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An application of Ramsey theorem to stopping games

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Abstract

We prove that every two-player nonzero-sum deterministic stopping game with uniformly bounded payoffs admits an ε-equilibrium, for every $\varepsilon > 0$. The proof uses Ramsey Theorem that states that for every coloring of a complete infinite graph by finitely many colors there is a complete infinite subgraph which is monochromatic.

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1. Introduction

Consider the following two-player nonzero-sum game, that is played in stages. At every stage n each of the two players has to decide whether to *quit* or to *continue* the game. If both players decide to continue, the game proceeds to stage $n+1$. Otherwise, the game terminates, and player i receives the payoff $r^i_{S,n}$, where $\emptyset \neq S \subseteq \{1,2\}$ is the set of players that decide to quit at stage n. If no player ever quits, the payoff is 0 to both players.

This game is a stopping game with deterministic payoff processes. Stopping games have been introduced by Dynkin (1969) as a generalization of optimal stopping problems,

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and later used in several models in economics and management science, such as optimal equipment replacement, job search, consumer behavior, research and development (see Mamer (1987) and the references therein), and the analysis of strategic exit (see Ghemawat and Nalebuff (1985) or Li (1989)). Dynkin was interested in zero-sum stopping games in which the sequences $(r_{S,n})_n$ are stochastic processes, where $r_{S,n} := r^1_{S,n} = -r^2_{S,n}$. He proved the existence of optimal pure strategies, under the assumption that at any stage, only one of the players is allowed to stop. Since then, a very extensive literature in the theory of stochastic processes has dealt with zero-sum stopping games, both in discrete and continuous time. Most contributions provide conditions on the sequences $(r_{S,n})$ under which each player has pure ε-optimal strategies (a pure strategy corresponds to the notion of stopping time in probability theory). The typical condition takes the form: $r_{\{1\},n} \leqslant r_{\{1,2\},n} \leqslant r_{\{2\},n}$ for each n. Rosenberg et al. (2001) removed this assumption and proved that every zero-sum stopping game admits a uniform value, when mixed strategies are allowed.

Nonzero-sum stopping games were studied, amongst others, by Mamer (1987), Morimoto (1986), Nagai (1987), and Ohtsubo (1987, 1991). They provided conditions on the payoff process under which ε-equilibria exist.

We prove that every two-player nonzero-sum deterministic stopping game with uniformly bounded payoffs admits an ε-equilibrium, for every $\varepsilon > 0$. The proof uses Ramsey Theorem that states that for every coloring of a complete infinite graph by finitely many colors there is a complete infinite subgraph which is monochromatic. Ramsey's (1930) original work has been extended in many directions, to become Ramsey Theory. This theory expresses a basic principle: every large set of objects necessarily contains a highly regular pattern. Put differently, if we partition a "large" system into "few" classes, one of these classes contains a "large" subsystem, whatever be the partition. The interested reader is referred to, e.g., Bollobás (1998) for more details on this theory.

An interesting feature of the proof is that it does not rely on the proof for zero-sum games.

We are not aware of any previous application of Ramsey Theorem to game theory, except for Ramsey games, which were designed to fit Ramsey theory. More information on Ramsey games, and on combinatorial game theory in general, can be found in http://www1.ics.uci.edu/~eppstein/cgt/.

2. The game and the result

A *deterministic* (*two-player*) *stopping game* Γ is described by a bounded sequence (r_n) in $\mathbf{R}^6$. The components of r_n are labeled $r^i_{S,n}$, where $i = 1, 2$ and $\emptyset \neq S \subseteq \{1, 2\}$. The game is played as follows. At every stage $n \geqslant 1$, each of the two players has to decide whether to *quit* or to *continue* the game. Let θ be the first stage, possibly infinite, in which at least one of the players decides to quit, and let S_* be the subset of players who decide to quit at stage θ (provided $\theta < +\infty$). The payoff to player i is $r^i_{S_*,\theta}$ if $\theta < +\infty$, and 0 if $\theta = +\infty$.

A (behavioral) *strategy* for player 1 is a function $x : \mathbf{N} \to [0, 1]$, $x(n)$ being the probability player 1 quits at stage n, provided no player quit before that stage. Strategies y of player 2 are defined analogously.

Every pair of strategies (x, y) induces a payoff to both players:

$$\gamma^i(x, y) = \mathbf{E}_{x,y}\big[r^i_{S_*,\theta} 1_{\theta<+\infty}\big],$$

where the expectation is taken w.r.t. the probability distribution $\mathbf{P}_{x,y}$ over plays induced by the strategies x and y.

