Strong Approachability*

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Abstract

We introduce the concept of strongly approachable sets in two-player repeated games with vector payoffs. A set in the payoff space is *strongly approachable* by a player if the player can guarantee that from a certain stage on the average payoff will be inside that set, regardless of the strategy that the other player implements. We provide sufficient conditions that ensure that a closed convex approachable set is also strongly approachable in the expected deterministic version of the game.

Keywords: approachability, repeated games, vector payoffs, strong approachability

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1. Introduction

Two-player repeated games with vector payoffs are repeated games in which the stage outcome is described by a vector in a Euclidean space $\mathbb{R}^d$. These games were introduced by Blackwell [3] as a generalization of two-player zero-sum repeated games. In a repeated game with vector payoffs, a set in $\mathbb{R}^d$ is *approachable* by a player if he has a strategy that ensures that the distance between the long-run average payoff and the set goes to 0, whatever the other player plays. In his seminal paper [3] Blackwell provides a geometric condition that guarantees that a set is approachable by a player. Hou [8] and later Spinat [17] proved that every approachable set contains a subset that satisfies Blackwell's sufficient condition, thus completing the characterization of approachable sets.

Repeated games with vector payoffs have been applied in various areas in game theory; they were used to construct optimal strategies for the uninformed player in two-player zero-sum repeated games with incomplete information on one side (Aumann and Maschler [1, 2] and subsequent work, e.g., Kohlberg [9] and Rosenberg, Solan, and Vieille [15]); to construct regret-free strategies (Foster and Vohra [5]); and to construct processes that converge to a correlated equilibrium in multiplayer strategic-form games (Hart and Mas-Colell [7]). The theory has been extended to repeated games in which the payoffs lie in an infinitely dimensional space (Lehrer [10]) and has been applied outside game theory, to provide an algorithm that generates normal numbers (Lehrer [11]).

In this paper we define a refinement of the concept of approachability termed "strong approachability." A set in the payoff space $\mathbb{R}^d$ is *strongly approachable* by a player if he has a strategy that guarantees that the long-run average payoff lies *inside* the set from some stage on, whatever the other player plays. Thus, every strongly approachable set is approachable, yet the converse need not hold. There are applications in which the concept of strong approachability is more natural than the concept of approachability. For example, a sales manager may need to meet projected sales in each region, and not only get close to projected sales. A second example is the House of Representatives; to have a majority in it a party must have the majority in majority of districts, and not only be close to the majority in majority of districts.

In the present paper we provide conditions that guarantee that a closed convex approachable set is strongly approachable in the expected deterministic version of the game, in which the mixed actions of the players are observed. The analog question for the original game seems to be much harder, see Remarks 2 and 24 below. Our results also hold when the repeated game is played in continuous-time. In this case the proofs are significantly simpler.

It is well known that in general the optimal rate of convergence of the average payoff to an approachable set is $O(n^{-1/2})$ (see, e.g., Cesa-Bianchi and Lugosi, [4], Remark 7.7). Strong approachability can thus be viewed as approachability in finite time. Mannor and Perchet [12] provide conditions on the target set that ensure that it is approachable at a rate $O(n^{-1})$.

The paper is arranged as follows. The model and the concept of strong approachability are presented in Section 2, where we also provide few simple sufficient conditions for strong approachability and introduce our main results. Proofs are provided in Section 3. In Section 4 we discuss the results and the assumptions they rely on, as well as open questions.

2. The Model and Main Results

2.1 Strong Approachability

We study two-player repeated games with d-dimensional vector payoffs. Such a game G is given by two finite sets of actions $\mathcal{I}$ and $\mathcal{J}$, one for each player, and a payoff function $u : \mathcal{I} \times \mathcal{J} \longrightarrow \mathbb{R}^d$. The function u is extended multilinearly[1] to $\Delta(\mathcal{I}) \times \Delta(\mathcal{J})$ by

$$u(p, q) := \mathbb{E}_{p,q}[u(i, j)] = \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} p_i u(i, j) q_j.$$

The game G is played as follows. At each stage $n \in \mathbb{N}$ the two players, simultaneously and independently, choose (mixed) actions $p^n \in \Delta(\mathcal{I})$ and $q^n \in \Delta(\mathcal{J})$. The corresponding payoff is $u(p^n, q^n)$ and the pair of actions (p^n, q^n) is announced to both players. Perfect recall is assumed for both players. The game G is called in [16] the *expected deterministic repeated game*.

For every $n \geq 1$ let $\mathcal{H}_{n-1} := (\Delta(\mathcal{I}) \times \Delta(\mathcal{J}))^{n-1}$ be the set of all possible *histories* at stage n. Let $\mathcal{H}_{\mathbb{N}} := (\Delta(\mathcal{I}) \times \Delta(\mathcal{J}))^{\mathbb{N}}$ be the set of all possible *plays* in G. A *strategy* of Player 1 is a function $\sigma_1 : \bigcup_{n=1}^{\infty} \mathcal{H}_{n-1} \longrightarrow \Delta(\mathcal{I})$ and a strategy of Player 2 is a function $\sigma_2 : \bigcup_{n=1}^{\infty} \mathcal{H}_{n-1} \longrightarrow \Delta(\mathcal{J})$. We will study pure strategies in the expected deterministic game G. The interpretation of strategies σ_1 and σ_2 is that at each stage n, given the

[1] For every finite set X denote by $\Delta(X)$ the set of probability distributions over X, that is, $\Delta(X) := \{p \in [0, 1]^{|X|} \mid \sum_{x\in X} p_x = 1\}$.

history $h_{n-1} \in \mathcal{H}_{n-1}$, Player 1 plays the action $p^n = \sigma_1(h_{n-1})$ and Player 2 plays the action $q^n = \sigma_2(h_{n-1})$.

Every pair of strategies (σ_1, σ_2) uniquely determines a play path. The average payoff in the first N stages is denoted by $\bar{g}^N$:

$$\bar{g}^N := \frac{1}{N} \sum_{n=1}^{N} u(p^n, q^n).$$

When $i \in \{1, 2\}$ is a player, $-i$ is the other player. Blackwell [3] introduced the notion of approachability. Translated to the expected deterministic game, a nonempty set $A \subseteq \mathbb{R}^d$ is *approachable* by player i if there exists a strategy σ_i of player i such that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every strategy σ_{-i} of the other player,[2]

$$d(\bar{g}^n, A) < \varepsilon, \quad \forall n \geq N.$$

The solution concept that we study in this paper is the following strong version of approachable sets.

Definition 1 *A nonempty set $A \subseteq \mathbb{R}^d$ is* strongly approachable *by player i if there is a strategy σ_i of player i and $N \in \mathbb{N}$ such that for every strategy σ_{-i} of the other player,*

$$\bar{g}^n \in A, \quad \forall n > N.$$

We say that such a strategy σ_i strongly approaches the set A.

In words, a set is strongly approachable by a player, if the player can guarantee that the average payoff will be in that set from a certain stage on, for every strategy of the other player.

Remark 2 *The concept of strong approachability can be defined for standard repeated games (rather than for the expected deterministic repeated game). However, the characterization of strongly approachable sets in this case is significantly harder since the expected stage payoff is uniquely determined by the players' mixed actions, while the realized stage payoff is random. For example, in the Matching Pennies that appears in Figure 1, the set $\{0\}$ is strongly approachable by Player 1 in the expected deterministic game using the stationary strategy $[\frac{1}{2}T, \frac{1}{2}B]$. However, this set is not strongly approachable in the original game, because the stage payoff is either 1 or -1, and therefore the*

[2] The distance and the norm referred to throughout the paper are defined by the Euclidean norm: $d(x, y) := \|x - y\|$, where $\|x\| := \sqrt{\langle x, x\rangle}$ and the inner product of two vectors $x, y \in \mathbb{R}^d$ is defined by $\langle x, y\rangle := \sum_{i=1}^{d} x_i y_i$. The distance between a vector x and a set A is defined by $d(x, A) := \inf_{a \in A} d(x, a)$.

average payoff cannot be equal to 0 from some point on. We elaborate on this issue in Remark 24.

		Player 2 L	Player 2 R
Player 1	T	1	-1
	B	-1	1

Figure 1: The payoff matrix in the Matching Pennies.

Example 3 *For* $d = 1$*, the model reduces to standard repeated games with scalar payoffs. Consider then a two-player zero-sum repeated game, and denote by* v *its value. The set* $[v, \infty)$ *is strongly approachable by Player 1 (with any strategy that plays at every stage an optimal strategy of the one-shot game), and the set* $(-\infty, v]$ *is strongly approachable by Player 2.*

Example 4 *Since in the definition of approachability the convergence to* A *is uniform over the strategies of the other player, if some set* $A \in \mathbb{R}^d$ *is approachable by a player then for every* $\varepsilon > 0$ *the set* $\mathcal{B}(A, \varepsilon) := \{x \in \mathbb{R}^d \mid d(x, A) < \varepsilon\}$ *is strongly approachable by that player.*

Plainly every set that is strongly approachable by a player is approachable by him. As the following example shows, the converse does not hold: there are sets that are approachable by a player but not strongly approachable by him. Below we identify conditions that ensure that an approachable set is strongly approachable.

Example 5 *Consider the game in Figure 2, where payoffs are one-dimensional.*

		Player 2 L	Player 2 R
Player 1	T	1	2
	B	-1	-1

Figure 2: The payoff matrix in Example 5.

By Theorem 8 below, the set $\{0\}$ *is a B-set (see Definition 7) and therefore approachable. Nevertheless, the set* $\{0\}$ *is not strongly approachable. Indeed, assume that there is a stage such that the average expected payoff is* 0*. Since Player 1 cannot guarantee the payoff* 0 *in the one-shot game, for every mixed action* $p \in \Delta(\mathcal{I})$ *that Player 1 would play* *at the next stage, there is a mixed action $q \in \Delta(\mathcal{J})$ of Player 2 such that the expected payoff is not* 0*. Therefore the average expected payoff at the next stage is not* 0 *and in particular not in the set* $\{0\}$*. That is, for every strategy of Player 1 there is a strategy of Player 2 such that if the average expected payoff at some stage is in* $\{0\}$*, then the average expected payoff at the next stage is not in* $\{0\}$*, so that the set* $\{0\}$ *is not strongly approachable.*

2.2 Sufficient Conditions for Strong Approachability

In this section we provide two simple sufficient conditions for strong approachability that are analogous to sufficient conditions for approachability provided by Blackwell [3]. These conditions will motivate the main results, which will be presented in Section 2.3.

For every action $p \in \Delta(\mathcal{I})$ of Player 1 denote $R_1(p) := \{u(p,q) \mid q \in \Delta(\mathcal{J})\}$, and for every action $q \in \Delta(\mathcal{J})$ of Player 2 denote $R_2(q) := \{u(p,q) \mid p \in \Delta(\mathcal{I})\}$. The set $R_1(p)$ (resp. $R_2(q)$) is the set of all payoffs that might be realized when Player 1 plays the action p (resp. q).

Let M be the greatest norm of a payoff vector in the game:

$$M := \max_{i \in \mathcal{I}} \max_{j \in \mathcal{J}} \|u(i,j)\|.$$

Denote by F the set of all feasible payoffs in the repeated game:[3]

$$F := Conv(\{u(i,j) \mid i \in \mathcal{I}, j \in \mathcal{J}\}).$$

The definitions imply that $\|\bar{g}^n\| \leq M$ and $\bar{g}^n \in F$ for every $n \in \mathbb{N}$.

A *hyperplane* H in $\mathbb{R}^d$ is a $(d-1)$-dimensional affine subset of $\mathbb{R}^d$. For $\alpha \in \mathbb{R}^d$ and $\beta \in \mathbb{R}$, the hyperplane $H(\alpha,\beta)$ is defined by

$$H(\alpha,\beta) := \{x \in \mathbb{R}^d \mid \langle \alpha, x \rangle = \beta\}.$$

A hyperplane divides $\mathbb{R}^d$ into two half-spaces. The *closed half-spaces* determined by $H(\alpha,\beta)$ are the two closed sets

$$\bar{H}^+(\alpha,\beta) := \{x \in \mathbb{R}^d \mid \langle \alpha, x \rangle \geq \beta\} \quad \textit{and}$$
$$\bar{H}^-(\alpha,\beta) := \{x \in \mathbb{R}^d \mid \langle \alpha, x \rangle \leq \beta\}.$$

[3]Denote by $Conv(X)$ the convex hull of the set X.

The *open half-spaces* determined by $H(\alpha,\beta)$ are the two open sets

$$H^+(\alpha,\beta) := \{x \in \mathbb{R}^d \mid \langle \alpha, x\rangle > \beta\} \quad \textit{and}$$
$$H^-(\alpha,\beta) := \{x \in \mathbb{R}^d \mid \langle \alpha, x\rangle < \beta\}.$$

For an hyperplane H, we denote by H^+ and H^- (resp. $\bar{H}^+$ and $\bar{H}^-$) the two open (resp. closed) half-spaces that H determines. As is well known, for every $x, y \in \mathbb{R}^d$ the hyperplane $H(x-y, \langle x-y, y\rangle)$ is the hyperplane through y perpendicular to the line[4] segment xy.

Definition 6 *Let A and B be two nonempty sets and $H = H(\alpha,\beta)$ be a hyperplane in $\mathbb{R}^d$.*

i. H is called a supporting hyperplane *of A if A is contained in one of the two closed half-spaces determined by H and there is at least one point of A in H.*

For a point $a \in A$, we say that H supports A at a *if H is a supporting hyperplane of A and $a \in H$.*

ii. H is called a separating hyperplane *for the sets A and B if one of the sets is contained in one of the open half-spaces determined by H and the other set is contained in the other closed half-space, e.g., $A \subseteq H^-$ and $B \subseteq \bar{H}^+$, or $A \subseteq \bar{H}^-$ and $B \subseteq H^+$.*

We say that H separates A and B *if H is a separating hyperplane for A and B.*

Blackwell [3] provided a geometric sufficient condition for approachability. Denote by $\pi_A(x)$ the set of closest points to x in A.

