}) + 3\varepsilon,
\end{aligned} \tag{29}$$

where the last inequality holds by (26) and because the payoff of player 1 on A_{63} is ξ_1.

- On the set $A_{61} \cap \{\mu^\eta \leq \lambda_1\}$ we have by the definition of φ_2^*

$$\gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_{\mu^\eta}) = \mathbf{E}[X_1(\lambda_1) 1_{\{\lambda_1 \leq \mu^\eta + \delta_1\}} + (v_1(\mu^\eta + \delta_1) + \varepsilon) 1_{\{\mu^\eta + \delta_1 < \lambda_1\}} | \mathcal{F}_{\mu^\eta}].$$

By (19), (21) and (13) we have

$$\mathbf{P}(\gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_{\mu^\eta}) > X_1(\mu^\eta) + 2\varepsilon) \leq 2\varepsilon.$$

By (27) we deduce that

$$\mathbf{P}(\gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_{\mu^\eta}) > \gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_{\mu^\eta}) + 3\varepsilon) \leq 3\varepsilon. \tag{30}$$

- On the set $A_{62} \cap \{\mu^\eta \leq \lambda_1\}$ we have by the definition of φ_2^*

$$\gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_{\mu^\eta}) = \mathbf{E}\left[X_1(\lambda_1) 1_{\{\mu^\eta \leq \lambda_1 < \varphi_2^*\}} + Y_1(\varphi_2^*) 1_{\{\varphi_2^* \leq \lambda_1\}} | \mathcal{F}_{\mu^\eta}\right].$$

By (19), (20), since $\mu_2^\eta < \mu_1^\eta$ on A_{62}, and by (15),

$$\mathbf{P}\left(\gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_{\mu^\eta}) > \mathbf{E}\left[(Y_1(\mu^\eta) + \varepsilon) 1_{\{\mu^\eta \leq \lambda_1 < \varphi_2^*\}} + (Y_1(\mu^\eta) + \varepsilon) 1_{\{\varphi_2^* \leq \lambda_1\}} | \mathcal{F}_{\mu^\eta}\right]\right) \leq 2\varepsilon.$$

By (28) we deduce that

$$\mathbf{P}(A_{62} \cap \{\gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_{\mu^\eta}) > \gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_{\mu^\eta}) + 2\varepsilon\}) \leq 3\varepsilon. \tag{31}$$

- On the set $A_{63} \cap \{\mu_2^\eta \leq \lambda_1\}$ we have $\mu^\eta = \lambda_1 = \infty$, so that

$$\gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_{\mu^\eta}) = \xi_1 = \gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_{\mu^\eta}) \quad \text{on } A_{63} \cap \{\mu_2^\eta \leq \lambda_1\}. \tag{32}$$

- On the set $A_{64} \cap \{\mu^\eta \leq \lambda_1\}$ we have

$$\begin{aligned}\gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_{\mu^\eta}) &= \mathbf{E}\big[Z_1(\mu^\eta) 1_{\{\lambda_1 = \mu^\eta\}} + Y_1(\mu^\eta) 1_{\{\mu^\eta < \lambda_1\}}\big] \\ &\leq Y_1(\mu^\eta) = \gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_{\mu^\eta}).\end{aligned} \tag{33}$$

- On the set $A_{65} \cap \{\mu^\eta \leq \lambda_1\}$ we have by the definition of φ_2^*

$$\gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_{\mu^\eta}) = \mathbf{E}\big[X_1(\lambda_1) 1_{\{\mu^\eta \leq \lambda_1 < \mu^\eta + \delta_1\}} + (v_1(\mu^\eta + \delta_1) + \varepsilon) 1_{\{\mu^\eta + \delta_1 \leq \lambda_1\}} | \mathcal{F}_{\mu^\eta}\big].$$

By (19), (21) and (13) we have

$$\mathbf{P}(A_{65} \cap \{\mu^\eta \leq \lambda_1\} \cap \{\gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_{\mu^\eta}) > X_1(\mu^\eta) + 2\varepsilon\}) \leq 2\varepsilon.$$

Because $\gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_{\mu^\eta}) = X_1(\mu^\eta)$ on A_{65}, we obtain

$$\mathbf{P}(A_{65} \cap \{\mu^\eta \leq \lambda_1\} \cap \{\gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_{\mu^\eta}) > \gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_{\mu^\eta}) + 2\varepsilon\}) \leq 2\varepsilon. \tag{34}$$

- On the set $A_{66} \cap \{\mu^\eta \leq \lambda_1\}$ we have

$$\begin{aligned}\gamma_1(\lambda_1, \varphi_2^* | \mathcal{F}_{\mu^\eta}) &= Z_1(\mu^\eta) 1_{\{\lambda_1 = \mu^\eta\}} + Y_1(\mu^\eta) 1_{\{\mu^\eta < \lambda_1\}} \\ &\leq Z_1(\mu^\eta) = \gamma_1(\varphi_1^*, \varphi_2^* | \mathcal{F}_{\mu^\eta}).\end{aligned} \tag{35}$$

From (30)–(35) we deduce that on $A_6 \cap \{\mu^\eta \leq \lambda_1\}$

$$\mathbf{P}(\gamma_1(\lambda_1, \varphi_2^* | \mu^\eta) > \gamma_1(\varphi_1^*, \varphi_2^* | \mu^\eta) + 3\varepsilon) \leq 10\varepsilon. \tag{36}$$

Because (29) and (36) hold for every stopping time λ_1 for player 1, it follows that $(\varphi_1^*, \varphi_2^*)$ is a 13ε-equilibrium on A_6, as desired.