Our main result is:

Theorem 1. *For every $\varepsilon \in (0, 1)$ the game admits an ε-equilibrium: there is a pair of strategies (x^*, y^*) such that*

$$\gamma^1(x, y^*) \leqslant \gamma^1(x^*, y^*) + \varepsilon \quad \textit{and} \quad \gamma^2(x^*, y) \leqslant \gamma^2(x^*, y^*) + \varepsilon,$$
for every x and y.

We conclude this section by an example, showing that a 0-equilibrium needs not exist, even if the sequence of possible payoffs is constant.

Example. Consider the zero-sum game defined by $r^1_{\{1\},n} = r^1_{\{2\},n} = 1$ and $r^1_{\{1,2\},n} = 0$ for every $n \in \mathbf{N}$. The strategy x_ε defined by $x_\varepsilon(n) = \varepsilon$ guarantees $1 - \varepsilon$: $\inf_y \gamma^1(x_\varepsilon, y) = 1 - \varepsilon$. Since payoffs are at most one, the value of the game is equal to one. However, player 1 has no optimal strategy. Indeed, let x be any strategy and let y be the strategy defined by

$$y(n) = \begin{cases} 0 & \text{if } x(n) = 0, \\ 1 & \text{if } x(n) > 0. \end{cases}$$

It is easy to verify that $\gamma^1(x, y) < 1$.

3. The proof

Since payoffs are uniformly bounded, we assume w.l.o.g. that payoffs are bounded by 1. Fix $\varepsilon > 0$ sufficiently small once and for all, and choose an ε-discretization A of the set $[-1, 1]^2$; that is, A is a finite set such that for every $u \in [-1, 1]^2$, there is $a \in A$ with $\|a - u\|_\infty < \varepsilon$.

Step 1. Periodic games

For every two positive integers $k < l$, we define a periodic stopping game $G(k, l)$ as follows:

$$r^i_{S,n}(k, l) = r^i_{S,k+(n-1 \bmod l-k)}.$$

We interpret this game as "the game that starts at stage k, and restarts at stage l (from stage k)." We denote by $\gamma_{k,l}(x, y)$ the payoff function in the game $G(k, l)$.

The game $G(k, l)$ may be analyzed as a stochastic game with finitely many states. The most convenient way is to define a stochastic game $\Gamma(k, l)$, where each stage of play corresponds to a period of play of $G(k, l)$. To be more formal, the set of actions of each player in $\Gamma(k, l)$ is $\{c, 1, 2, \ldots, l - k\}$. Action c corresponds to continuing in all stages of

	c	1	$\cdots$	$l-k$
c	0	$r_{\{2\},k}$ *		$r_{\{2\},l-1}$ *
1	$r_{\{1\},k}$ *	$r_{\{1,2\},k}$ *		$r_{\{1\},k}$ *
$\vdots$	$\vdots$			
$l-k$	$r_{\{1\},l-1}$ *	$r_{\{2\},k}$ *		$r_{\{1,2\},l-1}$ *

Fig. 1. The game $\Gamma(k,l)$.

the period. Action labeled p, $1 \leqslant p \leqslant l-k$, corresponds to continuing in the first $p-1$ stages of the period, and stopping in the pth stage. As is customary for stochastic games, we represent this game through the shown matrix in Fig. 1.

An entry is starred if the corresponding combination of actions leads to an absorbing state with the corresponding payoff; that is, the game terminates. Note that stationary strategies in $\Gamma(k,l)$ correspond to periodic strategies in $G(k,l)$, with period $l-k$. A stationary strategy of player i in $\Gamma(k,l)$ can be identified with a probability distribution π^i over the set $\{c, 1, \ldots, l-k\}$ of his actions, with the interpretation that π^i is used in every stage until at least one of the players chooses an action other than c (and the game terminates).

The game $\Gamma(k,l)$ is a recursive absorbing stochastic game: there is a unique nonabsorbing state, in which the reward function is identically zero.

By Flesch et al. (1996), such games have a stationary ε-equilibrium $\pi = (\pi^1, \pi^2)$. Moreover, it follows from their proof, or, alternatively, by the analysis of Vreize and Thuijsman (1989), that the profile π can be chosen such that one of the following alternatives holds:

A.1 $\pi^1(c) = \pi^2(c) = 1$.
A.2 (i) $\gamma^i_{k,l}(\pi) \geqslant 0$ for $i = 1, 2$,[1] and
(ii) $\pi^1(c) \leqslant 1-\varepsilon^2$ or $\pi^2(c) \leqslant 1-\varepsilon^2$.
A.3 If $\gamma^1_{k,l}(\pi) < 0$ then $\pi^2(c) \leqslant 1-\varepsilon^2$; if $\gamma^2_{k,l}(\pi) < 0$ then $\pi^1(c) \leqslant 1-\varepsilon^2$.