Definition 7 *A closed set $A \subseteq \mathbb{R}^d$ is a* B-set *for Player 1 (resp. Player 2) if for every $x \in F \setminus A$ there exists $y \in \pi_A(x)$ and an action $p \in \Delta(\mathcal{I})$ (resp. $q \in \Delta(\mathcal{J})$) such that the hyperplane $H(x-y, \langle x-y, y\rangle)$ separates x and $R_1(p)$ (resp. $R_2(q)$):*

$$R_1(p) \subseteq \bar{H}^-(x-y, \langle x-y, y\rangle),$$

and

$$x \in H^+(x-y, \langle x-y, y\rangle).$$

The following result follows from Blackwell [3], and either Hou [8] or Spinat [17].

[4] For two vectors $x, y \in \mathbb{R}^d$, denote by xy the line segment between x and y, that is, $xy := \{\alpha x + (1-\alpha)y \mid \alpha \in [0,1]\} \subseteq \mathbb{R}^d$.

Theorem 8 *A closed set is approachable by player i if and only if it contains a B-set.*

Let H be a supporting hyperplane of the set A. We say that H *is in direction* λ, and denote it by H_λ, if λ is an outer-pointing normal vector to A at a point $a \in A \cap H$. For a supporting hyperplane H of a B-set, we denote by $p^H \in \Delta(\mathcal{I})$ the action of Player 1 such that the set $R_1(p^H)$ is contained in $\bar{H}^-$.

If B is a closed set that is contained in the interior of some set A, then $d(B, A^c) = \inf_{b \in B} d(b, A^c) > 0$. The next result follows (see Example 4).

Proposition 9 *Let $A \subseteq \mathbb{R}^d$. If there exists a B-set $B \subseteq Int(A)$ for player i, then A is strongly approachable by player i.*

The case in Example 5 can be generalized. A set that contains only one point $x \in \mathbb{R}^d$ is strongly approachable by Player 1 if and only if Player 1 can guarantee the payoff x, that is, if there is an action $p \in \Delta(\mathcal{I})$ such that $R_1(p) = \{x\}$. Indeed, assume that Player 1 cannot guarantee the payoff x. Then at any stage n such that $\bar{g}^n = x$ and for every action of Player 1, Player 2 has an action such that the expected payoff at stage $n+1$ does not equal x, and therefore $\bar{g}^{n+1} \neq x$.

Similarly, if $A \subseteq \mathbb{R}^d$ is a set of dimension less than d, then to strongly approach A the player must play from some stage on actions p^n such that $R_1(p^n)$ is contained in the affine subspace that is spanned by A. For this reason we are interested in sets of full dimension in $\mathbb{R}^d$. For the rest of the paper we consider strong approachability of closed convex sets. Blackwell [3] further proved that a closed convex set A is approachable by Player 1 if and only if $A \cap R_2(q) \neq \emptyset$ for every $q \in \Delta(\mathcal{J})$.

Together with Proposition 9 this implies the following.

Corollary 10 *Let A be a closed convex set. If $Int(A) \cap R_2(q) \neq \emptyset$ for every $q \in \Delta(\mathcal{J})$, then A is strongly approachable by Player 1.*

For every $\lambda \in \mathbb{R}^d$, consider the two-player one-shot zero-sum auxiliary game G_λ where the sets of actions are $\mathcal{I}$ and $\mathcal{J}$ and the payoff function w is defined by $w(i,j) := \langle \lambda, u(i,j) \rangle, \forall i, j$. From the Minmax Theorem, this game has a value, denoted by V_λ.

Blackwell (Corollary 2 in [3]) proved that $A \cap R_2(q) \neq \emptyset$ for every $q \in \Delta(\mathcal{J})$ if and only if $V_\lambda \geq \inf_{a \in A} \langle \lambda, a \rangle$ for every $\lambda \in \mathbb{R}^d$. His arguments, together with Corollary 10, deliver the following.

Proposition 11 *Let A be a closed convex set with nonempty interior. If*

$$V_\lambda > \inf_{a \in A} \langle \lambda, a \rangle, \ \forall \lambda \in \mathbb{R}^d \setminus \{0\},$$

then A is strongly approachable by Player 1.

Since A is convex we abuse notations and for every x denote by $\pi_A(x)$ the unique point in A that is closest to x. Let A be a closed convex set, $x \notin A$, and $y = \pi_A(x)$. Then in particular we have $x \in H^+(x-y, \langle x-y, y \rangle)$.

Remark 12 *Let A be a closed convex set with nonempty interior. Standard continuity and compactness arguments imply that if $V_\lambda > \inf_{a \in A} \langle \lambda, a \rangle$, for every $\lambda \in \mathbb{R}^d \setminus \{0\}$, then there exists $\delta > 0$ such that for every supporting hyperplane H of A there exist $p \in \Delta(\mathcal{I})$ such that:*

1. *$R_1(p) \in H^-$.*
2. *$d(R_1(p), H) \geq \delta$.*

A set A that satisfies this condition is called a strict B-set *for Player 1.*[5]

Remark 13 *Blackwell's condition for approachability of closed convex sets requires that the condition $V_\lambda \geq \inf_{a \in A} \langle \lambda, a \rangle$ will hold for every $\lambda \in \mathbb{R}^d$. Since both V_λ and $\langle \lambda, a \rangle$ are homogeneous in λ, it is sufficient to require that this inequality holds for every $\lambda \in S^{d-1}$, where $S^{d-1} := \{x \in \mathbb{R}^d \mid \|x\| = 1\}$ is the unit sphere in $\mathbb{R}^d$.*

2.3 Main Results

Since every strongly approachable set is also approachable, a necessary condition for strong approachability of a closed convex set A is that $V_\lambda \geq \inf_{a \in A} \langle \lambda, a \rangle$, for every $\lambda \in \mathbb{R}^d$. As Example 5 shows, this condition is not sufficient. As shown in Proposition 11, if $V_\lambda > \inf_{a \in A} \langle \lambda, a \rangle$, for every $\lambda \in \mathbb{R}^d \setminus \{0\}$, then A is strongly approachable. It therefore remains to study the case that $V_\lambda \geq \inf_{a \in A} \langle \lambda, a \rangle$, for every $\lambda \in \mathbb{R}^d \setminus \{0\}$, yet a strict inequality does not always hold.

Theorem 14 provides a sufficient condition when there is exactly one direction $\lambda' \in S^{d-1}$ in which a weak inequality holds. Theorem 15 provides a sufficient condition when there is more than one such direction λ'.

[5] Notice that if $B \subset A$ and B is a strict B-set, then A is strict a B-set.

Theorem 14 *Let A be a closed convex approachable set with nonempty interior. Assume that there exist*

- *exactly one direction $\lambda' \in S^{d-1}$ that satisfies $V_{\lambda'} = \inf_{a \in A} \langle \lambda', a \rangle$;*
- *among all sets $\{R_1(p), p \in \Delta(\mathcal{I})\}$ there is exactly one*[6] *set $R_1(p^*)$ in the closed half-space $\bar{H}^-_{\lambda'}$; and*
- *every point in $R_1(p^*) \cap H_{\lambda'} \cap A$ is a smooth boundary point of A.*

The set A is strongly approachable by Player 1 if and only if the set $R_1(p^*)$ satisfies the following two conditions:

C1 $R_1(p^*) \cap H_{\lambda'} \cap A \neq \emptyset$, *and*

C2 *every point in $R_1(p^*) \cap H_{\lambda'}$ has a neighborhood $\mathcal{N}$ such that $R_1(p^*) \cap \mathcal{N} \subseteq A$.*

The generalization of Theorem 14 to the case where there is more than one direction $\lambda' \in S^{d-1}$ that satisfies the first bullet in the statement of Theorem 14 is the following.

Theorem 15 *Let A be a closed convex approachable set with nonempty interior. Denote by $\mathcal{S}$ the set of all $\lambda' \in S^{d-1}$ that satisfy $V_{\lambda'} = \inf_{a \in A} \langle \lambda', a \rangle$. Assume that for every $\lambda' \in \mathcal{S}$, among all sets $\{R_1(p), p \in \Delta(\mathcal{I})\}$ there is exactly one*[7] *set $R_1(p_{\lambda'})$ in the closed half-space $\bar{H}^-_{\lambda'}$. Moreover, assume that every point in $R_1(p_{\lambda'}) \cap H_{\lambda'} \cap A$ is a smooth boundary point of A. The set A is strongly approachable by Player 1 if and only if there exist a set $R_1(p^*)$ such that for every $\lambda' \in \mathcal{S}$ the following hold:*

C0 $R_1(p^*) = R_1(p_{\lambda'})$.

C1 $R_1(p^*) \cap H_{\lambda'} \cap A \neq \emptyset$.

C2 *Every point in $R_1(p^*) \cap H_{\lambda'}$ has a neighborhood $\mathcal{N}$ such that $R_1(p^*) \cap \mathcal{N} \subseteq A$.*

To keep the formulation of Theorems 14 and 15 rather simple, we imposed the smoothness condition in both theorems. In Section 4 we weaken the assumptions of Theorems 14 and 15.

[6] While we assume the uniqueness of $R_1(p^*)$ as the only set $R_1(p) \subseteq \bar{H}^-_{\lambda'}$, the action p^* need not be unique. That is, there might be $p \neq p^* \in \Delta(\mathcal{I})$ such that $R_1(p) = R_1(p^*)$.

[7] As in Footnote 6, the action $p_{\lambda'}$ does not have to be unique.

2.4 Examples

In this subsection we provide two examples. In the first, the conditions of Theorem 14 are not met and the set is not strongly approachable, while in the second the conditions of this theorem hold and the set is strongly approachable. These example illustrate the ideas that underlie the proof of the main results.

Example 16 *Consider the game in Figure 3.*

		Player 2	
		L	R
Player 1	T	$(1,0)$	$(0,0)$
	B	$(0,0)$	$(0,1)$

Figure 3: The payoff matrix in Example 16.

In this game the set of feasible payoff vectors is $F = Conv(\{(0,0),(1,0),(0,1)\})$ *(see Figure 4). By Proposition 14.37 in [13], the set* $A = \left[0, \frac{1}{4}\right]^2$ *is approachable by Player 1. Nevertheless, as we now show, the set* A *is not strongly approachable by Player 1.*

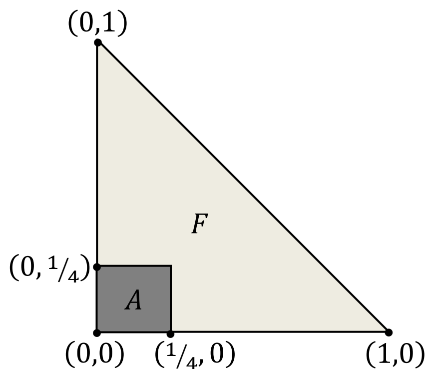

Figure 4: The sets F *and* A *in Example 16.*

Since each player has only two pure actions, we denote by p *(resp.* q*) the probability to play the pure action* T *(resp.* L*), and therefore the probability to play the pure action* B *(resp.* R*) is* $1-p$ *(resp.* $1-q$*).* $R_1(p)$ *is the set of all possible expected payoffs when Player 1 plays the mixed action* $[pT;(1-p)B]$*, that is,* $R_1(p) = Conv(\{(p,0),(0,1-p)\})$*. Three of the sets* $R_1(p)$ *are depicted in Figure 5.*

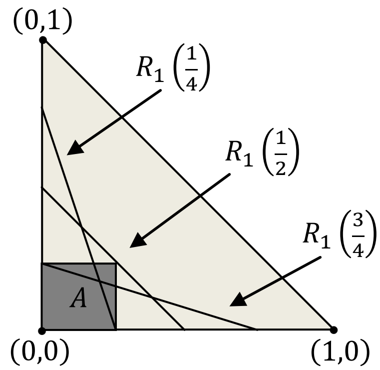

Figure 5: The sets A and three of the sets $R_1(p)$.

The set $R_2(\frac{1}{2})$ is the line segment $Conv(\{(0,\frac{1}{2}),(\frac{1}{2},0)\})$. If at stage N the average payoff is above *this line segment, that is, if $\bar{g}_1^N + \bar{g}_2^N > \frac{1}{2}$, then, by playing the mixed action $[\frac{1}{2}L;\frac{1}{2}R]$ in every subsequent stage Player 2 guarantees that the average payoff will remain above this line segment, and, in particular, will never get into A. We call the open half space $H^+ = \{x \in \mathbb{R}^2 \colon x_1 + x_2 > \frac{1}{2}\}$ a* dead zone*: once the average payoff gets into this half space, Player 2 can ensure that it will never get into A. Therefore, if A is strongly approachable, Player 1 must be able to ensure that the average payoff will never get into this dead zone.*

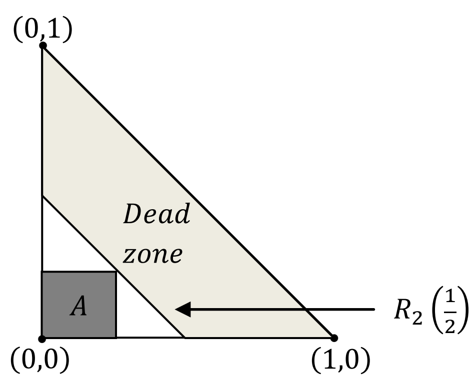

Figure 6: A dead zone determined by $R_2(\frac{1}{2})$.

We now argue that in this particular example, this goal is impossible. Indeed, for every $p \neq \frac{1}{2}$ the set $R_1(p)$ contains a point q'_p in the dead zone H^+. In addition, If Player 1 plays $[\frac{1}{2}T;\frac{1}{2}B]$, the pure action L of Player 2 guarantees a payoff $(\frac{1}{2},0)$ that is not in A. Therefore, the following strategy of Player 2 ensures that the long-run average payoff would always be outside of the set A:

- *As long as $p^n = [\frac{1}{2}T;\frac{1}{2}B]$, play L.*

- *For the first* $n \in \mathbb{N}$ *for which* $p^n \neq [\frac{1}{2}T; \frac{1}{2}B]$ *play* $q^n = q'_{p^n}$. *For* $m > n$ *play* $q^m = [\frac{1}{2}L; \frac{1}{2}R]$.

In particular, A *is not strongly approachable by Player 1. Note that in this example, Condition* ***C2*** *does not hold, and therefore by Theorem 14 the set* A *is not strongly approachable by Player 1.*

Remark 17 *The boundary of the set* A *in Example 16 is not smooth. One can provide in this example an approachable set* A *with a smooth boundary that is not strongly approachable.*

Example 18 *Consider the game in Figure 7 and let* A *be the closed disk with radius* 1 *around* $(0, 0)$.

