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Quitting games – An example

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Abstract. Quitting games are multi-player sequential games in which, at every stage, each player has the choice between *continuing* and *quitting*. The game ends as soon as *at least* one player chooses to quit; each player i then receives a payoff r_S^i, which depends on the set S of players that did choose to quit. If the game never ends, the payoff to each player is zero.

We exhibit a four-player quitting game, where the "simplest" equilibrium is periodic with period two. We argue that this implies that all known methods to prove existence of an equilibrium payoff in multi-player stochastic games are therefore bound to fail in general, and provide some geometric intuition for this phenomenon.

Key words: stochastic games, games of timing, cyclic equilibrium

1. Introduction

Quitting games are I-player sequential games in which, at any stage, each player has the choice between *continuing* and *quitting*. The game ends as soon as *at least* one player chooses to quit; the payoff to player $i \in I$ is r_S^i, where $S \subseteq I$ is the set of players that did choose to quit at that stage. If the game never ends, the payoff to each player is 0.

A quitting game is therefore a multi-player stochastic game of the simplest kind. There is only one history of play that does not lead to termination. A strategy of player i is a sequence $\mathbf{x}^i = (x_n^i)_{n \geq 0}$, where x_n^i is the probability that player i quits at stage n, provided the game has not terminated before. The strategy $\mathbf{x}$ is *stationary* if x_n^i is independent of n.

It is not known whether quitting games have an ε-equilibrium for every $\varepsilon > 0$. We briefly review existing results.

In the case of *two* players, stationary ε-equilibria do exist, for every $\varepsilon > 0$, see Flesch et al. (1996). A *three*-player example was devised by Flesch et al. (1997), with no stationary ε-equilibrium. In this example there exist equilibrium payoffs in the convex hull of the vectors $r_{\{i\}} \in \mathbf{R}^I$, $i \in I$. Moreover, there exist corresponding ε-equilibrium profiles $\mathbf{x}$ that are periodic (w.r.t. time) and such that the mixed move x_n^i is arbitrarily close to 0, for every stage n and every player i.

A complete analysis was provided in Solan (1999), for the more general class of three-player absorbing games.[1] Solan proved the existence of (uniform) ε-equilibrium profiles, by means of analyzing the limit behavior of stationary equilibria of a modified discounted game, when the discount factor goes to zero. This generalizes the method introduced by Vrieze and Thuijsman (1989) for the analysis of two-player absorbing games. In both of these proofs, the limit profile $\mathbf{x}$ is either a stationary equilibrium or is such that termination occurs with probability zero. In the latter case, an ε-equilibrium can be defined that plays a perturbation of $\mathbf{x}$. In all other known existence proofs of equilibrium payoffs for multi-player undiscounted stochastic games, see, e.g., Flesch et al. (1996), Thuijsman and Raghavan (1997), Solan (2000) or Vieille (2000a, 2000b), close inspection of the proofs reveals a similar dichotomy.

The main purpose of this note is to demonstrate that *all* these methods are bound to fail for four-player quitting games – hence for more complex stochastic games with more players. We provide a four-player example, where there is neither (1) a stationary ε-equilibrium for every $\varepsilon > 0$, nor (2) an equilibrium payoff in the convex hull of $\{r_{\{i\}}, i \in I\}$. Actually, the "simplest" equilibrium in this example is periodic with period 2, in which the probability of quitting in every stage is bounded away from zero.

The paper is organized as follows. In Section 2, we provide a geometric understanding of what is specific to two- and three-player games. Next, we define the game in Section 3 and prove our claims.

In a companion paper (Solan and Vieille (2001)) we introduced new tools and provided sufficient conditions on the payoffs under which quitting games admit an ε-equilibrium, for every $\varepsilon > 0$.

2. Two- and three-player quitting games

We here consider quitting games with at most three players and discuss the result below.

Proposition 1. *For every $\varepsilon > 0$ and every quitting game with at most three players, there exists an ε-equilibrium $\mathbf{x} = (x_n^i)_{i \in I, n \in \mathbf{N}}$ such that either $\mathbf{x}$ is a stationary profile or $x_n^i \leq \varepsilon$ for every $n \in \mathbf{N}$ and every $i \in I$.*

As discussed in the Introduction, this proposition follows immediately from Solan (1999). We shall here sketch a geometric proof. We discuss two-player and three-player games in turn.

We first introduce a few notations. We denote by c^i (continue) and q^i (quit) the two actions of player i. We let $\mathbf{a}_n^i$ be the action played by player i at

[1] An *absorbing game* is a stochastic game with a single non-absorbing state.

stage n, denote by $t = \min\{n \geq 1, \mathbf{a}_n^i = q^i$ for some player $i \in I\}$ the stage in which the game terminates,[2] and by S_t the set of players that choose to quit at that stage. Given a profile $\mathbf{x}$ of strategies, the expected payoff to player i is

$$\gamma^i(\mathbf{x}) = \mathbf{E}_{\mathbf{x}}[r^i_{S_t} \mathbf{1}_{t<+\infty}],$$

where $\mathbf{E}_{\mathbf{x}}$ stands for the expectation with respect to the probability distribution induced by $\mathbf{x}$ over the set of plays. We let c denote the profile of actions (c^i), and by $\mathbf{c}^i$ (resp. $\mathbf{q}^i$) the pure stationary strategy of player i that plays repeatedly c^i (resp. q^i).

2.1. Two-player quitting games

For notational convenience, we represent a two-player quitting game as

	c^2	q^2
c^1		(b_1, b_2)
q^1	(a_1, a_2)	(d_1, d_2)

If there is a pure stationary equilibrium we are done. Otherwise either $a_1 > 0$ or $b_2 > 0$ (otherwise $(\mathbf{c}^1, \mathbf{c}^2)$ is an equilibrium). Assume w.l.o.g. that $a_1 > 0$. Then $a_2 < d_2$ (otherwise $(\mathbf{q}^1, \mathbf{c}^2)$ is an equilibrium), which implies that $d_1 < b_1$ (otherwise $(\mathbf{q}^1, \mathbf{q}^2)$ is an equilibrium), which implies that $b_2 < 0$ (otherwise $(\mathbf{c}^1, \mathbf{c}^2)$ is an equilibrium).

If $a_2 \geq b_2$ then the stationary profile $(\mathbf{x}^1, \mathbf{c}^2)$ is an ε-equilibrium, where $x_n^1 = \eta$, and $\eta \in (0,1)$ is sufficiently small.

If $a_2 < b_2$ then the stationary profile $(\mathbf{x}^1, \mathbf{q}^2)$ is an ε-equilibrium, where $\mathbf{x}^1$ is defined as above.

Therefore, any two-player quitting games has a stationary ε-equilibrium. Note that equilibria need not exist, as e.g. in the zero-sum game

	c^2	q^2
c^1		$(1, -1)$
q^1	$(1, -1)$	$(0, 0)$

2.2. Three-player quitting games

A complete discussion would be both tedious and repetitive. We shall only deal with the case where $r^i_{\{i\}} > 0$ for each $i \in I$. It can be checked that the other cases do not involve additional ideas.