In particular, either the probability that both players continue is 1 or it is at most $1-\varepsilon^2$.

We denote by $(x_{k,l}, y_{k,l})$ the periodic profile in $G(k,l)$ which corresponds to a stationary ε-equilibrium π of $\Gamma(k,l)$ that satisfies one of A.1–A.3. It is a periodic ε-equilibrium of $G(k,l)$, with period $l-k$.

For every $k < l$ we choose $a(k,l) \in A$ such that

$$\left\| \gamma_{k,l}(x_{k,l}, y_{k,l}) - a(k,l) \right\|_\infty < \varepsilon.$$

Step 2. Application of Ramsey Theorem

To every pair of positive integers $k < l$ we attached in Step 1 an element in the finite set A—a color. By Ramsey Theorem there is an infinite subset of integers $K \subseteq \mathbf{N}$ and $a \in A$ such that $a(k,l) = a$ for every $k, l \in K$, $k < l$.

[1] With abuse of notations, $\gamma^i_{k,l}(\pi)$ is the payoff of player i in $\Gamma(k,l)$ under the stationary strategy pair π.

In particular, there exists an increasing sequence of positive integers $k_1 < k_2 < \cdots$ such that for every $j \in \mathbf{N}$, $a(k_j, k_{j+1}) = a$. For notational convenience, we write (x_j^*, y_j^*) for $(x_{k_j,k_{j+1}}, y_{k_j,k_{j+1}})$.

For every $k \in \mathbf{N}$, we let $G(k,\infty)$ denote the stopping game induced by Γ from stage k, i.e., $r^i_{S,n}(k,\infty) = r^i_{S,n+k-1}$ for every $n \in \mathbf{N}$. We denote by $\gamma_{k,\infty}(x,y)$ the payoff function in the game $G(k,\infty)$.

Let (x^*, y^*) be the profile in $G(k_1,\infty)$ obtained by concatenating the profiles (x_j^*, y_j^*):

$$x^*(n) = x_j^*(n - k_j + k_1) \quad \text{for } k_j - k_1 + 1 \leqslant n < k_{j+1} - k_1 + 1.$$

The definition of y^* is similar.

Step 3. $|\gamma^i_{k_1,\infty}(x^*, y^*) - a^i| \leqslant \varepsilon$ *for* $i = 1, 2$

Assume w.l.o.g. that $k_1 = 1$. If $|a^i| \leqslant \varepsilon$, then, for every j, either A.1 holds or $\mathbf{P}_{x^*,y^*}(\theta < k_{j+1} \mid \theta \geqslant k_j) \geqslant \varepsilon^2$. In the first case, $\mathbf{P}_{x^*,y^*}(k_j \leqslant \theta < k_{j+1}) = 0$, whereas in the second,

$$\left|\mathbf{E}_{x^*,y^*}\left[r^i_{S_*,\theta} \,\middle|\, k_j \leqslant \theta < k_{j+1}\right] - a^i\right| \leqslant \varepsilon.$$

By summing up over $j \in \mathbf{N}$ we get $|\gamma^i_{k_1,\infty}(x^*, y^*) - a^i| \leqslant \varepsilon$.

Assume now that $|a^i| > \varepsilon$. Then, for every j, $\mathbf{P}_{x^*,y^*}(\theta < k_{j+1} \mid \theta \geqslant k_j) \geqslant \varepsilon^2$ and $|\mathbf{E}_{x^*,y^*}[r^i_{S_*,\theta} \mid k_j \leqslant \theta < k_{j+1}] - a^i| \leqslant \varepsilon$. The first inequality yields $\mathbf{P}_{x^*,y^*}(\theta < +\infty) = 1$ while the second implies $|\mathbf{E}_{x^*,y^*}[r^i_{S_*,\theta} \mid \theta < +\infty] - a^i| \leqslant \varepsilon$. Therefore, $|\gamma^i_{k_1,\infty}(x^*, y^*) - a^i| \leqslant \varepsilon$.

Step 4. (x^*, y^*) *is a* 3ε*-equilibrium of the game* $G(k_1,\infty)$

We show that player 1 cannot profit more than 3ε by deviating from x^*.