		Player 2		
		L	N	R
	T	$(0, -1)$	$(-1, 4)$	$(1, 4)$
Player 1	M	$(11, -3)$	$(12, -5)$	$(13, -4)$
	B	$(-13, -7)$	$(-14, -5)$	$(-15, -7)$

Figure 7: The payoff matrix in Example 18.

The half space $H^- = \{x \in \mathbb{R}^2 \colon x_2 < -1\}$ *is a dead zone: once the payoff is in it, that is, once* $\bar{g}_2^N < -1$, *Player 2 can play* L *and ensure that the average payoff remains in* H^-. *Figure 8 depicts the sets* $R_1(T)$, $R_1(M)$, $R_1(B)$, A, *and the dead zone.*

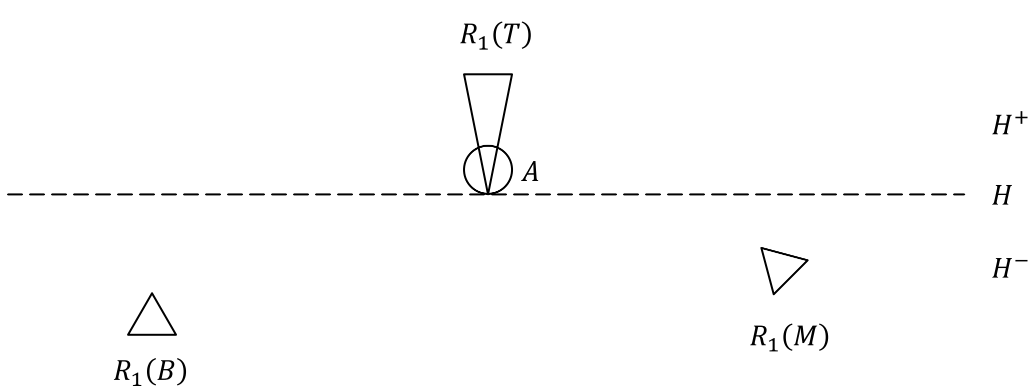

Figure 8: The set A *and the sets* $R_1(p)$ *in Example 18.*

We now explain how Player 1 can ensure that the average payoff will get into the set A and remain there. Assume that the stage N is large enough, so that the change in the average payoff in a single stage, which is of the order of $\frac{1}{N}$, is small. Once the payoff is in A, Player 1 can ensure that it will remain there, by properly playing the pure actions T, M, and B: if the average payoff is in the lower half of A, he plays T; if it is in the upper-left quarter, he plays M; and it is in the upper-right quarter, he plays B.

The question then is how to make the average payoff move into A when it is outside of this set. To this end we define the concept of the shade. Consider the set $R_1(p)$ as a source of light, and the set A as an opaque object. The shade *of A relative to $R_1(p)$ is the set of all points x in $\mathbb{R}^d$ such that all line segments that connect x and a point in $R_1(p)$ pass through A. The shade of A relative to the set $R_1(M)$ is denoted $\widetilde{A}^0_M$ in Figure 9.*

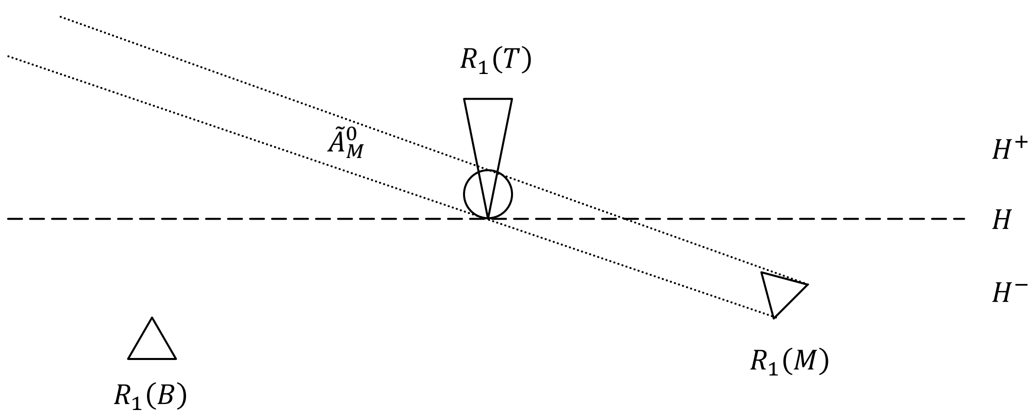

Figure 9: The set $\widetilde{A}^0_M$, which is the shade of A relative to $R_1(M)$.

If N is large enough, and if the average payoff is in the shade $\widetilde{A}^0_M$, then, by repeatedly playing the action M, Player 1 can make the average payoff move towards A, and eventually get into it. Unfortunately there is an exception to the last statement: the average payoff might "jump" over A, and, in fact, might get into the dead zone. To overcome this difficulty we will not consider the shade of A, but the shade of a subset of A that does not intersect the boundary of A.

Suppose now that the average payoff is not in the shade $\widetilde{A}^0_M$, but is in the shade of the union $A \cup \widetilde{A}^0_M$ relative to $R_1(B)$ (in fact, it is enough to consider the intersection of $A \cup \widetilde{A}^0_M$ with the set of feasible payoff vectors; see the set $\widetilde{A}^1_B$ in Figure 10). In this case, Player 1 can repeatedly play the action B until the average payoff moves inside $A \cup \widetilde{A}^0_M$, *and, in case the average payoff enters the set $\widetilde{A}^0_M$, play M until it enters A.*

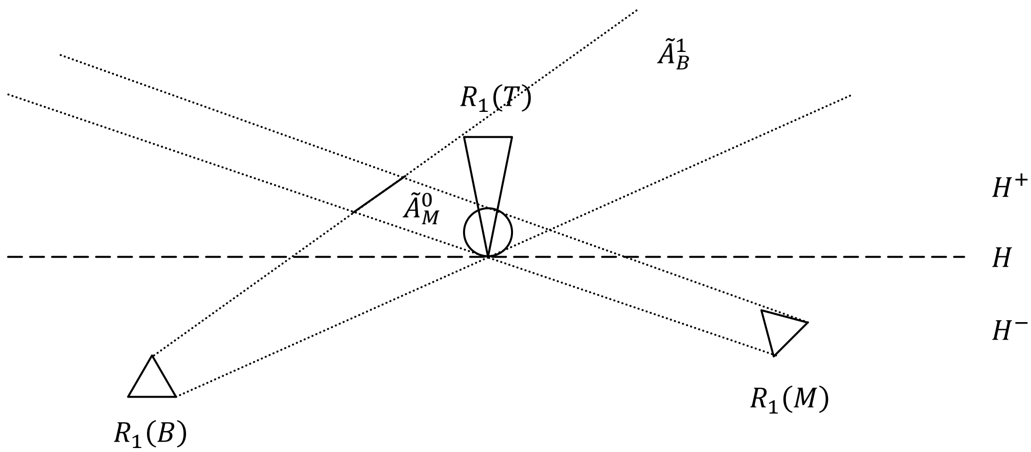

Figure 10: The set $\widetilde{A}^1_B$, which is the shade of $(A \cup \widetilde{A}^0_M) \cap F$ relative to $R_1(B)$.

To show that in general Player 1 can ensure that the average payoff enters A, we will construct a specific increasing sequence of sets, the first[8] *being A, such that each set in the sequence is the union of the previous one and its shade relative to one of the sets $R_1(p)$. We will also uniformly bound the number of stages required to move from one set to a set that precedes it in the sequence.*

To summarize, the strategy that Player 1 will use will be to play the action T for many stages, until the stage N is large enough, and then to play mixed actions that ensures that the average payoff moves to lower sets in the sequence of shades, until it reaches A. Once the average payoff reaches A, Player 1 will play in a way that ensures that the payoff remains within this set.

3. Proofs

Throughout this section we fix a repeated game G and a closed convex set A with nonempty interior that is approachable by Player 1.

To ensure that the average payoff will get inside the set A, we will use the following idea: for every $p \in \Delta(\mathcal{I})$ consider the set $R_1(p)$ as a source of light. Let $D \subseteq Int(A)$ be a full-dimensional set in $\mathbb{R}^d$. The set D is thought of as a body that casts shade. The line segment between any point in the shade of D (relative to the source of light $R_1(p)$)

[8]In fact, as mentioned before, the first set in the sequence will be a certain subset of A.

and any point in $R_1(p)$ intersects D, and therefore also $Int(A)$. Hence, if at some stage the average payoff is in the shade and Player 1 repeatedly plays the action p for a proper number of stages, then whatever Player 2 plays, provided the stage t is large enough, the average payoff will move into A. If the average payoff is not in the shade of D relative to $R_1(p)$, we can define the set $\widehat{D}$ to be the union of D and its shade relative to $R_1(p)$, and consider its shade relative to some other source of light $R_1(p')$ with $p' \in \Delta(\mathcal{I})$. If the average payoff is in the shade of $\widehat{D}$ relative to $R_1(p')$, then by playing repeatedly the action p' for a proper number of stages, Player 1 can ensure that the average payoff will get close to $\widehat{D}$, from where he can guarantee that it will get into A.

We will show that it is possible to form a sequence of shades, whose union covers all feasible vector payoffs that are not in D. Since $D \subseteq int(A)$ this will imply that Player 1 can ensure that the average payoff gets into A. Finally, we will show that Player 1 can ensure that the average payoff will stay inside the set A.

3.1 The Shades

Throughout this section we fix a full-dimensional convex strict[9] B-set $D \subseteq \mathbb{R}^d$. We now formally define the shade D_p of D relative to a "source of light" $R_1(p)$. For every $p \in \Delta(\mathcal{I})$ denote by D_p the set of all points, which are not in D and for which the line segment that joins them with any point in $R_1(p)$ intersects D. That is,

$$D_p := \{x \in \mathbb{R}^d \setminus D \mid xz \cap D \neq \emptyset, \forall z \in R_1(p)\}.$$

The set D_p is the shade that D casts when the unique source of light is $R_1(p)$.

We now prove that the union of a convex set with its shade is convex.

Lemma 19 *Let $D \subseteq \mathbb{R}^d$ be a convex set and let $p \in \Delta(\mathcal{I})$. The set $D \cup D_p$ is convex.*

Proof. Denote $\widehat{D} := D \cup D_p$ and let $x, y \in \widehat{D}$. Fix a point z in the line segment xy (see Figure 11). We will show that $z \in \widehat{D}$. To this end, we fix $r \in R_1(p)$ and show that the line segment rz intersects D. The three points x, y, r define a two-dimensional plane $P \subseteq \mathbb{R}^d$. Since $x, y \in \widehat{D}$ there are points $x' \in xr$ and $y' \in yr$ such that $x', y' \in D$. Since D is convex the line segment $x'y'$ is contained in D. The line segments xy, xr and yr are all contained in the plane P, so that $z, x', y' \in P$ as well, and therefore the line segment $x'y'$ is also in P. The line which is spanned by rz divides P into two half-planes, one

[9] A strict B-set is defined in Remark 12 in page 9.

contains x, x' and the other contains y, y'. Therefore the line segment $x'y'$ intersects the line segment zr. Since $x'y' \subseteq D$, the line segment zr intersects D, as desired. ■

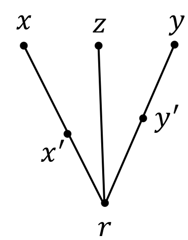

Figure 11: The points in the proof of Lemma 19.

Let $\varepsilon > 0$ and recall that $\mathcal{B}(F, \varepsilon)$ is the set of all points whose distance from at least one point in the set of feasible payoffs F is less than ε. Lemma 19 implies that the set $(D \cup D_p) \cap \mathcal{B}(F, \varepsilon)$ is convex since it is the intersection of two convex sets. Since $D \subseteq \mathcal{B}(F, \varepsilon)$, it follows that

$$(D \cup D_p) \cap \mathcal{B}(F, \varepsilon) = D \cup (D_p \cap \mathcal{B}(F, \varepsilon)).$$

We will be interested only in the intersection of the shade D_p with $\mathcal{B}(F, \varepsilon)$. The next lemma states that if D is full-dimensional and convex then there always exists $p \in \Delta(\mathcal{I})$ relative to which the shade is not empty.

Lemma 20 *Let $D \subset \mathcal{B}(F, \varepsilon)$ be a full-dimensional convex strict B-set. Then there exists $p \in \Delta(\mathcal{I})$ such that $D_p \cap \mathcal{B}(F, \varepsilon)$ is not empty.*

Proof. Since D is convex, the set of its smooth boundary points is dense[10] in ∂D (see, for example, [14], pp. 241–250). Let y be a smooth point of $\partial D \setminus \partial \mathcal{B}(F, \varepsilon)$, and denote by $\lambda \in S^{d-1}$ the direction of the supporting hyperplane $\widehat{H}$ of D through y. Since D is a strict B-set, by Remark 12, the distance between $\widehat{H}$ and $R_1(p^{\widehat{H}})$ is positive. We will show that the shade $D_{p^{\widehat{H}}}$ is not empty.

Let $\{y_k\}$ be a sequence of point on the outer-pointing normal to D at y such that $y_k \to y$. Assume that $y_k \notin D_{p^{\widehat{H}}}$ for every $k \in \mathbb{N}$, i.e., for every $k \in \mathbb{N}$ there is a point $r_k \in R_1(p^{\widehat{H}})$ such that the line segment $y_k r_k$ does not intersect D. A standard continuity argument implies that there is a point $r \in R_1(p^{\widehat{H}})$ such that the line segment yr intersects D only at y, that is, yr supports D at y. Since y is a smooth point of ∂D, there is only one supporting hyperplane of D at y, which is $\widehat{H}$. We deduce that the line segment ry

[10] ∂A is the boundary of A, that is, the set difference of the closure of A and the interior of A.

lies in $\widehat{H}$, and therefore $r \in \widehat{H}$. Thus, the distance between $\widehat{H}$ and $R_1(p^{\widehat{H}})$ is zero, which is a contradiction to the fact that the distance between $\widehat{H}$ and $R_1(p^{\widehat{H}})$ is positive. ■

We now construct inductively a nondecreasing sequence of convex sets $(\widetilde{D}^i)$ as follows. Denote $\widetilde{D}^0 = D$, and assume that the sets $\widetilde{D}^0, ..., \widetilde{D}^{i-1}$ are already defined. For every $p \in \Delta(\mathcal{I})$ denote by $\widetilde{D}^{i-1}_p$ the shade of $\widetilde{D}^{i-1}$ relative to $R_1(p)$. Choose $p = p_i \in \Delta(\mathcal{I})$ that maximizes[11] the volume of $\widetilde{D}^{i-1}_p \cap \mathcal{B}(F, \varepsilon)$. Denote this maximal volume by V^{i-1}. By Lemma 20 the maximal volume is positive, provided that $\widetilde{D}^{i-1} \subset \mathcal{B}(F, \varepsilon)$. Denote $\widetilde{D}^i := \widetilde{D}^{i-1} \cup (\widetilde{D}^{i-1}_{p_i} \cap \mathcal{B}(F, \varepsilon))$. That is, $\widetilde{D}^i$ is the union of $\widetilde{D}^{i-1}$ with its maximal shade restricted to $\mathcal{B}(F, \varepsilon)$. See Figure 12 for a schematic construction of some sets in this sequence.