We normalize the payoffs to have $r^i_{\{i\}} = 1$ for each $i \in I$. We organize the discussion according to the configuration of payoffs. The different cases are

[2] By convention, the minimum of an empty set is $+\infty$.

exhaustive, but not mutually exclusive. All strategies are stationary unless explicitly specified. Abusing notations, for every $x \in [0,1]$ and every player $i \in I$ we denote by $(1-x)\mathbf{c}^i + x\mathbf{q}^i$ the stationary strategy in which player i quits at every stage with probability x. For convenience, we sometimes refer to this stationary strategy simply as x.

Given $\varepsilon \in (0,1]$, set $T_\varepsilon = \{x \in [0,1]^3 \mid \sum_{i=1}^3 x^i = \varepsilon\}$, and $\Delta_\varepsilon = \{x \in [0,1]^3 \mid \sum_{i=1}^3 x^i \geq \varepsilon\}$. The set Δ_ε is a subset of the set of stationary profiles. It contains all profiles for which the probability of termination in any given stage is non-negligible.

Case 0: There exists $\varepsilon \in (0,1)$ such that, for every profile $x \in T_\varepsilon$, there is at least one player i whose unique best reply to x is $\mathbf{q}^i$.

We prove that the game has a stationary equilibrium. The proof is based on a standard fixed-point argument, applied to the best replies of a constrained game. Loosely speaking, on T_ε, the best-reply correspondence is pointing inwards Δ_ε. Hence, its restriction to Δ_ε has a fixed point, which is a stationary equilibrium of the game.

We let $C \in \mathbf{R}$ be an upper bound for all payoffs in the game.

For every $x \in T_\varepsilon$ let $I_x \subseteq I$ be the set of players i such that $\gamma^i(x^{-i}, q^i) - \gamma^i(x) > 0$. The assumption tells us that I_x is not empty for every $x \in T_\varepsilon$. Since $\gamma^i(x)$ and $\gamma^i(x^{-i}, q^i)$ are continuous over the compact set T_ε, $\rho = \min_{x \in T_\varepsilon} \max_{i \in I_x} \{\gamma^i(x^{-i}, q^i) - \gamma^i(x)\} > 0$.

It follows that there is $\varepsilon_0 > \varepsilon$ such that for every $\varepsilon_1 \in [\varepsilon, \varepsilon_0]$, and every $x \in T_{\varepsilon_1}$ there is a player i such that $\gamma^i(x^{-i}, q^i) - \gamma^i(x) > \rho/2$. Fix $\varepsilon_1 \in (\varepsilon, \min\{\varepsilon_1, \varepsilon + 1/C\})$ and define a continuous function $f : \Delta_\varepsilon \to \Delta_\varepsilon$ by

$$f^i(x)$$

$$= \begin{cases} x^i + (\varepsilon_1 - \varepsilon)(\gamma^i(x^{-i}, q^i) - \gamma^i(x)) & \gamma^i(x^{-i}, q^i) \geq \gamma^i(x) \\ x^i(1 + \min\{1, \rho/4\}(\varepsilon_1 - \varepsilon)(\gamma^i(x^{-i}, q^i) - \gamma^i(x))) & \gamma^i(x^{-i}, q^i) < \gamma^i(x). \end{cases}$$

Since f is continuous, it has a fixed point in Δ_ε, which is a stationary equilibrium.

Case 1: $r^2_{\{1\}}, r^3_{\{1\}} \geq 1$.

In that case, both players 2 and 3 are at worst indifferent between quitting alone or waiting for player 1 to quit. The stationary profile $((1-\eta)\mathbf{c}^1 + \eta\mathbf{q}^1, \mathbf{c}^2, \mathbf{c}^3)$ is an ε-equilibrium, provided η is sufficiently small.

This analysis remains valid when the roles of the players are permuted.

Case 2: There is no convex combination $\alpha_1 r_{\{1\}} + \alpha_2 r_{\{2\}} + \alpha_3 r_{\{3\}}$ of the three vectors $(r_{\{1\}}, r_{\{2\}}, r_{\{3\}})$ such that $\alpha_1 r_{\{1\}} + \alpha_2 r_{\{2\}} + \alpha_3 r_{\{3\}} \geq (1,1,1)$.

By compactness, there is $\rho > 0$ such that in every convex combination of $r_{\{1\}}$, $r_{\{2\}}$ and $r_{\{3\}}$, at least one player receives at most $1-\rho$. It follows that for $\varepsilon > 0$ sufficiently small, the assumption of **Case 0** holds. In particular, there is a stationary equilibrium.

Case 3: $r^1_{\{2\}}, r^1_{\{3\}} < 1$.

One can easily verify that the assumption of **Case 1** or **Case 2** is satisfied.

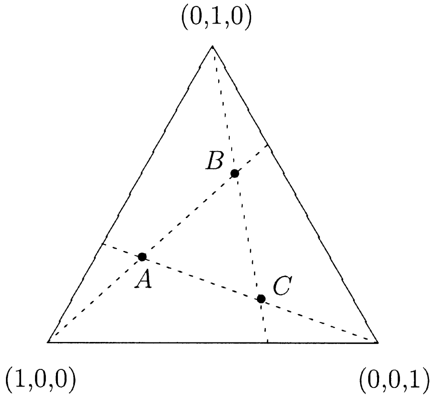

Fig. 1

Case 4: There is a convex combination $\alpha_1 r_{\{1\}} + \alpha_2 r_{\{2\}} + \alpha_3 r_{\{3\}}$ of the three vectors $(r_{\{1\}}, r_{\{2\}}, r_{\{3\}})$ such that $\alpha_1 r_{\{1\}} + \alpha_2 r_{\{2\}} + \alpha_3 r_{\{3\}} = (1,1,1)$.

The stationary profile $((1-\eta\alpha_1)\mathbf{c}^1 + \eta\alpha_1\mathbf{q}^1, (1-\eta\alpha_2)\mathbf{c}^2 + \eta\alpha_2\mathbf{q}^2, (1-\eta\alpha_3)\mathbf{c}^3 + \eta\alpha_3\mathbf{q}^3)$ is an ε-equilibrium, provided η is sufficiently small.

We next introduce a notational convention. For $i \neq j$, we shall write $r^i_{\{j\}} =' +'$ if $r^i_{\{j\}} \geq 1$ and $r^i_{\{j\}} =' -'$ if $r^i_{\{j\}} < 1$. If neither **Case 1** nor **Case 3** holds (nor their analogue symmetric cases), the triplet $(r_{\{1\}}, r_{\{2\}}, r_{\{3\}}) \in \mathbf{R}^9$ is either of the form $((1,+,-),(-,1,+),(+,-,1))$ or $((1,-,+),(+,1,-),(-,+,1))$. Each of these two situations is reducible to the other by a permutation of two players. We will proceed under the assumption that

$$(r_{\{1\}}, r_{\{2\}}, r_{\{3\}}) \text{ is of the form } ((1,+,-),(-,1,+),(+,-,1)).$$

Hence, player 2 (*resp.* player 3, player 1) is happy if player 1 (*resp.* player 2, player 3) quits, but gets a low payoff if player 3 (*resp.* player 1, player 2) quits.