Assume w.l.o.g. that $k_1 = 1$. Let x be a strategy in $G(k_1,\infty) = \Gamma$ and, for every $j \in \mathbf{N}$, let x_j be the corresponding periodic strategy in $G(k_j, k_{j+1})$: $x_j(n) = x(k_j + (n - 1 \bmod k_{j+1} - k_j))$. Since (x_j^*, y_j^*) is an ε-equilibrium in $G(k_j, k_{j+1})$, if $\mathbf{P}_{x,y^*}(k_j \leqslant \theta < k_{j+1}) > 0$ then

$$\mathbf{E}_{x,y^*}\left[r^1_{S_*,\theta} \,\middle|\, k_j \leqslant \theta < k_{j+1}\right] = \gamma^1_{k_j,k_{j+1}}\left(x_j, y_j^*\right) \leqslant \gamma^1_{k_j,k_{j+1}}\left(x_j^*, y_j^*\right) + \varepsilon \leqslant a^1 + 2\varepsilon.$$

Therefore,

$$\begin{aligned}\mathbf{E}_{x,y^*}\left[r^1_{S_*,\theta}1_{\theta<+\infty}\right] &= \sum_{j\in\mathbf{N}} \mathbf{P}_{x,y^*}(k_j \leqslant \theta < k_{j+1})\mathbf{E}_{x,y^*}\left[r^1_{S_*,\theta} \,\middle|\, k_j \leqslant \theta < k_{j+1}\right] \\ &\leqslant \mathbf{P}_{x,y^*}(\theta < +\infty)\left(a^1 + 2\varepsilon\right). \end{aligned} \tag{1}$$

- If $a^1 \geqslant -\varepsilon$, one has $a^1 + 2\varepsilon > 0$ hence $\mathbf{E}_{x,y^*}[r^1_{S_*,\theta}1_{\theta<+\infty}] \leqslant a^1 + 2\varepsilon$.
- If $a^1 < -\varepsilon$, then A.3 holds, and one has $\mathbf{P}_{x,y^*}(\theta < k_{j+1} | \theta \geqslant k_j) \geqslant \varepsilon^2$ for every j. Hence $\mathbf{P}_{x,y^*}(\theta < +\infty) = 1$, which yields $\mathbf{E}_{x,y^*}[r^1_{S_*,\theta}1_{\theta<+\infty}] \leqslant a^1 + 2\varepsilon$.

Therefore,

$$\gamma^1_{k_1,\infty}(x, y^*) \leqslant a^1 + 2\varepsilon \leqslant \gamma^1_{k_1,\infty}(x^*, y^*) + 3\varepsilon.$$

Step 5. Backward induction

Consider the following $(k_1 - 1)$-stage game $\overline{\Gamma}$. In $\overline{\Gamma}$, the two players play the first $k_1 - 1$ stages of Γ. If no player quits in the first $k_1 - 1$ stages, the payoff is $a = (a^1, a^2)$. Let $(\bar{x}, \bar{y})$ be an equilibrium in $\overline{\Gamma}$. Thus, $\bar{x}, \bar{y} : \{1, \ldots, k_1 - 1\} \to [0, 1]$. Denote by (x, y) the profile in Γ that coincides with $(\bar{x}, \bar{y})$ up to stage $k_1 - 1$, and with (x^*, y^*) from stage k_1 on. It is straightforward to deduce from Step 4 that (x, y) is a 3ε-equilibrium of Γ. This concludes the proof of the theorem.

Comment. The ε-equilibrium strategy pair that we constructed is uniform in a strong sense: it is a 2ε-equilibrium in every finite n-stage game, provided n is sufficiently large. This can be seen either by applying a general observation made by Solan and Vieille (2001, Proposition 2.13), or by the construction itself: if the expected payoff to player 1 under (x^*, y^*) is positive, he cannot profit by delaying the termination stage, whereas if it is negative, then with high probability termination occurs before stage k_m whatever he plays, for every $m > 1/\varepsilon^3$.

4. Extensions

We here discuss the extension to n-player games with $n > 2$, and to games with general payoff processes.

The proof we presented above is divided into three parts. First we define for every periodic game a color, by approximating an equilibrium payoff in the periodic game. Second, we apply Ramsey Theorem to the complete infinite graph. This way we get a sequence of periodic games. Third, we concatenate ε-equilibria in these periodic games to form a 3ε-equilibrium in the original infinite game.

When there are three players, the technique of Solan (1999) can be used to prove that periodic deterministic stopping games admit equilibrium payoffs. The ε-equilibria in the corresponding stochastic game $\Gamma(k, l)$ need not be stationary: they are either stationary or periodic. Nevertheless, one can still construct a 3ε-equilibrium by appropriately concatenating the ε-equilibrium strategies of the periodic games.

When there are more than three players, it is not known whether periodic deterministic stopping games admit equilibrium payoffs.

When the payoff processes are general, the periodic game is defined by its starting point, and by a stopping time that indicates when it restarts. The result of Flesch et al. (1996) can be applied to show that every such game admits an equilibrium payoff, and one can generalize Ramsey Theorem to this more general setup. However, it is not clear whether a concatenation of ε-equilibria in the periodic games forms a 3ε-equilibrium of the original game.

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