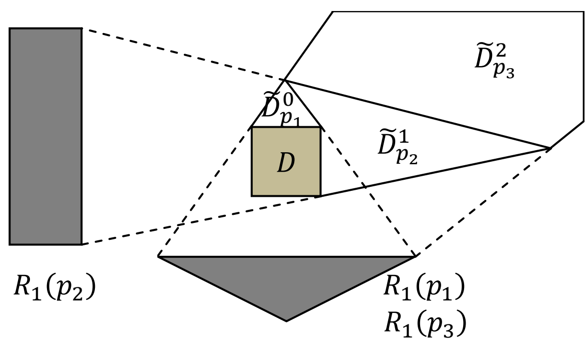

Figure 12: The set $\widetilde{D}^3 = D \cup \widetilde{D}^0_{p_1} \cup \widetilde{D}^1_{p_2} \cup \widetilde{D}^2_{p_3}$,
formed by $R_1(p_1)$, $R_1(p_2)$ and $R_1(p_3)$, with $p_3 = p_1$.

By Lemma 19, each set in the sequence $(\widetilde{D}^i)$ is convex. Using transfinite induction it follows that one can cover F, the set of feasible payoff vectors, with a transfinite sequence of shades. However, the proof of the main theorem requires this process to be finite. The following lemma shows that in fact the iterative process we presented covers F with finitely many shades.

Lemma 21 *There exists $K \in \mathbb{N}$ such that $F \subseteq \widetilde{D}^K$.*

Proof. Stage 1: $\bigcup_{k \in \mathbb{N}} \widetilde{D}^k = \mathcal{B}(F, \varepsilon)$.
The sequence $(\widetilde{D}^k)_{k \in \mathbb{N}}$ is a nondecreasing sequence of sets. Let $\widehat{D} := \bigcup_{k \in \mathbb{N}} \widetilde{D}^k$ and assume that $\widehat{D} \subsetneq \mathcal{B}(F, \varepsilon)$. As a union of a nondecreasing sequence of convex sets, $\widehat{D}$ is a convex set. Let $x' \in \mathcal{B}(F, \varepsilon) \setminus \widehat{D}$. By [6], the point x' can be separated from $\widehat{D}$ by a supporting

[11] Note that the function that assigns to each $p \in \Delta(\mathcal{I})$ the volume of $\widetilde{D}^{i-1}_p$ might have a discontinuity at a point $p_0 \in \Delta(\mathcal{I})$ if an extreme point of $R_1(p_0)$ lies on $\partial \widetilde{D}^{i-1}$.

hyperplane $\widehat{H}$ of $\widehat{D}$ at a smooth point $d \in \partial\widehat{D}$. Since $\widetilde{D}^0 = D$ is a strict B-set, so is $\widehat{D}$, and therefore as shown in the proof of Lemma 20, there is an action $p^{\widehat{H}} \in \Delta(\mathcal{I})$ such that the shade $\widehat{D}_{p^{\widehat{H}}}$ is not empty. In particular, the volume $V_{p^{\widehat{H}}}$ of $\widehat{D}_{p^{\widehat{H}}}$ is positive.

For every $k \in \mathbb{N}$ we have $\widetilde{D}^k \subseteq \mathcal{B}(F,\varepsilon)$. Since $\mathcal{B}(F,\varepsilon)$ is bounded, $\widehat{D}$ is also bounded. Because the sets $(\widetilde{D}^k_{p_{k+1}})_{k\in\mathbb{N}}$ are disjoint, the sequence of their volumes $(V^k)_{k\in\mathbb{N}}$ converges to zero. In particular there is some $N' \in \mathbb{N}$ such that $V^k < V_{p^{\widehat{H}}}$ for every $k > N'$. Consider the shade $\widetilde{D}^k_{p^{\widehat{H}}}$ of $\widetilde{D}^k$ relative to $R_1(p^{\widehat{H}})$, and denote its volume by $V^k_{p^{\widehat{H}}}$. For every $k > N'$, V^k is the maximal volume of a shade of $\widetilde{D}^k$, hence $V^k_{p^{\widehat{H}}} \leq V^k$. It follows that $\lim_{k\to\infty} V^k_{p^{\widehat{H}}} = 0$.

For every $p \in \Delta(\mathcal{I})$ the set $R_1(p)$ is convex. Hence, for a convex set E the shade of E relative to $R_1(p)$ can be defined by the set of all points such that the line segment that joins them with any extreme point of $R_1(p)$ intersects E. The set $R_1(p^{\widehat{H}})$ is convex and has finitely many extreme points, $u(p,j_1),...,u(p,j_{|\mathcal{J}|})$ where $\mathcal{J} = \{j_1,...,j_{|\mathcal{J}|}\}$. For every extreme point $u(p,j)$ of $R_1(p^{\widehat{H}})$, if $u(p,j) \in Int(\widehat{D})$, then there is $N_j \in \mathbb{N}$ such that $u(p,j) \in Int(\widetilde{D}^k)$ for every $k > N_j$. Therefore there exists $N' \in \mathbb{N}$ such that for every $1 \leq j \leq |\mathcal{J}|$, the vector $u(p,j)$ is either in $Int(\widetilde{D}^k)$ for every $k > N'$ or in the complement of $Int(\widetilde{D}^k)$, denoted by $(Int(\widetilde{D}^k))^c$, for every $k > N'$. If $u(p,j) \in Int(\widetilde{D}^k)$ then $u(p,j)$ has no effect on the shade $\widetilde{D}^k_{p^{\widehat{H}}}$, since the line segment that joins any point in $\mathbb{R}^d$ and $u(p,j)$ intersects $\widetilde{D}^k$; if $u(p,j)$ is in $(Int(\widetilde{D}^k))^c$, then it is either in $\partial\widetilde{D}^k$ or in $\mathcal{B}(F,\varepsilon) \setminus \widetilde{D}^k$. From the continuity of the line segments through each such $u(p,j)$ that intersect each $\widetilde{D}^k_{p^{\widehat{H}}}$, the sequence of shades $\widetilde{D}^k_{p^{\widehat{H}}}$ that is formed by $R_1(p^{\widehat{H}})$ converges to $\widehat{D}_{p^{\widehat{H}}}$, and therefore $V^k_{p^{\widehat{H}}} \to V_{p^{\widehat{H}}}$, which is a contradiction to the fact that $V_{p^{\widehat{H}}} > 0$.

Stage 2: There exists $K \in \mathbb{N}$ such that $F \subseteq \widetilde{D}^K$.
Assume to the contrary that for every $k \in \mathbb{N}$ there is a point $x_k \in F$ such[12] that $x_k \in D^{m_k}$ for $m_k \geq k$. The sequence of points x_k is bounded, and therefore it has a convergent subsequence. Let x' be the limit point of that subsequence. The set F is closed, and therefore $x' \in F$. In particular, $x' \in Int(\mathcal{B}(F,\varepsilon))$. Since $\lim_{k\to\infty} \widetilde{D}^k = \mathcal{B}(F,\varepsilon)$ it follows that there is $m \in \mathbb{N}$ such that $x' \in Int(\widetilde{D}^m)$. Therefore for a small enough neighborhood of x' all the points in the subsequence are in $\widetilde{D}^m$, which is a contradiction to the way we choose the points x_k. ■

[12]Since it is possible that for some k the set D^k is contained in $\mathcal{B}(F,\varepsilon) \setminus F$, we do not take $x_k \in D^k$ for every $k \in \mathbb{N}$.

3.2 The Dead Zone

Fix $x \in F \setminus A$, let $y = \pi_A(x)$, and denote $\lambda = y - x$. Assume that $V_\lambda = \inf_{a \in A}\langle \lambda, a\rangle$. Then, there exists an action $q^* = q^*_\lambda \in \Delta(\mathcal{J})$ such that $R_2(q^*) \subseteq \bar{H}^+(x - y, \langle x - y, y\rangle)$. Note that $R_2(q^*) \nsubseteq H^+(x-y, \langle x-y, y\rangle)$, otherwise A would not have been approachable. In other words, there exists $p \in \Delta(\mathcal{I})$ such that $u(p, q^*) \in H(x-y, \langle x-y, y\rangle)$ (See Figure 13).

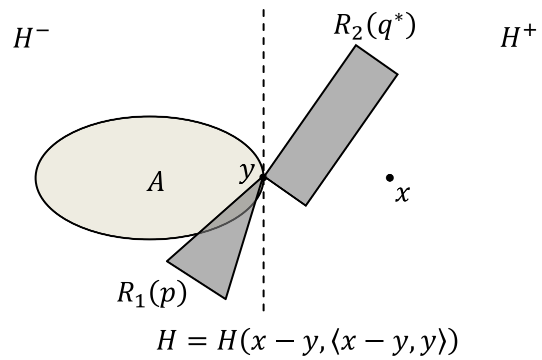

Figure 13: The hyperplane $H = H(x - y, \langle x - y, y\rangle)$
and the sets $R_1(p) \subseteq \bar{H}^-$ and $R_2(q^*) \subseteq \bar{H}^+$.

If $\bar{g}^n$, the average payoff at stage n, were in $H^+(x - y, \langle x - y, y\rangle)$, then by playing the action q^* in all stages after stage n, Player 2 could have guaranteed that for every $m \geq n$ the average payoff $\bar{g}^m$ would remain in $H^+(x - y, \langle x - y, y\rangle)$, and, in particular, outside A. Hence we deduce the following conclusion.

Corollary 22 *Let $A \subseteq \mathbb{R}^d$ be a closed convex set that is strongly approachable by Player 1, let $x \notin A$, let $y = \pi_A(x)$, and let $\lambda := y - x$. Assume that $V_\lambda = \inf_{a \in A}\langle \lambda, a\rangle$. Let σ_1 be a strategy of Player 1 that strongly approaches A. Then $\bar{g}^n \notin H^+(x-y, \langle x-y, y\rangle)$ for every strategy σ_2 of Player 2 and every $n \in \mathbb{N}$.*

With the notations of Corollary 22, we deduce that once the average payoff lies in the open half-space $H^+(x - y, \langle x - y, y\rangle)$, it will never get out of it, provided Player 2 plays properly. This observation leads us to the following definition.

Definition 23 *Let $A \subseteq \mathbb{R}^d$ be a closed convex set that is strongly approachable by Player 1, let $x \notin A$, let $y = \pi_A(x)$, let $\lambda := y - x$, and assume that $V_\lambda = \inf_{a \in A}\langle \lambda, a\rangle$. The open half-space $H^+(x - y, \langle x - y, y\rangle)$ is called a* dead zone *for Player 1.*

Remark 24 (Remark 2, continued) *One of the main reasons that the characterization of strongly approachable sets is significantly harder for standard repeated games than for the expected deterministic repeated game, is that in the standard game the dead zone loses its meaning: When Player 1 plays a mixed action p such that $R_1(p) \subset \bar{H}^-$, the stage payoff might not be in the set $R_1(p)$, and therefore the stage payoff might be in the dead zone, so that the average payoff might get into the dead zone. Similarly, when Player 2 plays the action q^*, the stage payoff might not be in the set $R_2(q^*)$, and therefore as above the average payoff might leave the dead zone.*

3.3 An Auxiliary Result

Let $x \in \mathbb{R}^d$, and let $t \in \mathbb{N}$. Denote by $G_{x,t}$ the variation of G in which the play starts at stage $t+1$ and the average payoff in the first t stages is taken to be x. That is, a strategy of Player 1 (resp. Player 2) is a function $\sigma_1 : \bigcup_{n=t+1}^{\infty} \mathcal{H}_{n-1} \longrightarrow \Delta(\mathcal{I})$ (resp. $\sigma_2 : \bigcup_{n=t+1}^{\infty} \mathcal{H}_{n-1} \longrightarrow \Delta(\mathcal{J})$), and if $(p^n, q^n)_{n=t+1}^{N}$ are the actions taken by the players at stages $t+1, t+2, ..., N$, then

$$\bar{g}^N = \frac{tx + \sum_{n=t+1}^{N} u(p^n, q^n)}{N}.$$

Assume for a moment that there is exactly one dead zone $H^+_{\lambda'}$, defined by a supporting hyperplane $H_{\lambda'}$ of A. Roughly speaking, we will show that it is possible to form a sequence of shades, whose union covers all feasible vector payoffs that are in the half-space $H^-_{\lambda'}$ that contains A. We will then show that we can control the pace of progress inside each shade, in such a way that the average payoff does not 'jump' over A. Thus, if the average payoff is not inside the dead zone, Player 1 can ensure that it will eventually get into A. We will finally prove that Player 1 has a strategy that ensures that the average payoff remains inside the set A.

It turns out that this procedure also works when there are several dead zones, that is, several $\lambda \in \mathbb{R}^d \setminus \{0\}$ for which $V_\lambda = \inf_{a \in A} \langle \lambda, a \rangle$. This observation will be the main ingredient of the proof of Theorem 15.

The following auxiliary result plays a major role in the proof of Theorem 14; its proof is the most challenging part of the proof of this theorem.