Case 5: There is a convex combination $\alpha_1 r_{\{1\}} + \alpha_2 r_{\{2\}} + \alpha_3 r_{\{3\}}$ of the three vectors $(r_{\{1\}}, r_{\{2\}}, r_{\{3\}})$ such that $\alpha_1 r_{\{1\}} + \alpha_2 r_{\{2\}} + \alpha_3 r_{\{3\}} \geq (1,1,1)$.

The set of such $(\alpha_1, \alpha_2, \alpha_3)$ is defined by three halfspaces and by the conditions $\alpha_i \geq 0$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$. It is therefore a triangle (reduced to a singleton if and only if **Case 4** holds).

The vertices of this triangle are labelled A, B, C in such a way that players 1 and 3 (resp. 1 and 2, 2 and 3) get a payoff 1 under the convex combination $A = (\alpha_1^A, \alpha_2^A, \alpha_3^A)$ (resp. B, C) (see Figure 1).

We write A (resp. B, C) as a convex combination of $(1,0,0)$ and B (resp. of $(0,1,0)$ and C, of $(0,0,1)$ and A):

$$A = \beta_1(1,0,0) + (1-\beta_1)B$$

$$B = \beta_2(0,1,0) + (1-\beta_2)C$$

$$C = \beta_3(0,0,1) + (1-\beta_3)A.$$

Fix $M \in \mathbf{N}$, large enough. Define a non-stationary profile σ as follows. Players 1, 2 and 3 (in that order) alternate indefinitely as follows. During M stages, player i quits with probability $\frac{\beta_i}{M}$ (while the other two players continue). Depending on who starts first, the payoff induced by σ is close to the payoff associated with the convex combination A, B or C respectively. Moreover, the profile σ is an ε-equilibrium of the quitting game.

3. The example

Here we analyze the following four player quitting game:

4

3 (top), **4** (left) — players **1** (rows), **2** (columns):

1	*continue*	$4,1,0,0$
	$1,4,0,0$	$1,1,1,1$

3 (top), **4** (right) — players **1** (rows), **2** (columns):

1	$0,0,4,1$	$1,1,0,1$
	$1,0,1,1$	$0,1,0,0$

3 (bottom), **4** (left) — players **1** (rows), **2** (columns):

1	$0,0,1,4$	$0,1,1,1$
	$1,1,1,0$	$1,0,0,0$

3 (bottom), **4** (right) — players **1** (rows), **2** (columns):

1	$1,1,1,1$	$0,0,1,0$
	$0,0,0,1$	$-1,-1,-1,-1$

Fig. 2

In this game player 1 chooses a row (top row = continue), player 2 chooses a column (left column = continue), player 3 chooses either the top two matrices or the bottom two matrices, (top two matrices = continue) and player 4 chooses either the left two matrices or the right two matrices (left two matrices = continue).

Note that there are the following symmetries in the payoff function: for every 4-tuple of actions (a,b,c,d) we have:

$$v^1(a,b,c,d) = v^2(b,a,d,c),$$

$$v^1(a,b,c,d) = v^4(c,d,b,a) \quad \text{and}$$

$$v^2(a,b,c,d) = v^3(c,d,b,a),$$

where $v^i(a,b,c,d)$ is the payoff to i if the action combination is (a,b,c,d) $(v^i(c^1,c^2,c^3,c^4) = 0)$.

In Solan and Vieille (2001), it is proven that this game admits a cyclic equilibrium profile $\mathbf{y}$ with period 2 and with the following structure:

$$y_n = \begin{cases} (x,0,z,0) & n \text{ odd} \\ (0,x,0,z) & n \text{ even} \end{cases}$$

where $x, z \in]0,1[$ are independent of n; that is, at odd stages players 2 and 4 continue, while 1 and 3 quit with positive probability, whereas at even stages 1 and 3 continue, while 2 and 4 quit with positive probability.

We shall now prove the following:

Proposition 2. *The game does not admit a stationary equilibrium.*

Proposition 3. *For every $\varepsilon > 0$ small enough, the game does not admit an ε-equilibrium* $\mathbf{x}$ *such that $\|x_n - c\| < \varepsilon$ for every $n \in \mathbf{N}$.*

It follows from Propositions 2 and 3 that the game does not admit a stationary ε-equilibrium, provided ε is small enough. Indeed, let us argue by contradiction, and assume that for every $\varepsilon > 0$ there exists a stationary ε-equilibrium x_ε. Let x_* be an accumulation point of $\{x_\varepsilon\}$ as $\varepsilon \to 0$. If x_* is terminating ($x_* \neq c$) then it is a stationary 0-equilibrium, which is ruled out by Proposition 2. Otherwise, $x_* = c$, and then, for $\varepsilon > 0$ sufficiently small, there is an ε-equilibrium $\mathbf{x}$ where $\|x_n - c\| < \varepsilon$, which is ruled out by Proposition 3.

Proposition 2 is proved in section 3.1, while Proposition 3 is proved in section 3.2.

3.1. No stationary equilibria

We check here that the game has no stationary equilibrium. We organize the discussion according to the number of players who play both actions with positive probability.

3.1.1. No non-fully mixed stationary equilibrium

We prove here that there is no stationary equilibrium in which at least one player plays a pure strategy.

It is immediate to check that there is no stationary equilibrium in which at least three players play pure stationary strategies.

We shall now verify that there is no stationary equilibrium where two players play pure stationary strategies. Using the symmetries in the payoff function, it is enough to consider the cases where either player 3 and 4 play pure strategies, or players 2 and 4 play pure strategies.

Assume first that there is an equilibrium in which players 3 and 4 play pure stationary strategies. The strategies of players 1 and 2 form then an equilibrium of a 2×2 game. We will see that these two-player games have only pure equilibria. The four-player game would thus have an equilibrium in pure stationary strategies – a contradiction. In the first three cases, the induced game is equivalent to a one-shot game. In the last case, it is a quitting game.

Case 1: Players 3 and 4 play (q^3, q^4): the unique equilibrium in the induced game is (c^1, c^2).

Case 2: Players 3 and 4 play (c^3, q^4): the unique equilibrium is (c^1, q^2).

Case 3: Players 3 and 4 play (q^3, c^4): symmetric to case 2.

Case 4: Players 3 and 4 play (c^3, c^4): the unique equilibria are (q^1, c^2) and (c^1, q^2).

We shall now see that there is no stationary equilibrium where players 2 and 4 play pure actions, by analyzing the induced game between players 1 and 3.

Case 1: Players 2 and 4 play (c^2, c^4): the induced game has a unique equilibrium (q^1, q^3).

Case 2: Players 2 and 4 play (q^2, c^4): the unique equilibrium in the induced game is $\left(\frac{1}{2}c^1 + \frac{1}{2}q^1, \frac{1}{4}c^3 + \frac{3}{4}q^3\right)$. Player 2 would receive $\frac{5}{8}$, but he would get 1 by playing c^2.

Case 3: Players 2 and 4 play (c^2, q^4): the unique equilibrium is (q^1, c^3).

Case 4: Players 2 and 4 play (q^2, q^4): the unique equilibrium is (c^1, q^3).