Proof of Theorem 8. To prove that if $Z_1(t) \in \mathrm{co}\{X_1(t), Y_1(t)\}$ and $Z_2(t) \in \mathrm{co}\{X_2(t), Y_2(t)\}$ for every $t \geq 0$, then there is a pair of non-randomized stopping times that form an ε-equilibrium, we are going to check where randomized stopping times were used in the proof of Theorem 7, and we will see how in each case one can use non-randomized stopping times instead of randomized stopping times.

1. In Part 1 (and in the analogue part 3) we used a punishment strategy $\psi_1(\delta_0, \varepsilon)$ that in general is a non-randomized stopping time. However, by Lemma 17, when $Z_2(t) \in \mathrm{co}\{X_2(t), Y_2(t)\}$ for every $t \geq 0$, this randomized stopping time can be taken to be non-randomized.
2. In Part 4 we used, in addition to the punishment strategy $\psi_2(\delta_0, \varepsilon)$, a simple randomized stopping time for player 1. The set that we were concerned with in part 4 was the set $A_4 := \{X_1(0) \geq v_1(0)\} \cap \{X_2(0) > Y_2(0)\}$. Because $Z_2(0) \in \mathrm{co}\{X_2(0), Y_2(0)\}$

$$X_2(0) \geq Z_2(0) \geq Y_2(0).$$

But then the following pair of non-randomized stopping times is a 3ε-equilibrium when restricted to A_4:
- $\varphi_1^* := 0$: player 1 stops at time 0.

- $\varphi_2^* := \psi_2(\delta_0, \varepsilon)$: if player 1 does not stop before time δ_0, player 2 punishes him (with a non-randomized stopping time; see the first item) in the game $\Gamma_1(\delta_0)$.

3. In Part 6 randomization was used both for punishment (on A_{61}, A_{62}, A_{64} and A_{65}) and for stopping (on A_{61} and A_{62}). As mentioned above, under the assumptions of Theorem 8, for punishment one can use non-randomized stopping times. We now argue that one can modify the definition of $(\varphi_1^*, \varphi_2^*)$ on A_{61} and A_{62} so as to obtain a non-randomized equilibrium. Because of the symmetry between A_{61} and A_{62}, we show how to modify the construction only on A_{61}.
On A_{61} we have $\mu_1^\eta < \mu_2^\eta$, so that by (15) we have $Y_2(\mu_1^\eta) < X_2(\mu_2^\eta)$. Because $Z_2(\mu_1^\eta) \in \mathrm{co}\{X_2(\mu_1^\eta), Y_2(\mu_1^\eta)\}$ it follows that $Y_2(\mu_1^\eta) \leq Z_2(\mu_1^\eta) \leq X_2(\mu_2^\eta)$. But then the following pair of non-randomized stopping times is a 3ε-equilibrium on A_{61}:

- $\varphi_1^* := \mu_1^\eta$: player 1 stops at time μ_1^η.
- $\varphi_2^* := \psi_2(\mu_1^\eta + \delta_1, \varepsilon)$: if player 1 does not stop before time $\mu_1^\eta + \delta_1$, player 2 punishes him (with a non-randomized stopping time; see the first item) in the game $\Gamma_1(\mu_1^\eta + \delta_1)$. □

5. The result of Laraki and Solan

As mentioned earlier, Laraki and Solan [20] proved Theorem 7 for two-player zero-sum Dynkin games. We need the stronger version that is stated in Lemma 9, where the payoff is conditioned on the σ-algebra $\mathcal{F}_t$. It turns out that the arguments used by Laraki and Solan [20] prove this case as well, when one uses the following Lemma instead of Lemma 4 in Laraki and Solan [20].

LEMMA 18. Let X be a right-continuous process. For every stopping time λ and every positive $\mathcal{F}_\lambda$-measurable function ε there is a positive $\mathcal{F}_\lambda$-measurable and bounded function δ such that

$$|X(\lambda) - \mathbf{E}[X(\rho)|\mathcal{F}_\lambda]| \leq \varepsilon, \tag{37}$$

for every stopping time ρ that satisfies $\lambda \leq \rho \leq \lambda + \delta$.

PROOF. Because the process X is right continuous, the function $w \mapsto \mathbf{E}[X(\lambda + w)|\mathcal{F}_\lambda]$ is right continuous at $w = 0$ on Ω, and it is equal to $X(\lambda)$ at $w = 0$. By defining

$$\delta' = \frac{1}{2}\sup\{w > 0 : |X(\lambda) - \mathbf{E}[X(\lambda + w)|\mathcal{F}_\lambda]| \leq \varepsilon\}$$

we obtain a positive $\mathcal{F}_\lambda$-measurable function such that (37) is satisfied for every stopping time ρ, $\lambda \leq \rho \leq \lambda + \delta'$. The proof of the Lemma is complete by setting $\delta = \min\{\delta', 1\}$. □

This Lemma can also be used to adapt the proof of Proposition 7 in Laraki and Solan [20] in order to prove Lemma 10, which states that the value process is right continuous.

One can use Lemma 18 to improve some of the bounds given in Section 4. We chose not to use this Lemma in the paper, so as to unify the arguments given for the various bounds.

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Notes

1. Email: rida.laraki@polytechnique.edu.
2. Our results hold for the larger class of $\mathcal{D}$ payoff processes defined by Dellacherie and Meyer [7, §II-18]. This class contains in particular integrable processes.
3. A statement *holds* on a measurable set A if and only if the set of points in A that do not satisfy the statement has probability 0.
4. The additional ε arises because in Part 1 we had $\gamma_1(\varphi_1^*, \varphi_2^*) = X_1(0)$, whereas in Part 4 we have $\mathbf{P}(A_4 \cap \{\gamma_1(\varphi_1^*, \varphi_2^*) < X_1(0) - \varepsilon\}) \leq \varepsilon$.

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