Proposition 25 *Let A be an approachable convex set with nonempty interior. Assume that there exist exactly one direction $\lambda' \in S^{d-1}$ such that $V_{\lambda'} = \inf_{a \in A} \langle \lambda', a \rangle$. Denote* *by p^* an optimal strategy (action) in the auxiliary*[13] *game $G_{\lambda'}$, so that $R_1(p^*)$ is in the closed half-space $\bar{H}^-_{\lambda'}$. Assume that every point in $R_1(p^*) \cap H_{\lambda'} \cap A$ is a smooth boundary point of A; and*

C1 *$R_1(p^*) \cap H_{\lambda'} \cap A \neq \emptyset$; and*

C2 *every point in $R_1(p^*) \cap H_{\lambda'}$ has a neighborhood $\mathcal{N}$ such that $R_1(p^*) \cap \mathcal{N} \subseteq A$.*

Then for every $x \in (F \cap H^-_{\lambda'}) \setminus A$ and every $t \in \mathbb{N}$ there exist a strategy σ_1 of Player 1 and $\widehat{n} \in \mathbb{N}$ such that under σ_1 the average payoff $\bar{g}^{\widehat{n}}$ in the game $G_{x,t}$ belongs to $Int(A)$ for every strategy σ_2 of Player 2.

In words, if there is only one dead zone, then Player 1 has a strategy such that, regardless of the stage of the game or the strategy of Player 2, and provided the current average payoff is not in the dead zone, the average payoff will get inside A after a finite time.

Remark 26 *Proposition 25 is more general than what we really need, since the claim is true for every $x \in (F \cap H^-_{\lambda'}) \setminus A$. In the proofs of Theorem 14 and Theorem 15, the point x will be some point in $R_1(p^*)$.*

3.4 Proof of Proposition 25

Define $C := Conv(\{x\} \cup R_1(p^*))$, i.e., $C = Conv(\{x, u(p^*, j_1), ..., u(p^*, j_{|\mathcal{J}|})\})$ where $\mathcal{J} = \{j_1, ..., j_{|\mathcal{J}|}\}$ is Player 2's set of pure actions. Since $x \notin H_{\lambda'}$ it follows by Assumption **C2** that

$$C \cap H_{\lambda'} = R_1(p^*) \cap H_{\lambda'} \subseteq A. \tag{1}$$

Fix $\varepsilon > 0$ and denote by $\mathcal{N}_{\lambda'}$ the open ball of radius $\varepsilon > 0$ around λ' in the unit sphere S^{d-1}, that is, $\mathcal{N}_{\lambda'} = \mathcal{B}(\{\lambda'\}, \varepsilon) \cap S^{d-1}$.

Definition 27 *Denote by $Y = Y(\mathcal{N}_{\lambda'})$ the set of all points $y \in F$ for which there exists a separating hyperplane for y and A in some direction $\lambda \in \mathcal{N}_{\lambda'}$.*

Notice that $H_{\lambda'}$ separates A and $H^+_{\lambda'}$, and therefore $H^+_{\lambda'} \subseteq Y$. It follows that any point $x' \in F \setminus Y$ satisfies $x' \in \bar{H}^-_{\lambda'}$ (see Figure 14). From Assumptions **C1** and **C2** it follows that $R_1(p^*) \cap H_{\lambda'} \subseteq A$ and since every point in $R^*_1(p) \cap H_{\lambda'} \cap A$ is a smooth boundary point of A it also follows that $\delta_1 := d(C \setminus A, Y) > 0$.

[13] Recall that the game $G_{\lambda'}$ is the one-shot game in which the payoff function is u, projected in the direction λ'; see page 8.

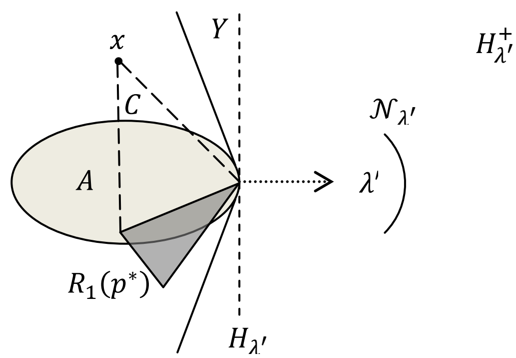

Figure 14: The sets C and $Y = Y(\mathcal{N}_{\lambda'})$.

Lemma 28 *For any open ball $\mathcal{N}_{\lambda'} \subseteq S^{d-1}$ around λ' there exists $\delta_2 > 0$ such that if H is a supporting hyperplane of A in direction $\lambda \in S^{d-1} \setminus \mathcal{N}_{\lambda'}$, then the distance between H and $R_1(p^H)$ is at least δ_2.*

Proof. Since the direction λ of H is different than λ', it follows that $V_\lambda > \inf_{a \in A} \langle \lambda, a \rangle$. Therefore, from Remark 12, there exists $\delta_\lambda > 0$ such that the distance between H and $R_1(p^H)$ is at least δ_λ. Since the set $S^{d-1} \setminus \mathcal{N}_{\lambda'}$ is closed, standard continuity and compactness arguments imply that there exists $\delta_2 > 0$ such that for every supporting hyperplane H of A in direction $\lambda \in S^{d-1} \setminus \mathcal{N}_{\lambda'}$, the distance between H and $R_1(p^H)$ is at least δ_2. ■

Lemma 29 *There exists an open ball $\mathcal{N}_{\lambda'} \subseteq S^{d-1}$ around λ' such that $C \setminus A \subseteq F \setminus Y(\mathcal{N}_{\lambda'})$.*

Proof. Eq. (1) implies that for any $x' \in C \setminus A$ the distance between x' and $H_{\lambda'}$ is greater then zero, so that $x' \in H^-_{\lambda'}$. Since $A \subseteq \bar{H}^-_{\lambda'}$ it follows that $H_{\lambda'}$ does not separate x' and A. From continuity of the supporting hyperplanes of the convex set A it follows that there exists an open ball $\mathcal{N}_{\lambda',x'} \subseteq S^{d-1}$ around λ' such that for every $\lambda \in \mathcal{N}_{\lambda',x'}$ the supporting hyperplane of A in the direction λ does not separate x' and A. The set C is the convex hull of a finite set of points. Denote by $\{z_i\}$ those points which are not in A. Every point z_i determines a corresponding ball $\mathcal{N}_{\lambda',z_i}$ of λ'. Define $\mathcal{N}_{\lambda'} := \bigcap_{z_i} \mathcal{N}_{\lambda',z_i}$. The set $\mathcal{N}_{\lambda'}$ satisfies that for every $\lambda \in \mathcal{N}_{\lambda'}$ the supporting hyperplane of A in direction λ does not separate A and each z_i. From the convexity of C, and since the extreme points of C that are not $\{z_i\}$ are in A, it follows that for every $\lambda \in \mathcal{N}_{\lambda'}$ the supporting hyperplane of A in direction λ does not separate any $x' \in C \setminus A$ and A. It follows that $C \setminus A \subseteq F \setminus Y(\mathcal{N}_{\lambda'})$. ■

Let $\delta \leq \min\{\frac{\delta_1}{2}, \frac{\delta_2}{2}\}$, and let $a \in Int(A)$. For every $\delta' \in (0,1)$ define a set $B = B_{\delta'}$ by shrinking the set A relatively to the point a by a scale factor of $1-\delta'$, that is, $B = B_{\delta'} := \{x(1-\delta') + \delta' a \mid x \in A\}$. For every $y \in \partial A$ let $y' := (1-\delta')y + \delta' a \in \partial B$. Since the set A is convex, $B = B_{\delta'} \subseteq A$. From continuity, there exists a sufficiently small $\delta' > 0$ such that for every $y \in \partial A$, the distance between y and y' is not greater than δ, that is, $\sup_{y \in \partial A} d(y, y') \leq \delta$.

Fix a neighborhood $\mathcal{N}_{\lambda'}$ of λ' that satisfies Lemma 29 and denote by $\widehat{Y} = \widehat{Y}(\mathcal{N}_{\lambda'}, \delta')$ the set of all points $y \in F$ such that there exists a separating hyperplane for y and B in direction $\lambda \in \mathcal{N}_{\lambda'}$. The direction of every supporting hyperplane $\widehat{H}$ of the set B that separates B and any point $x' \in F \setminus \widehat{Y}$ is in $S^{d-1} \setminus \mathcal{N}_{\lambda'}$. Assume $\widehat{H}$ supports B at the point $(1-\delta')y$. The direction of the parallel hyperplane H that supports A at the point y is also in $S^{d-1} \setminus \mathcal{N}_{\lambda'}$, and therefore by Lemma 28 the distance between H and $R_1(p^H) \subseteq H^-$ is greater than or equals to $\delta_2 \geq 2\delta$. The distance between H and $\widehat{H}$ is less than δ, so that $R_1(p^H) \subseteq \widehat{H}^-$, and the distance between $\widehat{H}$ and $R_1(p^H)$ is greater than or equals to δ.

Lemma 30 $C \setminus A \subseteq F \setminus \widehat{Y}(\mathcal{N}_{\lambda'}, \delta')$.

Proof. Since $\sup_{y \in \partial A} d(y, y') \leq \delta$, the set $\widehat{Y}$ contains points whose distance from Y is not greater than $\delta \leq \frac{\delta_1}{2}$. The distance between $C \setminus A$ and Y is δ_1, and therefore there are no points of $C \setminus A$ in $\widehat{Y}$. ■

Similarly to the way we have constructed the sequence of shades over a strict B-set,[14] we now construct inductively a nondecreasing sequence of convex sets $(\widetilde{B}^i)$ where $\widetilde{B}^0 = B$ and each $\widetilde{B}^i$ is the union of $\widetilde{B}^{i-1}$ with its maximal shade restricted to $\mathcal{B}(F, \varepsilon)$, that is, $\widetilde{B}^i := \widetilde{B}^{i-1} \cup (\widetilde{B}^{i-1}_{p_i} \cap \mathcal{B}(F, \varepsilon))$. From Lemma 19 we get that each set in the sequence $(\widetilde{B}^i)$ is convex. From Lemma 20 we deduce that as long as $\widetilde{B}^{i-1} \not\supseteq \mathcal{B}(F, \varepsilon) \setminus \widehat{Y}$, each shade B^i is not empty. The proof of the latter claim follows from the proof of Lemma 20, where instead of choosing any smooth point of $\partial \widetilde{B}^{i-1} \setminus \partial \mathcal{B}(F, \varepsilon)$, we choose a smooth point y of $\partial \widetilde{B}^{i-1} \setminus \partial(\mathcal{B}(F, \varepsilon) \setminus \widehat{Y})$ such that the supporting hyperplane $\widehat{H}$ through y is in the direction λ for some $\lambda \in S^{d-1} \setminus \mathcal{N}_{\lambda'}$. Such a smooth boundary point exists since $\widetilde{B}^{i-1}$ is convex, and therefore, as mentioned before, the set of its smooth boundary points is dense (see, for example, [14], pp. 241–250). From Lemma 21 we deduce that there exists $K \in \mathbb{N}$ such that $F \setminus \widehat{Y} \subseteq \widetilde{B}^K$. The proof of this claim is similar to the proof of Lemma 21, when replacing $\mathcal{B}(F, \varepsilon)$ by $\mathcal{B}(F, \varepsilon) \setminus \widehat{Y}$ and F by $F \setminus \widehat{Y}$.

[14]See page 17. Recall that $\widetilde{B}^i$ depends on ε.

The sets A and B are convex and satisfy that $B \subseteq Int(A)$ and $\partial B \cap \partial(A) = \emptyset$. Hence there exists a minimal distance η' between ∂A and ∂B. Denote $\eta = \frac{\eta'}{2K}$. Since the distance between two points (payoffs) in F is bounded by $2M$, it follows that for $N' > \frac{2M}{2\eta}$, the distance between the average payoff at stage n and the average payoff at stage $n+1$ is at most 2η for every $n > N'$.

We now describe a strategy σ_1 of Player 1 in the game $G_{x,t}$. We will then show that this strategy guarantees that the average payoff will get inside A. Let N' be the first integer that is greater than or equal to $\frac{2M}{2\eta}$. For every $n \geq t+1$, the strategy σ_1 plays at stage n as follows:

- If $n \leq N'$, the strategy σ_1 plays the action p^*.
- If $n > N'$, let k, $1 \leq k \leq K$, be the minimal natural number such that $\bar{g}^n \in Cl(\mathcal{B}(\widetilde{B}^k, k \cdot \eta))$. The strategy σ_1 plays the action p_k.

If $t \leq N'$, the action p^* is played (at least) $N' - t + 1$ times. By playing the action p^* at the first $N' - t + 1$ times, the average payoff stays in $C = Conv(\{x\} \cup R_1(p^*))$, and by Lemma 30, it is in A or in $F \setminus \widehat{Y}$. We will show that if the average payoff is in $F \setminus \widehat{Y}$, then the strategy σ_1 of player 1 ensures that there exists n sufficiently large such that $\bar{g}^n$ is in A.

To simplify the notations, denote by B^k the shade of $\widetilde{B}^{k-1}$ relative to $R_1(p_k)$. Notice that if the average payoff is very close to the boundary (or in the boundary) of B^k for some $1 \leq k \leq K$, then by playing the action p_k Player 1 cannot guarantee that the average payoff would not 'jump' over $\widetilde{B}^{k-1}$. For example, in Figure 15, if Player 1 plays the action p_4 and Player 2 plays the pure action j_1, the average payoff might jump to the other side of the set $\widetilde{B}^3 = B \cup B^1 \cup B^2 \cup B^3$.

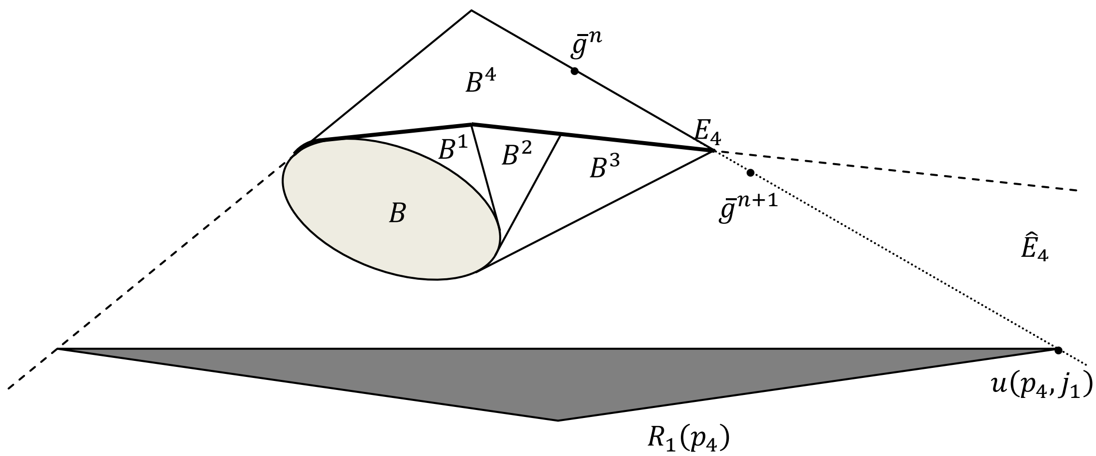

Figure 15: A case where the average payoff $\bar{g}^n$ is in the boundary of $\widetilde{B}^4$, and the sets E_4 (thick curve) and $\widehat{E}_4$ (bounded by the dashed lines).