Next, we check that there is no stationary equilibrium where one player, say player 4, plays a pure strategy, and all the other players play a fully mixed strategy. We denote by (x, y, z) the fully mixed stationary equilibrium in the three-player game when player 4 plays some pure stationary strategy.

Assume first that player 4 plays q^4. Then, in order to have player 2 indifferent, we should have

$$x(1 - z) = z - (1 - x)(1 - z),$$

which implies that $z = 1/2$. In order to have player 1 indifferent, we should have

$$(1 - y)z + y(1 - z) = yz - (1 - y)(1 - z),$$

which solves to $yz = 1/2$, and therefore $y = 1$, which is pure.

Assume now that player 4 plays c^4. First we note that $x < 1/2$, otherwise player 3 prefers to play q^3 over c^3. Next, if player 2 is indifferent between his actions, then

$$\frac{(1 - x)(1 + 3z)}{1 - xz} = x + (1 - x)z,$$

or, equivalently,

$$(1 - x)(1 + 2z + xz^2) = (1 - xz)x.$$

Since $x < 1/2$, it follows that $1 - x > x$. Therefore it follows that

$$1 + 2z + xz^2 < 1 - xz,$$

which is clearly false.

3.1.2. No fully mixed stationary equilibrium

We prove now that there is no fully mixed stationary equilibrium. We shall first write the best-reply conditions. Next, we shall check that these can not be satisfied simultaneously.

We focus on player 1. Let $(y, z, t) \in (0, 1)^3$ be a given fully mixed profile of players 2, 3 and 4.

By playing c^1 at stage 1 and the mixed action $x \in (0, 1)$ in all subsequent stages, player 1's expected payoff is

$$\alpha(y, z, t) := yzt(\gamma^1(x, y, z, t) - 2) - 2yz + 3zt - yt + y + z.$$

On the other hand, by playing q^1 at stage 1, player 1's expected payoff is

$$\beta(y, z, t) := t + (1 - t)(y + z - 1).$$

If $x \in (0, 1)$ is a stationary best reply to (y, z, t), the two payoffs are equal, and equal to $\gamma^1(x, y, z, t)$:

$$\alpha(y, z, t) = \beta(y, z, t) = \gamma^1(x, y, z, t).$$

In particular, the polynomial Δ_1 that is defined by

$$\Delta_1(y, z, t) := \alpha(\beta(y, z, t); y, z, t) - \beta(y, z, t)$$

vanishes at (y, z, t). Observe that facing (y, z, t), the stationary strategy c^1 yields a payoff in $[0, 1]$. Defining $\Delta_2(x, z, t), \Delta_3(x, y, t)$ and $\Delta_4(x, y, z)$ in a symmetric way, we have thus proved the next result.

Lemma 4. *If* $(x, y, z, t) \in (0, 1)^4$ *is a fully mixed stationary equilibrium, then,* $\Delta_1(y, z, t) = \Delta_2(x, z, t) = \Delta_3(x, y, t) = \Delta_4(x, y, z) = 0$ *and* $\gamma^i(x, y, z, t) \in [0, 1]$ for each $i = 1, 2, 3, 4$.

We shall prove (see Lemmas 6, 7 and 8 below) that there is no $(x, y, z, t) \in (0, 1)^4$ such that (i) $y = \min\{x, y, z, t\}$, (ii) $\Delta_1(y, z, t) = \Delta_4(x, y, z) = 0$ and (iii) $\gamma^1(x, y, z, t), \gamma^4(x, y, z, t) \in [0, 1]$. By symmetry, condition (i) is w.l.o.g. By Lemma 4, this will therefore imply that the game has no fully mixed stationary equilibrium.

For simplicity of notations we sometimes omit the arguments in β.

Lemma 5. $\Delta_1(t, t, t) > 0$ *for every* $t \in [0, 1]$.

Proof: $t \mapsto \Delta_1(t, t, t)$ is a polynomial in one variable. The result follows by using any method for counting the number of zeroes of a polynomial in a compact interval, e.g. Sturm's method. ■

We state a useful observation.

Fact 1: β is separately increasing on the set $\{y \leq \min\{z, t\}\}$.

Fact 2: Δ_1 is decreasing in y and increasing in z on the set $\{y \leq \min\{z, t\}\}$.

The proofs of the **Fact 1** and the first assertion in **Fact 2** are obtained by elementary algebraic manipulations, and are therefore omitted. For the second assertion in **Fact 2**, observe that $\frac{\partial \Delta_1}{\partial z}$ is decreasing in y. Since $y \leq t$, this yields $\frac{\partial \Delta_1}{\partial z}(y, z, t) \geq \frac{\partial \Delta_1}{\partial z}(t, z, t) = (\beta - 2)t^2 + t^2 z(1 - t) + 2t > 0$.

Lemma 6. $\Delta_1 > 0$ *on the set* $\{y \le t \le z\}$.

Proof: By **Fact 2**, $\Delta_1(y,z,t) \ge \Delta_1(t,t,t) > 0$ if $y \le t \le z$. ■

Lemma 7. $\Delta_4 > 0$ *on the set* $\left\{y \le z \le \frac{1}{2}\right\} \cap \{\gamma^4 \ge 0\}$.

Proof: Indeed, if $\gamma^4 \ge 0$, one has

$$\Delta_4(x,y,z) \ge -2xyz - 2xz + 4xy + 1 - 2y = y(-2xz + 4x - 2) + 1 - 2xz.$$

Denote by $f_{x,z}(y)$ the affine function in y that appears in the right-hand side. Since $f_{x,z}(0) = 1 - 2xz > 0$ and $f_{x,z}(z) = (1-2z) + 2xz(1-z) > 0$, it is positive on $\left\{y \le z \le \frac{1}{2}\right\}$. ■

Lemma 8. $\Delta_1 > 0$ *on the set* $\left\{\max\left\{y, \frac{1}{2}\right\} \le z \le t\right\}$.

Proof: We split the discussion into several steps.

Step 1: $\Delta_1 > 0$ on $\left\{y < \frac{1}{2} \le z \le t\}\right\}$.

Indeed, by **Fact 2**, $\Delta_1(y,z,t) \ge \Delta_1\left(\frac{1}{2},\frac{1}{2},t\right) = \frac{1}{2} - \frac{t}{2} + \beta\frac{t}{4} > 0$.

Step 2: $\Delta_1 > 0$ on $\left\{\frac{1}{2} \le y \le z \le t \le \frac{2}{3}\right\}$.

Indeed, by **Fact 2**, $\Delta_1(y,z,t) \ge \Delta_1(z,z,t)$. We prove below that $\Delta_1(z,z,t)$ is decreasing in z. This will imply $\Delta_1(z,z,t) \ge \Delta_1(t,t,t) > 0$, hence the claim.