Nevertheless, as we argue now, by repeatedly playing the action p_k, Player 1 can guarantee that the average payoff would not be very far from $\widetilde{B}^{k-1}$. We will show that Player 1 can guarantee that the average payoff would be in a closed $\frac{M}{n+1}$-neighborhood of $\widetilde{B}^{k-1}$, that is, in $Cl(\mathcal{B}(\widetilde{B}^{k-1}, \frac{M}{n+1}))$.

The set B^k was 'built upon' $\widetilde{B}^{k-1}$, and therefore the set $E_k := \partial B^k \cap \partial \widetilde{B}^{k-1}$ is not empty, and every point in this set is on the line segment that joins a point in B^k and a point in $R_1(p_k)$ (see the thick curve in Figure 15). The set E_k is a subset of $\partial \widetilde{B}^{k-1}$, and therefore every point in this set is contained in a hyperplane[15] that supports $\widetilde{B}^{k-1}$, and every such hyperplane divides the space into two half-spaces, one contains $\widetilde{B}^{k-1}$. Denote by $\widehat{E}_k$ the intersection of all these half-spaces that contain $\widetilde{B}^{k-1}$. Notice that $R_1(p_k) \subseteq \widehat{E}_k$ (see Figure 15).

Assume $\bar{g}^{N'} \notin A$. As mentioned above it follows that $\bar{g}^{N'} \in F \setminus \widehat{Y}$. By Lemma 21, there exists $k \leq K$ such that $\bar{g}^{N'}$ is in the shade B^k. Without loss of generality assume that $\bar{g}^{N'} \in B^K$. Since $R_1(p_K) \subseteq \widehat{E}_K$, there is $\bar{n}_1 \in \mathbb{N}$, independent of $\bar{g}^{N'}$, such that by playing n_1 times the action p_K, for some $n_1 \leq \bar{n}_1$, the average payoff $\bar{g}^{N'+n_1}$ would be in an η-neighborhood of E_K. Since $E_K \subseteq \partial \widetilde{B}^{K-1}$, the average payoff would be in an η-neighborhood of $\widetilde{B}^{K-1}$, i.e., in $Cl(\mathcal{B}(\widetilde{B}^{K-1}, \eta))$.

The point $\pi_{\widetilde{B}^{K-1}}(\bar{g}^{N'+n_1})$ is the closest point in $\widetilde{B}^{K-1}$ to the average payoff at stage $N' + n_1$. Assume without loss of generality that $\pi_{\widetilde{B}^{K-1}}(\bar{g}^{N'+n_1})$ is in B^{K-1}. Then the

[15] Recall that for each $k \in \mathbb{N}$ the set $\widetilde{B}^k$ is convex.

average payoff is in an η-neighborhood of B^{K-1}. For every $1 \leq k \leq K$ denote by S_k the set of all points that lie on some line segment from the shade B^k to $R_1(p^k)$, that is, $S_k := \{x \in br \mid b \in B^k, r \in R_1(p^k)\}$. The set B^k is a subset of S_k, for every $1 \leq k \leq K$. Since the set B^{K-1} is the shade of $\widetilde{B}^{K-2}$ relative to $R_1(p_{K-1})$, for every payoff in $R_1(p_{K-1})$ that is obtained at stage $N' + n_1 + 1$, the $(N' + n_1 + 1)$-stage average payoff will get closer to S_{K-1}, that is, if the action p_{K-1} is played at stage $N' + n_1 + 1$, then

$$d\left(\bar{g}^{N'+n_1+1}, S_{K-1}\right) \leq d\left(\bar{g}^{N'+n_1}, S_{K-1}\right).$$

Indeed, for a given payoff in $R_1(p_{K-1})$ the line segment that joins it with the average payoff $\bar{g}^{N'+n_1}$, and the line segment that joins it with the point in S_{K-1} closest to the average payoff $\pi_{S_{K-1}}(\bar{g}^{N'+n_1})$ intersect at that payoff, so 'walking along' that lines, the points get closer to each other (see Figure 16).

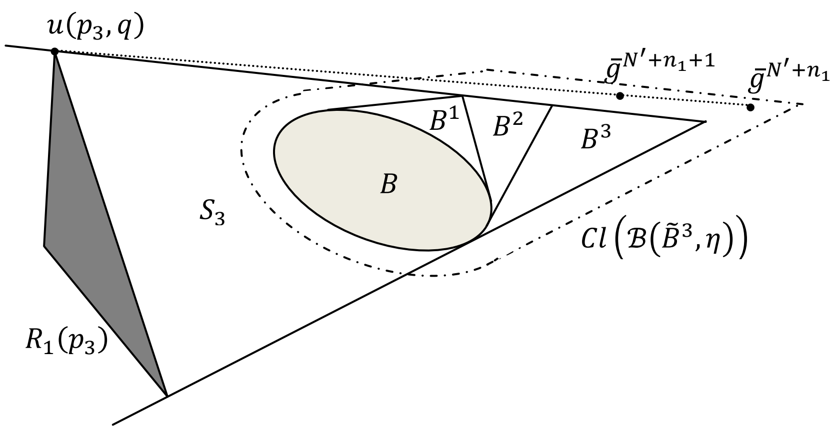

Figure 16: For any $u(p_3, q)$ played at stage $N' + n_1 + 1$, $\bar{g}^{N'+n_1+1}$ would be on the (dashed) line segment that joins $\bar{g}^{N'+n_1}$ and $u(p_3, q)$.

It follows that for any $s \in \mathbb{N}$, by playing s times the action p_{K-1} the distance between the $(N' + n_1 + s)$-stage average payoff and S_{K-1} is at most η:

$$d\left(\bar{g}^{N'+n_1+s}, \pi_{S_{K-1}}(\bar{g}^{N'+n_1+s})\right) \leq \eta. \tag{2}$$

Since for every $m \geq N'$ the distance between the average payoff at stage m and the average payoff at stage $m+1$ is at most 2η, and since $R_1(p_{K-1}) \subseteq \widehat{E}_{K-1}$, there is $\bar{n}_2 \in \mathbb{N}$, independent of $\bar{g}^{N'+n_1}$, such that by playing n_2 times the action p_{K-1}, for some $n_2 \leq \bar{n}_2$, the closest point in S_{K-1} to the $(N' + n_1 + n_2)$-stage average payoff would be in an

η-neighborhood of E_{K-1}, i.e.,

$$d\left(\pi_{S_{K-1}}(\bar{g}^{N'+n_1+n_2}), E_{K-1}\right) \leq \eta. \tag{3}$$

From Eqs. (2) and (3) and the triangle inequality we deduce that the distance between the $(N'+n_1+n_2)$-stage average payoff and E_{K-1} satisfies

$$\begin{aligned} d\left(\bar{g}^{N'+n_1+n_2}, E_{K-1}\right) &\leq d\left(\bar{g}^{N'+n_1+n_2}, \pi_{S_{K-1}}(\bar{g}^{N'+n_1+n_2})\right) + d\left(\pi_{S_{K-1}}(\bar{g}^{N'+n_1+n_2}), E_{K-1}\right) \\ &\leq \eta + \eta = 2\eta. \end{aligned}$$

Since $E_{K-1} \subseteq \partial \widetilde{B}^{K-2}$, this implies that $\bar{g}^{N'+n_1+n_2} \in Cl(\mathcal{B}(\widetilde{B}^{K-2}, 2\eta))$. Proceeding inductively we deduce that there is a collection $(\bar{n}_k)_{k=1}^K$ of natural numbers and there exists $t < \sum_{k=1}^K \bar{n}_k$ such that the average payoff $\bar{g}^{N'+t}$ is in a $(K\cdot\eta)$-neighborhood of $\widetilde{B}^{K-K} = B$.

Since $\eta = \frac{\eta'}{2K}$ this implies that there is a finite number $\widehat{n} \in \mathbb{N}$ such that $\bar{g}^{\widehat{n}} \in Cl(\mathcal{B}(B, \frac{\eta'}{2})) \subseteq Int(A)$, as desired.

This concludes the proof of Proposition 25.

Proposition 25 shows that for any point $x \in F \setminus \widehat{Y}$ we can reach the interior of A in the game $G_{x,t}$ after a finite number of stages. The proof of Proposition 25 shows that the time period until getting into A is uniformly bounded by $\bar{n} := \bar{n}_1 + ... + \bar{n}_K$. Hence, we get the following result.

Proposition 31 *Let A be an approachable convex set with nonempty interior that satisfies the assumptions of Proposition 25, let $\delta > 0$, and*[16] *let $B = B_\delta$. Let η' and $\widehat{Y}$ be as in the proof of Proposition 25. For every $t \in \mathbb{N}$ there exists $\bar{n} \in \mathbb{N}$ such that for any point $x \in F \setminus \widehat{Y}$, the average payoff $\bar{g}^n$ in the game $G_{x,t}$ is in an $\frac{\eta'}{2}$-neighborhood of B, for some $t \leq n \leq \bar{n}$, when Player 1 plays the strategy described in the proof of Proposition 25 and for every strategy of Player 2.*

3.5 Proof of Theorem 14

In this section we prove Theorem 14, that takes care of the case in which there is exactly one direction $\lambda' \in S^{d-1}$ for which $V_{\lambda'} = \inf_{a\in A}\langle \lambda', a\rangle$. In this case, Player 1 should ensure that the average payoff stays away of only one half-space (dead zone).

Assume that Conditions **C1** and **C2** are satisfied. It follows that $R_1(p^*) \cap H_{\lambda'} \subseteq A$. The following lemma shows that there exists $\delta' > 0$ such that all the points in $R_1(p^*)$

[16] Recall that $B_\delta := \{x(1-\delta') + \delta' a \mid x \in A\}$, where a is a predetermined point in the interior of A.

whose distance from $R_1(p^*) \cap H_{\lambda'}$ is at most δ' are in A, and the points that are not in $R_1(p^*) \cap H_{\lambda'}$ are in $Int(A)$.

Lemma 32 *There exists $\delta' > 0$ such that for every point x in $R_1(p^*) \cap H_{\lambda'}$ the ball $\mathcal{B}(x, \delta')$ satisfies $(\mathcal{B}(x, \delta') \cap R_1(p^*)) \setminus (R_1(p^*) \cap H_{\lambda'}) \subseteq Int(A)$.*

Proof. $R_1(p^*)$ is a convex polytope whose extreme points are $u(p^*, j_1), \ldots, u(p^*, j_{|\mathcal{J}|})$, where $\mathcal{J} = \{j_1, ..., j_{|\mathcal{J}|}\}$ is Player 2's pure action set, and so it has a finite number of faces and edges. Since the set A is convex, it is sufficient to show that every face or edge of $R_1(p^*)$ that intersects $H_{\lambda'}$, but is not contained in it, intersects $Int(A)$. Such faces and edges satisfy that they support $R_1(p^*)$ at an extreme point in $H_{\lambda'}$ and at an extreme point not in $H_{\lambda'}$. Since $R_1(p^*)$ has finitely many faces and edges, the lemma will follow.

Let $u(p^*, j_k)$ and $u(p^*, j_{k'})$ be extreme points of $R_1(p^*)$ such that $u(p^*, j_k) \in H_{\lambda'}$ and $u(p^*, j_{k'}) \notin H_{\lambda'}$. By Condition **C2**, the point $u(p^*, j_k)$ has a neighborhood $\mathcal{N}$ such that $R_1(p^*) \cap \mathcal{N} \subseteq A$. Recall that by assumption every point in $R_1(p^*) \cap H_{\lambda'} \cap A$ is a smooth boundary point of A. Therefore, $u(p^*, j_k)$ is a smooth boundary point of A, so there is only one supporting hyperplane of A through $u(p^*, j_k)$, which is $H_{\lambda'}$. Since $u(p^*, j_{k'}) \notin H_{\lambda'}$, the line segment that joins $u(p^*, j_j)$ and $u(p^*, j_{j'})$ is not contained in $H_{\lambda'}$ and therefore $R_1(p^*) \cap \mathcal{N} \setminus (R_1(p^*) \cap H_{\lambda'}) \subseteq Int(A)$. ■

From Condition **C2** we get that every face or edge of the polytope $R_1(p^*)$ that intersects $H_{\lambda'}$ also intersects A in a sufficiently small neighbourhood. Furthermore by Lemma 32, every such face or edge either lies in $\partial A \cap H_{\lambda'}$ or intersects $Int(A)$. Since $R_1(p^*)$ has finitely many faces and edges, there is $\delta > 0$ such that every such face or edge that intersects $Int(A)$ also intersects $B = B_\delta$. An example is provided in Figure 17.

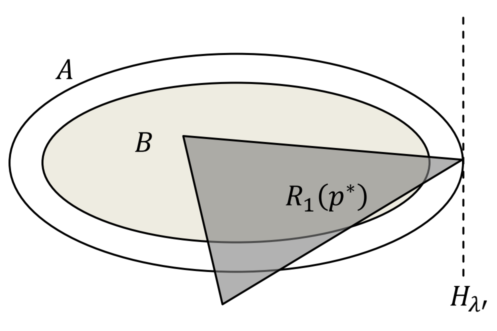

Figure 17: $R_1(p^*)$ intersects B.

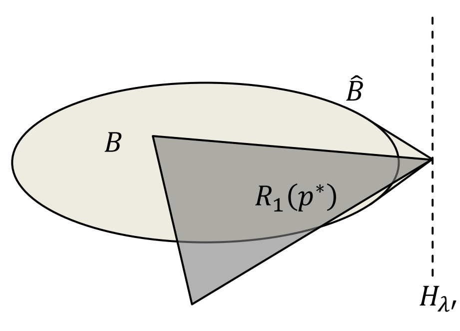

Figure 18: The set $\widehat{B}$.

In the proof of Proposition 25 we have shown that for every open ball $\mathcal{N}_{\lambda'} = \mathcal{B}(\lambda', \varepsilon)$ around λ' there exist $\delta, \delta' > 0$ such that the set $B = B_{\delta'}$ satisfies the following condition:

for any supporting hyperplane H of B in a direction in $S^{d-1} \setminus \mathcal{N}_{\lambda'}$ there exists an action $p^H \in \Delta(\mathcal{I})$ such that $R_1(p^H) \subseteq H^-$, and the distance between H and $R_1(p^H)$ is greater than or equals to δ. Consider the set $\widehat{B} = Conv(B, R_1(p^*) \cap H_{\lambda'})$ (see Figure 18).

Lemma 33 *Provided ε is sufficiently small, any supporting hyperplane H of $\widehat{B}$ in direction $\lambda \in \mathcal{N}_{\lambda'}$ satisfies that $R_1(p^*) \subseteq \bar{H}^-$.*

Proof. If ε is sufficiently small,[17] then any supporting hyperplane of $\widehat{B}$ in direction $\lambda \in \mathcal{N}_{\lambda'}$ and any facet of $R_1(p^*)$ that intersects $H_{\lambda'}$ but not contained in it do not intersect in $H^-_{\lambda'}$. Let H be a supporting hyperplane of $\widehat{B}$ in direction $\lambda \in \mathcal{N}_{\lambda'}$. The hyperplane H supports $\widehat{B}$ either only at points of $R_1(p^*) \cap H_{\lambda'}$, at a point of $R_1(p^*) \cap H_{\lambda'}$ and also at a point of B, or only at points of B.

- If H supports $\widehat{B}$ only at points of $R_1(p^*) \cap H_{\lambda'}$, the claim clearly holds.
- Suppose now that H supports $\widehat{B}$ at a point of $R_1(p^*) \cap H_{\lambda'}$ and also at a point of B, and assume that $R_1(p^*) \nsubseteq \bar{H}^-$. Then there is an extreme point $u(p^*, j_k)$ of $R_1(p^*)$ in H^+. In particular, there is an extreme point of $R_1(p^*)$ that belongs to $R_1(p^*) \cap H_{\lambda'}$ such that the line segment that joins it with $u(p^*, j_k)$ does not intersect B, which is a contradiction to the fact that every facet of $R_1(p^*)$ that intersects $H_{\lambda'}$ but not contained in it also intersects B.
- Suppose now that H supports $\widehat{B}$ only at points of B, and assume that $R_1(p^*) \nsubseteq \bar{H}^-$. As above, there must be an extreme point $u(p^*, j_k)$ of $R_1(p^*)$ in H^+. Since $R_1(p^*) \cap H_{\lambda'} \subseteq \widehat{B}$ and since H supports $\widehat{B}$, each extreme point of $R_1(p^*)$ that belongs to $R_1(p^*) \cap H_{\lambda'}$ is contained in H^-. Therefore there is an extreme point of $R_1(p^*)$ that belongs to $R_1(p^*) \cap H_{\lambda'}$ such that the line segment that joins it with $u(p^*, j_k)$ intersects H. Since $R_1(p^*)$ is convex, it follows that there exists a facet of $R_1(p^*)$ that intersects $H_{\lambda'}$ but not contained in it such that the hyperplane it induces intersects H in $H^-_{\lambda'}$. Therefore H is in direction $\lambda \notin \mathcal{N}_{\lambda'}$, a contradiction to the way have selected H.

■

[17] Recall that an angle between two hyperplanes is the angle between their two normal unit vectors. Denote by γ_1 the minimal angle between $H_{\lambda'}$ and a facet of $R_1(p^*)$ that intersects $H_{\lambda'}$ but not contained in it, and by γ_2 the maximal angle between $\lambda \in \mathcal{N}_{\lambda'}$ and λ'. Any ε such that γ_2 is smaller than γ_1 would be sufficient.

It follows that the set $\widehat{B}$ is a B-set for Player 1, since each supporting hyperplane of $\widehat{B}$ is either in direction $\lambda \in S^{d-1} \setminus \mathcal{N}_{\lambda'}$ or $\lambda \in \mathcal{N}_{\lambda'}$. In particular, the set $\widehat{B}$ is approachable by Player 1.

Let $\eta = d(A, B)$ be the distance between ∂A and ∂B. The set $\widehat{B}$ is approachable by Player 1, and therefore there is a strategy σ_1' and $N' \in \mathbb{N}$ such that if Player 1 implements σ_1', then whatever Player 2 plays, the average payoff at each stage $n \geq N'$ is in $Cl(\mathcal{B}(\widehat{B}, \frac{\eta}{2}))$ (see Figure 19). The strategy σ_1' may play actions that cause the average payoff to enter the dead zone. To ensure that this does not happen, we define a strategy σ_1 that slightly differs from σ_1'. By σ_1', at each stage n, the average payoff $\bar{g}^n$ has a corresponding hyperplane H^n that supports $\widehat{B}$ at the closest point to $\bar{g}^n$. The strategy σ_1 plays as follows at each stage n: if the hyperplane H^n is in direction $\lambda \in \mathcal{N}_{\lambda'}$, then σ_1 plays p^*; otherwise it plays the same action that σ_1' would play. By Lemma 33, for every supporting hyperplane H^n of $\widehat{B}$ in direction $\lambda \in \mathcal{N}_{\lambda'}$, the set $R_1(p^*)$ is in $\bar{H}^{n-}$. Since $R_1(p^*)$ is in the intersection of $\bar{H}^{n-}$ and $\bar{H}^-_{\lambda'}$, the (expected) payoff when Player 1 plays the action p^* is also in the intersection of $\bar{H}^{n-}$ and $\bar{H}^-_{\lambda'}$. Thus, the strategy σ_1 approaches $\widehat{B}$, and therefore there is $N' \in \mathbb{N}$ such that if Player 1 implements σ_1, then whatever Player 2 plays, the average payoff after stage N' is in $\widehat{B}' := Conv(\mathcal{B}(B, \frac{\eta}{2}), R_1(p^*) \cap H_{\lambda'})$ (see Figure 20). This set is contained in A.

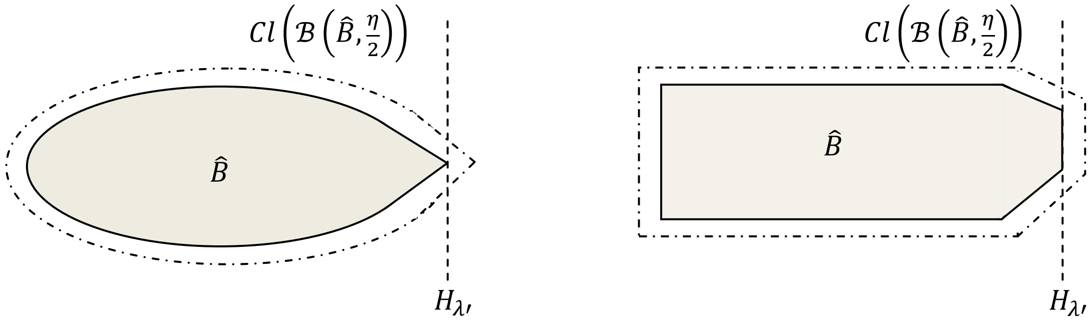

Figure 19: Two examples of a $\frac{\eta}{2}$-neighborhood of $\widehat{B}$ calculated with $R_1(p^*)$ from Figure 20.

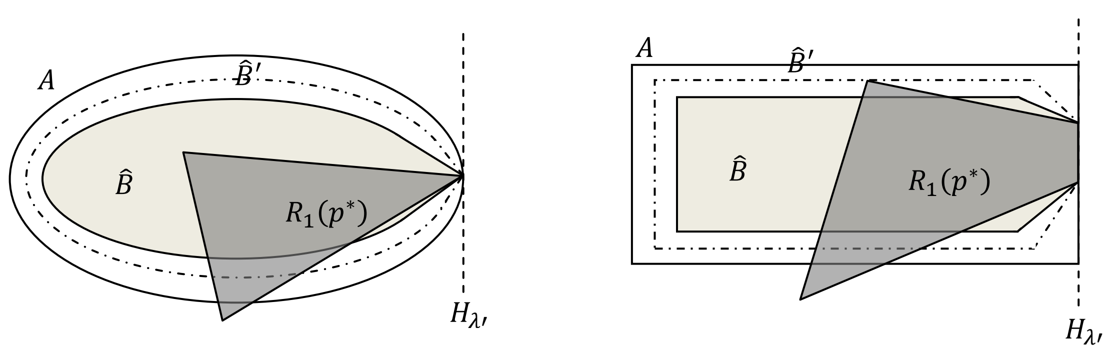

Figure 20: The set $\widehat{B}'$, which is contained in A.

Let σ_1^* be the following strategy for Player 1.

- In the first N' stages play the action p^*.
- For $n > N'$; at stage n play as follows:
  - If $\bar{g}^n \in F \setminus \widehat{B}'$, play the corresponding action from the strategy in Proposition 25, with $x = \bar{g}^{N'}$ and $t = N'$;
  - If $\bar{g}^n \in \widehat{B}' \setminus \widehat{B}$, play the corresponding action from the strategy σ_1 that approaches $\widehat{B}$;
  - If $\bar{g}^n \in \widehat{B}$, play p^*.

The strategy σ_1^* plays the action p^* in the first N' stages, so that from that stage on the difference between the average payoff at a stage and the average payoff in the following stage is relatively small. Then, if the average payoff is not in $\widehat{B}'$ (in particular not in $H_{\lambda'}$) the strategy plays according to the strategy in Proposition 25, so that the average payoff gets into $\mathcal{B}(B, \frac{\eta}{2})$ and in particular into $\widehat{B}'$; if the average payoff is in $\widehat{B}'$ but not in $\widehat{B}$, the strategy plays according to σ_1, so that the average payoff stays in $Cl(\mathcal{B}(\widehat{B}, \frac{\eta}{2}))$, but never gets into the dead zone; if the average payoff is in $\widehat{B}$, the strategy plays the action p^*.

The following lemma shows that the set A is strongly approachable by Player 1.

Lemma 34 *Under the strategy σ_1^*, there exist $N \in \mathbb{N}$ such that $\bar{g}^n \in A$ for every $n \geq N$ and every strategy σ_2 of Player 2.*

Proof. Set $N = N' + \bar{n}$, where $\bar{n}$ is given in Proposition 31. After playing N' times the action p^*, the average payoff is either in $\widehat{B}'$ or in $F \setminus \widehat{B}'$.

- If $\bar{g}^{N'} \in F \setminus \widehat{B}'$, then from Proposition 31, there exists $n \leq \bar{n}$ such that $\bar{g}^{N'+n} \in \mathcal{B}(B, \frac{\eta}{2})$ when Player 1 implements the strategy from Proposition 25.
- If $\bar{g}^n \in \widehat{B}' \setminus \widehat{B}$ for $n \geq N'$, then by playing the strategy σ_1 that ensures that the average payoff stays in $Cl(\mathcal{B}(\widehat{B}, \frac{\eta}{2}))$, but never gets into the dead zone, we deduce that $\bar{g}^{n+1} \in \widehat{B}' \subseteq A$.

- If $\bar{g}^n \in \widehat{B}$ for $n \geq N'$, then the strategy that approaches $\widehat{B}$ allows Player 1 to choose any action. Playing the action p^* ensures that the average payoff does not get into the dead zone. In particular, $\bar{g}^{n+1} \in \widehat{B}'$, since the difference between $\bar{g}^n$ and $\bar{g}^{n+1}$ is less than $\frac{\eta}{2}$.

That is, from stage N onward the average payoff is in A. ■

The first bullet shows that until the $N' + \bar{n}$-th stage, the average payoff will be in $\widehat{B}' \subseteq A$, and the other two bullets shows that after the average payoff gets into $\widehat{B}'$, it will stay there.

We deduce that for every $n \geq N$, $\bar{g}^n \in A$, so A is strongly approachable. This completes the proof of one direction of Theorem 14.

We now prove the other direction of Theorem 14. First notice that if Player 1 does not play the action p^* in the first stage, then Player 2 can play an action such that the expected payoff is in $H^+_{\lambda'}$. Moreover, as long as the average payoff is in $H_{\lambda'}$, if Player 1 does not play the action p^*, then Player 2 can play an action such that the expected payoff is in $H^+_{\lambda'}$, and then the average payoff up to that stage would be in $H^+_{\lambda'}$. As mentioned above, if the average payoff at some stage n is in $H^+_{\lambda'}$, Player 2 has a continuation strategy such that the long-run average payoff stays in $H^+_{\lambda'}$, and in particular it does not get into the set A. We can therefore assume that $p^1 = p^*$, and $p^{n+1} = p^*$ as long as $\bar{g}^n \in H_{\lambda'}$.

- Assume that $R_1(p^*) \cap H_{\lambda'} \cap A = \emptyset$. From the definition of a dead zone, $R_1(p^*) \cap H_{\lambda'} \neq \emptyset$. Consider the strategy of Player 2 in which he plays an action q such that $u(p^*, q) \in H_{\lambda'}$ as long as $\bar{g}^n \in H_{\lambda'}$. As long as Player 1 plays p^* the average payoff is $u(p^*, q) \in H_{\lambda'} \setminus A$, and therefore the long-run average payoff would not get into A. If Player 1 plays a different action p, Player 2 has an action q' such that $u(p, q') \in H^+_{\lambda'}$. From that stage on Player 2 will play the optimal strategy in the auxiliary game $G_{\lambda'}$, so that the average payoff will stay in $H^+_{\lambda'}$, and in particular would not get into A.