An elementary computation gives

$$\frac{\partial}{\partial t}\frac{\partial}{\partial z}\{\Delta_1(z,z,t)\} = 2z(\beta-2) + 2zt(2-y-z) + 2z^2(1-t) - 2z^2t + 4.$$

Therefore, $\frac{\partial}{\partial z}\{\Delta_1(z,z,t)\}$ is increasing in t and

$$\frac{\partial}{\partial z}\Delta_1(z,z,t) \le \frac{\partial}{\partial z}\Delta_1\left(z,z,\frac{2}{3}\right) = \frac{4}{3}z(\beta-2) + \frac{4}{9}z^2 - 4z + \frac{8}{3}. \tag{1}$$

The right-hand side in (1) is decreasing in z. It is therefore maximal for $z = \frac{1}{2}$. It is then equal to $\frac{2}{3}(\beta-1) + \frac{1}{9} < 0$ since, by **Fact 1**, $\beta(y,z,t) \le \beta\left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right) = \frac{7}{9}$.

Step 3: $\Delta_1 > 0$ on $\left\{\frac{1}{2} \le y < \frac{2}{3} \le z \le t\right\}$.

By **Fact 1**, $\beta\left(\frac{2}{3},\frac{2}{3},t\right) \ge \beta\left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right) \ge \frac{3}{4}$. Hence, by **Fact 1**,

$$\Delta_1(y,z,t) \ge \Delta_1\left(\frac{2}{3},\frac{2}{3},t\right) = \left(\beta\left(\frac{2}{3},\frac{2}{3},t\right) - 2\right)\frac{4}{9}t + \frac{1}{9} + \frac{2}{3}t$$
$$\ge \frac{t+1}{9}.$$

Step 4: $\Delta_1 > 0$ on $\left\{\frac{2}{3} \le y \le z \le t\right\}$.

By **Fact 1**, $\beta \geq \frac{3}{4}$. Plainly,

$$\frac{\partial \Delta_1}{\partial t} = (\beta - 2)yz + yzt(2 - y - z) + 4z - 2 \geq (\beta - 2)yz + 4z - 2. \tag{2}$$

The right-hand side of (2) is increasing in z. Therefore, it is minimal when $z = y$, hence at least $-\frac{5}{4}y^2 + 4y - 2$. This latter expression is itself minimized at $y = \frac{2}{3}$ where it equals $\frac{1}{9}$. Thus, Δ_1 is increasing in t. By **Fact 2**, this implies $\Delta_1(y, z, t) \geq \Delta_1(z, z, z) > 0$.

Step 5: $\Delta_1 > 0$ on $\left\{\frac{1}{2} \leq y \leq z \leq \frac{2}{3} \leq t\right\}$.

By **Facts 1** and **2**, $\beta(y, z, t) \geq \frac{2}{3}$ and $\Delta_1(y, z, t) \geq \Delta_1(z, z, t)$. Therefore,

$$\Delta_1(y, z, t) \geq -\frac{4}{3}z^2 t - 2z^2 + 4zt + 1 - 2t. \tag{3}$$

Let f be the function defined by the right-hand side of (3), and $t \geq \frac{2}{3}$. The function $f(\cdot, t)$ is a quadratic concave function in z. Since

$$f\left(\frac{1}{2}, t\right) = \frac{1}{2} - \frac{t}{3} > 0 \quad \text{and} \quad f\left(\frac{2}{3}, t\right) = \frac{2}{27}t + \frac{1}{9},$$

it is positive on $[1/2, 2/3]$. ■

3.2. No perturbed ε-equilibrium

We first present a sketch of the proof. The proof goes by contradiction. Let $\mathbf{x} = (x_n)$ be an ε-equilibrium such that $\|x_n - c\| < \varepsilon$ for each n. Since each player gets a positive payoff when quitting alone, the probability that the game terminates in finite time is close to one. Moreover, since x_n is close to 0, the quitting coalition is a singleton with high probability. In particular, the sum of the payoffs of all four players under $\mathbf{x}$ is close to 5. Hence, at least one player gets a payoff substantially higher than 1 under $\mathbf{x}$, while no player receives a payoff that is much below one. There is no convex combination of r_1, r_2 and r_3 which satisfies these conditions. Therefore, the probability that player 4 belongs to the quitting coalition is bounded away from zero. By symmetry, the same holds for each player $i \in I$.

Next, we claim that there is no such ε-equilibrium that gives to players 1 and 2 (or 3 and 4) a payoff substantially higher than one. Indeed, assume such an equilibrium were to exist. In the first stage of the game, both players 1 and 2 would choose to continue with very high probability, since the payoff obtained by quitting is approximately 1. Moreover, by the ε-equilibrium property, they will do so in every stage n such that their expected payoff, starting from stage n, is higher than one, unless the probability that the game reaches stage n is close to zero. (This is specific to the class of quitting games.) Therefore, as long as their continuation payoff exceeds 1 and the probability of surviving is not too small, players 1 and 2 will not contribute to the quitting coalition. However, as long as players 1 and 2 do not contribute, their continuation payoff increases. Indeed, the expected payoff starting from today is a weighted average of the payoff received if someone quits today and of the expected payoff starting from tomorrow. Since the payoff to players 1 and 2 is below one if only player 3 or 4 quits, the expected payoff starting from tomorrow must exceed the continuation payoff starting from today.

Assume now that player 1, but not player 2, gets a payoff substantially higher than 1. Let n_1 be the first stage such that the continuation payoff of player 1 is close to one. Since the continuation payoff of player 1 decreases between stages 1 and n_1, the probability that player 2 quits before stage n_1 is non-negligible. Since player 1 hardly contributes to the probability of quitting before stage n_1, the continuation payoffs of player 2 do not decrease over time up to stage n_1. Since player 2 quits with non-negligible probability, his continuation payoffs must remain close to one for a while. In particular, players 3 and 4 should not quit in those stages. This implies that the continuation payoffs of player 3 and 4 increase in these stages. After a while (stage n_1 at the latest), both continuation payoffs of players 3 and 4 are higher than one, a situation that has been ruled out above.

We now proceed to the formal proof. We let $\rho = 8$ be twice the maximal payoff in absolute value, and $N = 4$ be the number of players.

It is convenient to assume that, in any given stage, at most one player quits with positive probability. This assumption entails no loss of generality, as shown by the next lemma.

Lemma 9. *Let $\varepsilon \leq 1/8$ and let* $\mathbf{x}$ *be an ε-equilibrium such that $\|x_n - c\| < \varepsilon$ for every n. Then there exists a $12N\rho\varepsilon$-equilibrium* $\mathbf{y}$ *such that, for every $n \in \mathbf{N}$, $\|y_n - c\| < \varepsilon$ and $|\{i \in I, y_n^j > 0\}| \leq 1$.*

Proof: We define $\mathbf{y}$ by dividing each stage into four substages, and by letting each player quit in turn with the probability specified by $\mathbf{x}$. Formally, for $n \in \mathbf{N}$ and $j \in I$, we set

$$y^i_{(n-1)N+j} = \begin{cases} x_n^i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

We first compare the payoff vectors under the two profiles $\mathbf{x}$ and $\mathbf{y}$. Plainly,

$$\begin{aligned}\mathbf{P}_{\mathbf{y}}(t > nN \mid t > (n-1)N) &= \prod_{i=1}^{N}(1 - x_n^i) \\ &= \mathbf{P}_{\mathbf{x}}(t > n \mid t > n-1) \quad \text{for every } n \in \mathbf{N}.\end{aligned}$$

Observe next that, for each $j \in I$,

$$\mathbf{P}_{\mathbf{y}}(S_t = \{i\} \mid (n-1)N < t \leq nN) = \frac{x_n^i \prod_{j<i}(1 - x_n^j)}{1 - \prod_{j \in I}(1 - x_n^j)}, \quad \text{and}$$

$$\mathbf{P}_{\mathbf{x}}(S_t = \{i\} \mid t = n) = \frac{x_n^i \prod_{j \neq i}(1 - x_n^j)}{1 - \prod_{j \in I}(1 - x_n^j)}$$

The denominator is at least $1 - 4\varepsilon \geq 1/2$, hence the difference between these two probabilities is at most 2ε.