- Assume that there is a point $z \in R_1(p^*) \cap H_{\lambda'}$ that does not have a neighborhood $\mathcal{N}$ such that $R_1(p^*) \cap \mathcal{N} \subseteq A$, i.e., there is a point $z' \in R_1(p^*)$ such that the line segment zz' intersects A only at z. Assume to the contrary that the set A is strongly approachable, i.e., there exists $N \in \mathbb{N}$ such that for every strategy of Player 2 and for every $n > N$, one has $\bar{g}^n \in A$. As mentioned above, Player 1 needs to play the action p^* as long as $\bar{g}^n \in H_{\lambda'}$. Consider the strategy of Player 2 in which he plays in the first N stages an action q^* satisfying $u(p^*, q^*) = z$. Then the average payoff at stage N is z, so that Player 1 needs to play the action p^* at stage $N+1$. At stage $N+1$ the strategy of Player 2 indicates to play an action q' such that $u(p^*, q') = z'$. Therefore, the average payoff at stage $N+1$ will be in the open line segment zz', which is outside of A, a contradiction to the assumption that A is strongly approachable.

This concludes the proof of Theorem 14.

Remark 35 *As mentioned above, if Condition **C1** does not hold, Player 2 can prevent Player 1 from making the average payoff get into the set A at any stage of the game: Player 2 has a strategy the guarantees that the average payoff is either in $H_{\lambda'} \setminus A$ or in $H^+_{\lambda'}$, so it never gets into A. On the other hand, if Condition **C1** holds and Condition **C2** does not hold, then Player 1 can eventually make the average payoff get into A: Indeed, as mentioned above, Player 1 can ensure that the average payoff stays in $\bar{H}^-_{\lambda'}$. Once it is in $H^-_{\lambda'} \setminus A$ she can use the strategy described in the proof of Theorem 14 to ensure that it gets into A and remains there.*

Remark 36 *Does the fact that a player can strongly approach a certain set in the game $G_{x,t}$ depend on x? The answer to this question is positive: It is possible that a certain set will be strongly approachable by a player in the game $G_{x,t}$ but not strongly approachable by him in the game $G_{x',t}$, for $x' \neq x$. This happens, for example, if x is not in a dead zone while x' is in a dead zone.*

3.6 The General Case

In the general case, where there are several directions $\lambda \in S^{d-1}$ for which

$$V_\lambda = \inf_{a \in A} \langle \lambda, a \rangle \tag{4}$$

Player 1 should ensure that the average payoff stays away of more than one half-space (dead zone).

As follows from the proof of the second direction of Theorem 14, if the set A is strongly approachable by Player 1, then the action p that he plays in the first stage must satisfy that $R_1(p) \subseteq \bar{H}^-_\lambda$ for every λ that satisfies Eq. (4). In other words, there is $p^* \in \Delta(\mathcal{I})$ that is optimal in the game G_λ for every λ that satisfies Eq. (4). The condition in the statement of the Theorem implies that this p^* is unique.

A dead zone is defined by a direction λ for which $V_\lambda = \inf_{a \in A}\langle \lambda, a\rangle$. Since p^* is unique, each such direction is determined by an extreme point of $R_1(p^*)$. The number of such directions can be finite, for example, if the set A has a smooth boundary, or infinite (uncountable), if one of the extreme points of $R_1(p^*)$ is on a non-smooth boundary point of A.

To prove Theorem 15 we need a generalization of Proposition 25, which is the following.

Proposition 37 *Let A be a closed convex approachable set with nonempty interior. Denote by $\mathcal{S}$ the set of all $\lambda' \in S^{d-1}$ that satisfy $V_{\lambda'} = \inf_{a \in A}\langle \lambda', a\rangle$ and let*

$$\mathcal{H} := \bigcap_{\lambda' \in \mathcal{S}} H^-_{\lambda'}.$$

For every $\lambda' \in \mathcal{S}$ denote by $p_{\lambda'}$ an optimal strategy (action) in the auxiliary game $G_{\lambda'}$, so that $R_1(p_{\lambda'})$ is in the closed half-space $\bar{H}^-_{\lambda'}$. Assume that every point in $R_1(p_{\lambda'}) \cap H_{\lambda'} \cap A$ is a smooth boundary point of A; and

C1 *$R_1(p_{\lambda'}) \cap H_{\lambda'} \cap A \neq \emptyset$, and*

C2 *every point in $R_1(p_{\lambda'}) \cap H_{\lambda'}$ has a neighborhood $\mathcal{N}$ such that $R_1(p_{\lambda'}) \cap \mathcal{N} \subseteq A$.*

Then for every $x \in (F \cap \mathcal{H}) \setminus A$ and every $t \in \mathbb{N}$ there exists a strategy σ_1 of Player 1 and $\widehat{n} \in \mathbb{N}$ such that under σ_1 the average payoff $\bar{g}^{\widehat{n}}$ in the game $G_{x,t}$ belongs to $Int(A)$ for every strategy σ_2 of Player 2.

In Proposition 25 and in Theorem 14 we did not rely on the condition that there is only one direction λ satisfying Eq. (4). Therefore the generalizations of Proposition 25 and Theorem 14 to the case of more than one direction λ satisfying Eq. (4) (i.e. more than one dead zone), which are Proposition 37 and Theorem 15 respectively, hold with similar proofs.

4. Discussion

4.1 Exactly One Set $R_1(p_{\lambda'})$ in Every Complement of a Dead Zone

The main results are valid when we assume that for every direction $\lambda' \in S^{d-1}$ for which $V_{\lambda'} = \inf_{a \in A}\langle \lambda', a\rangle$, among all sets $\{R_1(p), p \in \Delta(\mathcal{I})\}$ there exists exactly one set $R_1(p_{\lambda'})$ in the closed half-space $\bar{H}^-_{\lambda'}$. We used this assumption to show that each of the two following conditions is necessary:

1. $R_1(p_{\lambda'}) \cap H_{\lambda'} \cap A \neq \emptyset$.

2. Every point in $R_1(p_{\lambda'}) \cap H_{\lambda'}$ has a neighborhood $\mathcal{N}$ such that $R_1(p_{\lambda'}) \cap \mathcal{N} \subseteq A$.

We do not know whether these conditions are necessary in the case where the number of sets among all sets $\{R_1(p), p \in \Delta(\mathcal{I})\}$ that are contained in the closed half-space $\bar{H}^-_{\lambda'}$ is not restricted to one. While exploring this issue we encountered the following obstacle. From Proposition 25 and its generalization, Proposition 37, we know that if the average payoff is in the open half-space $H^-_{\lambda'}$, then Player 1 has a strategy such that the average payoff would get into the desired set, regardless of the number of sets $R_1(p_{\lambda'})$. Therefore, it seems that one should study a restricted game in which Player 1 is restricted to mixed actions that ensure that the expected stage payoff is in the closed half space $\bar{H}^-_{\lambda'}$, and for every such mixed action of Player 1, Player 2 is restricted to mixed actions that ensure that the expected stage payoff is on the hyperplane $H_{\lambda'}$. Regarding this restricted game we should ask whether Player 1 can strongly approach the set $A \cap H_{\lambda'}$. The problem with this approach is that there are examples in which, when Player 1 plays in this way, for any stage n Player 2 can ensure that the average payoff remains in $A \cap H_{\lambda'}$ until stage n, moves to $H^-_{\lambda'} \setminus A$ in stage $n+1$, from where Player 1 can ensure that eventually the average payoff would get into A. However, the definition of strong approachability requires that the average payoff gets into A in a bounded time, and this approach does not delivers such a uniformity property.

4.2 Smoothness of $R_1(p^*) \cap H_{\lambda'}$ in ∂A

To keep the formulation of Theorems 14 and 15 rather simple, we assumed that every point in $R_1(p^*) \cap H_{\lambda'} \cap A$ is a smooth boundary point of A. In fact, we could have assume a weaker assumption on $R_1(p^*)$, which is that every face or edge of $R_1(p^*)$ that intersects $H_{\lambda'}$, but not contained in it, is not a subset of ∂A in any small enough neighborhood of the intersection point.

The original assumption was used in the proof of Lemma 32. We now argue that the weaker assumption also satisfies Lemma 32. Recall that by Condition **C2** the point $u(p^*, j_k)$ has a neighborhood $\mathcal{N}$ such that $R_1(p^*) \cap \mathcal{N} \subseteq A$. Therefore, if $u(p^*, j_k)$ is a vertex (nonsmooth boundary point) of A, then in a small enough neighborhood of $u(p^*, j_k)$, the line segment that joins $u(p^*, j_k)$ and $u(p^*, j_{k'})$ intersects either $Int(A)$ or ∂A. From the assumption on the faces and edges of $R_1(p^*)$, and since $u(p^*, j_{k'}) \notin H_{\lambda'}$, in a small enough neighborhood of $u(p^*, j_k)$, the intersection of that line and the neighborhood $\mathcal{N}$ is not contained in ∂A, and therefore the line intersects $Int(A)$.

The above weaker assumption is not the weakest we can assume. It is possible to weaken the assumption on the faces and edges of $R_1(p^*)$ by the following iterative process.

i. Let f be a face or an edge of $R_1(p^*)$ that intersects $H_{\lambda'}$ but not contained in it. If $(f \cap \partial A) \setminus H_{\lambda'} \neq \emptyset$, then f is contained in ∂A.

ii. The condition in (i) is satisfied when replacing the hyperplane $H_{\lambda'}$ with f.

iii. Bullet (ii) holds for every replaced face or edge, and so on.

Since $R_1(p^*)$ has finitely many faces (and therefore edges), this iterative process is finite. The proof of Theorems 14 and 15, with proper adjustments, remain valid under the new assumptions.

Figure 21 provides a graphic example of the weaker assumption; The boundary of A is not smooth in $f_0 := R_1(p^*) \cap H_{\lambda'} \cap A$. The faces f_0 and f_1 of $R_1(p^*)$ intersect at some point. The face f_1 is contained in ∂A in every neighborhood of the intersection point, and therefore the face f_1 is contained in ∂A. Similarly, the face f_2 is contained in ∂A, since it is contained in ∂A in every neighborhood of the intersection point of the faces f_1 and f_2. The set $\widehat{B}$ would be as in Figure 22. It still holds that $\widehat{B} \subseteq A$.

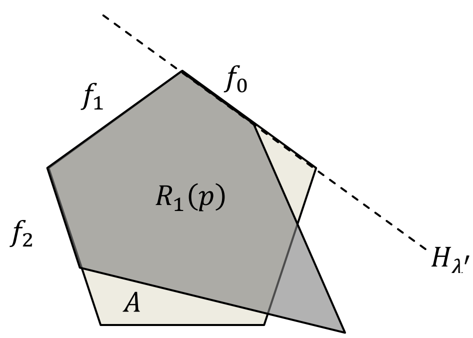

Figure 21: A set $R_1(p)$ that satisfies the weaker assumption.

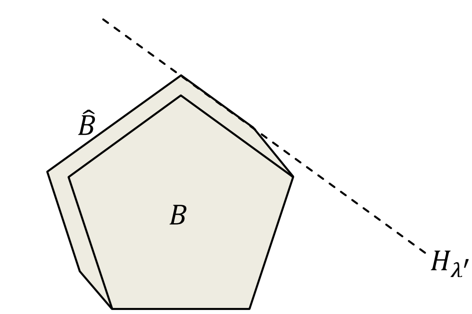

Figure 22: The set $\widehat{B}$ when $R_1(p)$ satisfies the weaker assumption.

The question whether this assumption is necessary is open and left for future research.

Remark 38 *As mentioned before (see Remark 26), Proposition 25 (resp. Proposition 37) is more general than what we need, since in the proof of Theorem 14 (resp. Theorem 15) the point x is restricted to the unique set $R_1(p^*) = R_1(p_{\lambda'})$. We assumed that every point in $R_1(p_{\lambda'}) \cap H_{\lambda'} \cap A$ is a smooth boundary point of A, and used this assumption to show that $\delta_1 := d(C \setminus A, Y) > 0$. Since in this section we no longer make the smoothness assumption, we cannot use it to show that $\delta_1 > 0$. Nevertheless, since the point x is in $R_1(p_{\lambda'})$, this claim still holds.*

References

[1] Aumann, R.J., and Maschler, M. (1967) "Repeated Games of Incomplete Information: a Survey of Recent Results," in Reports of the U.S. Arms Control and Disarmament Agency ST-116, Washingtom, D.C., Chapter III, 287-403.

[2] Aumann, R.J., and Maschler, M. (1995) "Repeated Games with Incomplete Information," The MIT Press.

[3] Blackwell, D. (1956) "An Analog of the Minmax Theorem for Vector Payoffs," *Pacific Journal of Mathematics*, **6**, 1–8.

[4] Cesa-Bianchi, N. and Lugosi, G. (2006) "Prediction, Learning and Games," Cambridge University Press.

[5] Foster, D.P. and Vohra, R.V. (1997) "Calibrated Learning and Correlated Equilibrium," *Games Economics Behavior*, **21**, 40–55.

[6] Goberna, M.A., Gonzalez, E., Martinez-Legaz, J.E., and Todorov, M.I. (2010) "Motzkin Decomposition of Closed Convex Sets," *Journal of Mathematical Analysis and Applications*, **364**, 209–221.

[7] Hart, S. and Mas-Colell, A. (2000) "A Simple Adaptive Procedure Leading to Correlated Equilibrium," *Econometrica*, **68**, 1127–1150.

[8] Hou, T.F. (1971) "Approachability in a Two-Person Game," *The Annals of Mathematical Statistics*, **42**, 735–744.

[9] Kohlberg E. (1975) "Optimal Strategies in Repeated Games with Incomplete Information," *International Journal of Game Theory*, **4**, 7–24.

[10] Lehrer, E. (2002) "Approachability in Infinitely Dimensional Spaces," *International Journal of Game Theory*, **31**, 255–270.

[11] Lehrer, E. (2004) "The Game of Normal Numbers," *Mathematics of Operations Research*, **29**, 259–265.

[12] Mannor, S. and Perchet, V. (2013) "Approachability, Fast and Slow," *JMLR Workshop and Conference Proceedings*, **30**, 474–488.

[13] Maschler, M., Solan, E., and Zamir, S. (2013) "Game Theory." Cambridge University Press.

[14] Rockafellar, R.T. (1970) "Convex Analysis," Princeton University Press.

[15] Rosenberg, D., Solan, E., and Vieille, N. (2004) "Stochastic Games with a Single Controller and Incomplete Information," *SIAM Journal on Control and Optimization*, **43**, 86–110.

[16] Sorin, S. (2011) "Zero-Sum Repeated Games: Recent Advances and New Links with Differential Games," *Dynamic Games and Applications*, **1**, 172–207.

[17] Spinat, X. (2002) "A Necessary and Sufficient Condition for Approachability," *Mathematics of Operations Research*, **27**, 31–44.