By summation over n, this yields

$$|\mathbf{P}_{\mathbf{x}}(S_t = \{i\} \mid t < +\infty) - \mathbf{P}_{\mathbf{y}}(S_t = \{i\} \mid t < +\infty)| \leq 2\varepsilon. \tag{4}$$

Under $\mathbf{y}$, no two players quit simultaneously with positive probability, therefore $\sum_{i \in I} \mathbf{P}_{\mathbf{y}}(S_t = \{i\} \mid t < +\infty) = 1$. Using (4), it follows that $\mathbf{P}_{\mathbf{x}}(|S_t| > 1 \mid t < +\infty) \leq 2N\varepsilon$.

Since $\|\gamma(\mathbf{x}) - \gamma(\mathbf{y})\| \leq \rho \sum_{S \subseteq I} |\mathbf{P}_{\mathbf{x}}(S_t = S) - \mathbf{P}_{\mathbf{y}}(S_t = S)|$, one gets

$$\|\gamma(\mathbf{x}) - \gamma(\mathbf{y})\| \leq 4N\rho\varepsilon. \tag{5}$$

Next, we prove that player i has no pure profitable deviation from $\mathbf{y}^i$.

Consider first the strategy $\mathbf{c}^i$. The above argument does not rely on the ε-equilibrium property of $\mathbf{x}$ and applies to any profile $\mathbf{x}$ such that $\|x_n - c\| \leq \varepsilon$ for every $n \in \mathbf{N}$. When applied to the profile $(\mathbf{x}^{-i}, \mathbf{c}^i)$, it yields $|\gamma^i(\mathbf{x}^{-i}, \mathbf{c}^i) - \gamma^i(\mathbf{y}^{-i}, \mathbf{c}^i)| \leq 4N\rho\varepsilon$. Since $\mathbf{x}$ is an ε-equilibrium, this yields, by (5),

$$\gamma^i(\mathbf{y}^{-i}, \mathbf{c}^i) \leq \gamma^i(\mathbf{y}) + \varepsilon + 8N\rho\varepsilon.$$

Consider next the strategy $\mathbf{q}^i_{(n-1)N+k}$ that quits at stage $(n-1)N + k$ for the first time. We compare the payoffs to player i under the two profiles $(\mathbf{y}^{-i}, \mathbf{q}^i_{(n-1)N+k})$ and $(\mathbf{x}^{-i}, \mathbf{q}^i_n)$. When mimicking the above argument, one obtains

$$\mathbf{P}_{\mathbf{y}^{-i}, \mathbf{q}^i_{(n-1)N+k}}(t \leq (n-1)N) = \mathbf{P}_{\mathbf{x}^{-i}, \mathbf{q}^i_n}(t \leq n-1), \quad \text{and}$$

$$|\mathbf{E}_{\mathbf{y}^{-i}, \mathbf{q}^i_{(n-1)N+k}}[r^i_{S_t} \mid t \leq (n-1)N] - \mathbf{E}_{\mathbf{x}^{-i}, \mathbf{q}^i_n}[r^i_{S_t} \mid t \leq (n-1)]| \leq 4N\rho\varepsilon.$$

Moreover,

$$|\mathbf{E}_{\mathbf{y}^{-i}, \mathbf{q}^i_{(n-1)N+k}}[r^i_{S_t} \mid t > (n-1)N] - 1| \leq (N-1)\rho\varepsilon + \rho\varepsilon,$$

where $(N-1)\rho\varepsilon$ accounts for the probability that someone may quit in the first $k-1$ substages of stage n, and $\rho\varepsilon$ accounts for the probability that some player other than i may quit in substage k. Also, $|\mathbf{E}_{\mathbf{x}^{-i}, \mathbf{q}^i_n}[r^i_{S_t} \mid t > (n-1)] - 1| \leq \rho N\varepsilon$. Collecting these inequalities yields

$$\gamma^i(\mathbf{y}^{-i}, \mathbf{q}^i_{(n-1)N+k}) \leq \gamma^i(\mathbf{x}^{-i}, \mathbf{q}^i_n) + 6N\rho\varepsilon \leq \gamma^i(\mathbf{x}) + 7N\rho\varepsilon \leq \gamma^i(\mathbf{y}) + \varepsilon + 11N\rho\varepsilon.$$

This concludes the proof. ■

We henceforth assume that $\mathbf{x}$ is an ε-equilibrium such that $|\{i \in I, x_n > 0\}| \leq 1$ and $\|x_n - c\| < \varepsilon$ for every $n \in \mathbf{N}$. We refer to such a profile as a *perturbed ε-equilibrium*.

Lemma 10. *For every perturbed ε-equilibrium* $\mathbf{x}$ *one has*

1. $\mathbf{P}_{\mathbf{x}}(t < +\infty) \geq 1 - \varepsilon$.
2. $\gamma^i(\mathbf{x}) \geq 1 - \rho\varepsilon - \varepsilon$ *for every* $i \in I$, *and* $\gamma^i(\mathbf{x}) \geq \frac{5}{4} - 2\varepsilon$ *for some* $i \in I$.
3. $\mathbf{P}_{\mathbf{x}}(S_t = \{i\}) \geq \frac{2}{15} - \rho\varepsilon$ *for every* $i \in I$.

The first claim holds for any ε-equilibrium, whether it is a perturbed ε-equilibrium of not.

Proof: Given $n \in \mathbf{N}$, let $\mathbf{y}^{i,n}$ be the strategy of player i that coincides with $\mathbf{x}^i$ in the first n stages and plays q^i at stage $n+1$. The sequence of payoffs $(\gamma^i(\mathbf{x}^{-i}, \mathbf{y}^{i,n}))_{n \in \mathbf{N}}$ converges to $\gamma^i(\mathbf{x}) + \mathbf{P}_{\mathbf{x}}(t = +\infty)$, as n goes to infinity. Since $\gamma^i(\mathbf{x}^{-i}, \mathbf{y}^{i,n}) \le \gamma^i(\mathbf{x}) + \varepsilon$ for every n, claim **1** follows.

By quitting at the first stage, player i obtains at least $1 - \rho\varepsilon$. The first part of claim **2** follows. Whenever the quitting set is a singleton the payoffs to the players sum up to $0 + 0 + 1 + 4 = 5$. Therefore,

$$\sum_{i \in I} \gamma^i(\mathbf{x}) = 5\mathbf{P}_{\mathbf{x}}(t < +\infty) \ge 5 - 5\varepsilon.$$

In particular, there exists i such that $\gamma^i(\mathbf{x}) \ge \frac{5}{4} - \frac{5}{4}\varepsilon$. The second part of claim **2** follows.

We turn to the proof of claim **3**. Set $p^i := \mathbf{P}_{\mathbf{x}}(S_t = \{i\})$. Note that

$$\gamma^1(\mathbf{x}) = p^1 + 4p^2,$$

and that analogous identities hold for players 2, 3 and 4. In particular, by claim **2**, one has

$$p^1 + 4p^2 \ge 1 - 2\rho\varepsilon \quad \text{and} \quad 4p^1 + p^2 \ge 1 - 2\rho\varepsilon,$$

which implies $p^1 + p^2 \ge \frac{2}{5} - \frac{4}{5}\rho\varepsilon$. By exchanging the roles of the players, one gets $p^3 + p^4 \ge \frac{2}{5} - \rho\varepsilon$. Therefore, $p^1 + p^2 \le \frac{3}{5} + \rho\varepsilon$. Thus, (p^1, p^2) satisfy

$$p^1 + 4p^2 \ge 1 - 2\rho\varepsilon, \quad 4p^1 + p^2 \ge 1 - 2\rho\varepsilon, \quad \text{and} \quad p^1 + p^2 \le \frac{3}{5} + \rho\varepsilon. \tag{6}$$

Any solution to the system (6) satisfies $p^1, p^2 \ge \frac{2}{15} - \rho\varepsilon$. ■

Given a profile $\mathbf{x}$, a player $i \in I$, and a stage $n \in \mathbf{N}$, we let $\mathbf{x}^i(n)$ be the strategy which plays c^i up to stage n, and coincides with $\mathbf{x}^i$ after stage n. We denote by $\mathbf{x}_n = (x_n, x_{n+1}, \ldots)$ the profile induced by $\mathbf{x}$ in the subgame starting from stage n. Finally, we let $p_n^i := \mathbf{P}_{\mathbf{x}}(t < n, S_t = \{i\})$. Note that $p^i = \lim_{n \to \infty} p_n^i$. Though p_n^i depends on the profile, this is not made explicit in the notation.

We prove now that, as long as the continuation payoff of player i exceeds 1, player i's contribution to the probability of termination is small.

Lemma 11. *Let* $\mathbf{x}$ *be a strategy profile such that* $|\{i : x_n^i > 0\}| \le 1$ *for every* n. *Assume that* $\gamma^i(\mathbf{x}_n) \ge 1 + \sqrt{\varepsilon}$ *for some player* i *and every* $n \le n_0$. *Then*

$$\gamma^i(\mathbf{x}^{-i}, \mathbf{x}^i(n)) \ge \gamma^i(\mathbf{x}) + \sqrt{\varepsilon} \times p_n^i, \quad \textit{for every } n \le n_0.$$

In particular, by the ε-equilibrium property, this yields $p_n^i \le \sqrt{\varepsilon}$.

Proof: We proceed by induction. Assume $n = 1$. If $x_1^i = 0$, then $\mathbf{x}^i(1) = \mathbf{x}^i$ and $p_1^i = 0$, and the result holds. Otherwise, $p_1^i = 1 - x_1^i$, hence

$$\gamma^i(\mathbf{x}) = p_1^i + (1 - p_1^i)\gamma^i(\mathbf{x}^{-i}, \mathbf{x}^i(1)).$$

Then

$$\gamma^i(\mathbf{x}^{-i},\mathbf{x}^i(1)) = \gamma^i(\mathbf{x}) + \frac{p_1^i}{1-p_1^i}(\gamma^i(\mathbf{x})-1) \geq \gamma^i(\mathbf{x}) + \sqrt{\varepsilon}p_1^i.$$

Assume now that $1 < n \leq n_0$. If $x_n^i = 0$, then $\mathbf{x}^i(n) = \mathbf{x}^i(n-1)$ and $p_n^i = p_{n-1}^i$. In particular, by the induction hypothesis,

$$\gamma^i(\mathbf{x}^{-i},\mathbf{x}^i(n)) = \gamma^i(\mathbf{x}^{-i},\mathbf{x}^i(n-1)) \geq \gamma^i(\mathbf{x}) + \sqrt{\varepsilon}p_{n-1}^i = \gamma^i(\mathbf{x}) + \sqrt{\varepsilon}p_n^i,$$

and the result holds.

If $x_n^i > 0$ then, applying the case $n = 1$ to the profile $\mathbf{x}_{n-1}$ we get

$$\gamma^i(\mathbf{x}_{n-1}^{-i},\mathbf{x}^i(n)_{n-1}) \geq \gamma^i(\mathbf{x}_{n-1}^{-i},\mathbf{x}^i(n-1)_{n-1}) + \sqrt{\varepsilon}(1-x_n^i).$$

Using the induction hypothesis we get:

$$\begin{aligned}\gamma^i(\mathbf{x}^{-i},\mathbf{x}^i(n)) &\geq \gamma^i(\mathbf{x}^{-i},\mathbf{x}^i(n-1)) + \mathbf{P}_{\mathbf{x}^{-i},\mathbf{c}^i}(t \geq n-1)\sqrt{\varepsilon}(1-x_n^i)\\ &\geq \gamma^i(\mathbf{x}) + \sqrt{\varepsilon}(p_{n-1}^i + \mathbf{P}_{\mathbf{x}^{-i},\mathbf{c}^i}(t \geq n-1)(1-x_n^i))\\ &\geq \gamma^i(\mathbf{x}) + \sqrt{\varepsilon}p_n^i. \quad \blacksquare\end{aligned}$$

We say that players 1 and 2 (resp. 3 and 4) are *partners*. The partner of player i is denoted by $\tilde{\imath}$.

We next prove that, whenever player i gets a payoff higher than one in a perturbed ε-equilibrium, player i will not contribute to the probability of termination, while the partner of i will contribute, until a stage is reached in which the continuation payoff of player i is close to one.

Lemma 12. *Let $a > 0$, $\varepsilon \in (0, 1/900)$ and $i \in I$. Let $\mathbf{x}$ be a perturbed ε-equilibrium such that $\gamma^i(\mathbf{x}) \geq 1 + a$. Then there exists $n_1 > 1$ such that (i) $\gamma^i(\mathbf{x}_{n_1}) < 1 + \sqrt{\varepsilon}$, (ii) $p_{n_1}^i \leq 2\sqrt{\varepsilon}$, and (iii) $3p_{n_1}^{\tilde{\imath}} \geq a - \sqrt{\varepsilon}$.*

Proof: For convenience, assume $i = 1$. Since $p^1 \geq 2/15 - 3\varepsilon$, by Lemma 11, there is a stage n such that $\gamma^1(\mathbf{x}_n) < 1 + \sqrt{\varepsilon}$. Let $n_1 = \inf\{n \in \mathbf{N}, \gamma^1(\mathbf{x}_n) < 1 + \sqrt{\varepsilon}\}$ be the first such stage. Note that $n_1 > 1$. By definition, claim (i) holds and $\gamma^1(\mathbf{x}_n) \geq 1 + \sqrt{\varepsilon}$ for each $n \leq n_1 - 1$ hence, by Lemma 11, $p_{n_1-1}^i \leq \sqrt{\varepsilon}$. Since $p_{n_1}^1 \leq p_{n_1-1}^1 + x_{n_1}^1$ and $x_{n_1}^1 \leq \varepsilon$, claim (ii) follows.

We now prove (iii). Since $\gamma^1(\mathbf{x}_{n_1}) < 1 + \sqrt{\varepsilon}$ one has

$$\begin{aligned}1 + a \leq \gamma^1(\mathbf{x}) &= p_{n_1}^1 + 4p_{n_1}^2 + (1 - p_{n_1}^1 - p_{n_1}^2 - p_{n_1}^3 - p_{n_1}^4)\gamma^1(\mathbf{x}_{n_1})\\ &\leq p_{n_1}^1 + 4p_{n_1}^2 + (1 - p_{n_1}^1 - p_{n_1}^2) + \sqrt{\varepsilon}\\ &\leq 1 + 3p_{n_1}^2 + \sqrt{\varepsilon},\end{aligned}$$

and (iii) follows. $\blacksquare$

We next prove that there is no perturbed ε-equilibrium in which two partners get a payoff substantially higher than one.

Corollary 13. *Let* $\varepsilon \in (0, 1/900)$ *and* $a > 7\sqrt{\varepsilon}$. *There is no perturbed* ε-*equilibrium* $\mathbf{x}$ *such that*

$$\gamma^i(\mathbf{x}), \gamma^{\tilde{i}}(\mathbf{x}) \geq 1 + a \quad \textit{for some } i \in I.$$

Proof: We argue by contradiction. Let $\mathbf{x}$ be such a perturbed ε-equilibrium, and assume w.l.o.g. $i = 1$. Apply Lemma 12 twice, to players 1 and 2. Call n_1 and n_2 the corresponding two stages, and assume w.l.o.g that $n_1 \leq n_2$, so that $p^2_{n_1} \leq p^2_{n_2}$. Thus, one has both $p^2_{n_1} \geq a/3 - \sqrt{\varepsilon}/3$, and $p^2_{n_2} \leq 2\sqrt{\varepsilon}$. Hence $a - \sqrt{\varepsilon} \leq 6\sqrt{\varepsilon}$ – a contradiction. ■

We now proceed to the proof of Proposition 3.

Proof of Proposition 3: Let $\varepsilon > 0$ be small enough, and let $\mathbf{x}$ be a perturbed ε-equilibrium. We assume w.l.o.g. that $\gamma^1(\mathbf{x}) \geq 5/4 - 2\varepsilon$. We will exhibit a stage n_2 such that $\mathbf{x}_{n_2}$ is a 8ε-equilibrium, and $\gamma^3(\mathbf{x}_{n_2}), \gamma^4(\mathbf{x}_{n_2}) \geq 1 + 1/12$, contradicting Corollary 13.

Apply Lemma 12 to $\mathbf{x}$ and $i = 1$, and denote n_1 the corresponding stage. Thus, $p^1_{n_1} \leq 2\sqrt{\varepsilon}$ and $p^2_{n_1} \geq \frac{1}{3} \times \frac{1}{4} - \sqrt{\varepsilon}$. By Lemma 11, there exists a stage $n \leq n_1$ with $\gamma^2(\mathbf{x}_n) < 1 + \sqrt{\varepsilon}$. We set

$$n_2 = \max\{n \leq n_1, \gamma^2(\mathbf{x}_n) \leq 1 + \sqrt{\varepsilon}\}.$$

Since $p^1_{n_2} \leq p^1_{n_1} \leq 2\sqrt{\varepsilon}$, and $p^1 \geq \frac{2}{15} - \rho\varepsilon$, one obtains

$$\mathbf{P}_{\mathbf{x}}(t < n_2) \leq 1 - \mathbf{P}(t \geq n_2 \text{ and } S_t = \{1\}) \leq \frac{13}{15} + \rho\varepsilon + 2\sqrt{\varepsilon} \leq \frac{7}{8}.$$

Since $\mathbf{x}$ is an ε-equilibrium, $\mathbf{x}_{n_2}$ is a 8ε-equilibrium.

Our next goal is to prove that $p^2_{n_2} \geq \frac{1}{12} - 17\sqrt{\varepsilon}$. If $n_2 = n_1$ there is nothing to prove. Assume $n_2 < n_1$, so that $\gamma^2(\mathbf{x}_{n_1}) > 1 + \sqrt{\varepsilon}$. By the definition of n_2, $\gamma^2(\mathbf{x}_k) > 1 + \sqrt{\varepsilon}$ for every $n_2 < k \leq n_1$. Apply Lemma 11 with $\mathbf{y} = \mathbf{x}_{n_2}$ (thus $y_k = x_{n_2+k}$, for each k) and $n = n_1 - n_2$. Since $\mathbf{x}_{n_2}$ is a 8ε-equilibrium, the conclusion, rephrased in terms of $\mathbf{x}$, is that $\mathbf{P}_{\mathbf{x}}(t < n_1, S_t = \{2\} \mid t \geq n_2) \leq 8\varepsilon/\sqrt{\varepsilon} = 8\sqrt{\varepsilon}$. In particular $p^2_{n_1} - p^2_{n_2} \leq 8\sqrt{\varepsilon}$, and therefore $p^2_{n_2} \geq \frac{1}{12} - 9\sqrt{\varepsilon}$.

We use this result to prove that $\gamma^3(\mathbf{x}_{n_2}), \gamma^4(\mathbf{x}_{n_2}) \geq 1 + 1/12$. As previously, one has

$$1 - 2\rho\varepsilon \leq \gamma^2(\mathbf{x}) = 4p^1_{n_2} + p^2_{n_2} + \left(1 - \sum_{i \in I} p^i_{n_2}\right)\gamma^2(\mathbf{x}_{n_2}). \tag{7}$$

By definition of n_2, $\gamma^2(\mathbf{x}_{n_2}) \leq 1 + \sqrt{\varepsilon}$. Since $p^1_{n_2} \leq p^1_{n_1} \leq 2\sqrt{\varepsilon}$, one deduces from (7) that $p^3_{n_2} + p^4_{n_2} \leq (7 + 2\rho\varepsilon)\sqrt{\varepsilon} + 4\rho\varepsilon \leq 8\sqrt{\varepsilon}$.

On the other hand,

$$1 - 2\rho\varepsilon \leq \gamma^3(\mathbf{x}) = 4p^4_{n_2} + p^3_{n_2} + \left(1 - \sum_{i} p^i_{n_2}\right)\gamma^3(\mathbf{x}_{n_2}). \tag{8}$$

Since $p^2_{n_2} \geq 1/12 - 17\sqrt{\varepsilon}$, (8) yields $\gamma^3(\mathbf{x}_{n_2}) \geq 1 + \frac{1}{11} - \varepsilon^{1/4} \geq 1 + 1/12$. Similarly, $\gamma^4(\mathbf{x}_{n_2}) \geq 1 + \frac{1}{12}$. Since $\mathbf{x}_{n_2}$ is a 8ε-equilibrium, we get a contradiction to Corollary 13. ■

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