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Absorbing Team Games[1]

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A team is a group of people having the same motives but possibly different available actions. A team game is a game where two teams face each other. An absorbing game is a repeated game where some of the entries are absorbing, in the sense that once they are chosen the play terminates, and all future payoffs are equal to the payoff at the stage of termination. We prove that every absorbing team game has an equilibrium payoff and that there are ϵ-equilibrium profiles with cyclic structure. *Journal of Economic Literature* Classification Numbers: C72, C73.

Key Words: stochastic games; repeated games with absorbing states; absorbing games; Nash equilibrium; undiscounted payoff; team games.

1. INTRODUCTION

A team is a group of people having the same motives but possibly different available actions. The members of the team may be connected by some contract or by the mere fact that they happen to have the same occupation and therefore the same motives. An economic theory of teams was developed by Marschak and Radner (1972). The theory mainly deals with the problem of information distribution among the team members. An annotated bibliography on the theory of teams that concentrates on stochastic models can be found in Başar and Bansal (1989).

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A *team game* is a competitive model where two teams face each other. One could attempt to analyze a team game as a two-player game by considering each team as a single player. However, such a reduction misses an important feature of team games—players on the same team cannot necessarily correlate their actions. Therefore, viewing a team as a single player adds new strategies to the game.

In particular, an equilibrium in a team game is not necessarily an equilibrium in the same game, where each team is considered as a single player, and vice versa. Moreover, existence of an equilibrium in a team game does not imply and neither is implied by the existence of an equilibrium in the corresponding two-player game.

Team games were first studied by Palfrey and Rosenthal (1983) in the context of voting games. Recently von Stengel and Koller (1997) studied a model where a team faces a single adversary and characterized the "best" equilibrium from the point of view of the team. Team games were studied also in the context of Rendezvous–Evasion games [see, e.g., Kim and Roush (1987), Lim (1997), and Alpern and Lim (1998)] and in experiments—the effect of repetition on cooperation among players and on free riding [see, e.g., Bornstein, *et al*., (1994, 1996, 1997)].

In sequential games, the equilibrium path is usually sustained by threats of punishment. In team games members of the same team receive the same payoff; hence, punishing a deviator means punishing every member of his or her team. In many situations, a team will be punished by the other team if one of its members deviates, and the deviator suffers as part of the team. For example, a war between two families may begin if a member of one family attacks a member of the other family, and all the shepherds hunt a pack of wolves if one wolf captured a single sheep.

In the present paper we are interested in a model where a deviator is punished *both* by his or her team and by the opposing team. Thus, the team of the deviator suffers from its decision to punish the deviator and from the reaction of the opposing team. As an example, consider a key star of a basketball team who does not show up for a practice. The key star is suspended from the next game, even though the chances of winning the game without him are virtually zero. The opponent in the next game can react in two ways. It can either let its own star rest as well and win the game by a small margin or play as hard as it can, humiliating the first team. In this example, the first team punishes itself by suspending its star, and the opponent can exploit this move in various ways, some of which are more damaging than others.

The model that we study is a class of stochastic games called *absorbing games*. In these games, at every stage each player chooses an action, independently of the other players. The action combination that was chosen determines three things: (a) a daily payoff for each player, (b) a probability that the game terminates (is absorbed), and (c) a terminal payoff that each player receives at every future stage if the game has terminated by this combination.

If the players are divided into two teams, and the players in each team have the same payoffs, both daily payoffs and terminal payoffs, then we face an absorbing team game.

Absorbing games arise naturally in the context of repeated games with symmetric incomplete information and signalling. Kohlberg and Zamir (1974) proved that if there exists a value in any two-player zero-sum absorbing game then there exists a value in any repeated game with symmetric incomplete information and deterministic signalling. This result was generalized by Neyman and Sorin (1998) for existence of equilibrium payoff in n-player repeated games with symmetric incomplete information and nondeterministic signalling.

A (uniform) *equilibrium payoff* is a payoff vector (v^i) such that for every $\epsilon > 0$ there is a finite horizon $t_0 \in \mathbf{N}$ and a strategy profile $\sigma_\epsilon = (\sigma_\epsilon^i)$ that satisfy the following conditions:

- If the players follow the profile σ_ϵ, then the expected average payoff of each player i in every t-stage game (for $t \geq t_0$) is at least $v^i - \epsilon$.
- If player i deviates to any strategy then his or her expected average payoff in every t-stage game (for $t \geq t_0$), is at most $v^i + \epsilon$.
- If the players follows σ_ϵ in the infinite game then for every player i the expected value of the lim inf of his or her average payoffs is at least $v^i - \epsilon$.
- If player i deviates to any strategy in the infinite game then the expected value of the lim sup of his or her average payoffs is at most $v^i + \epsilon$.

The strategy profile σ_ϵ is *an ϵ-equilibrium profile*. One can show that σ_ϵ is a 2ϵ-equilibrium profile in every discounted game, provided the discount factor is sufficiently close to 1.

We prove that in every absorbing team game there exists an equilibrium payoff v. Moreover, the ϵ-equilibrium profiles have a very special (and simple) structure.

1.1. *The ϵ-Equilibrium Profiles*

In general n-player stochastic games, existence of a uniform equilibrium payoff is not known, and in the classes where existence was proven, the ϵ-equilibrium profiles might be complex.

Blackwell and Ferguson (1968) gave an example of a 2-player zero-sum absorbing game where no 0-equilibrium profiles exist, the only ϵ-equilibrium profiles are history dependent, and they require infinite recall.

Kohlberg (1974) proved the existence of a uniform equilibrium payoff in 2-player zero-sum absorbing games. This result was generalized by Mertens and Neyman (1981) for general zero-sum stochastic games.

Vrieze and Thuijsman (1989) proved the existence of a uniform equilibrium payoff in 2-player non-zero-sum absorbing games, where the ϵ-equilibrium profiles are *"almost" stationary*—they are given by a stationary profile and a statistical test: the players follow the stationary profile as long as no deviation is detected, and once a deviation is detected, both players switch to a punishment strategy.

Flesch *et al.* (1997) provided an example of a 3-player absorbing game where no "almost" stationary ϵ-equilibrium profile exists. Nevertheless, Solan (1999) proved that every 3-player absorbing game has a uniform equilibrium payoff, where the ϵ-equilibrium profiles are *perturbed*—they are given by a stationary profile, small perturbations, and statistical tests: as long as no deviation is detected, the players play mainly the stationary profile, but perturb to other actions with a small probability. Once a deviation of one of the players is detected, the deviator is punished by the other two players for the rest of the game. The small perturbations have a "nice" structure: they are cyclic, but the length of the cycle may depend on ϵ.

In particular, it follows that every absorbing team game, where one team consists of a single player and the other consists of two players, admits a uniform equilibrium payoff.

Recently Vieille (1997a,b) proved the existence of a perturbed ϵ-equilibrium profile in general 2-player non-zero-sum stochastic games. However, Solan and Vieille (1998) provided an example of a 4-player absorbing game where no perturbed ϵ-equilibrium profile exists. Existence of a uniform equilibrium payoff in n-players stochastic games was established only for irreducible games (Sobel, 1971; Federgruen, 1978) and for games with additive reward and additive transitions [which includes the class of games with complete information (Thuijsman and Raghavan, 1997)]. In both cases there are 0-equilibrium profiles—in the former they are stationary and in the latter "almost" stationary.

In the present paper we prove that every absorbing team game admits a uniform equilibrium payoff, where the corresponding ϵ-equilibrium profiles are perturbed. Moreover, the perturbations are cyclic, and the length of the cycle is either 1 or 2.

The paper is arranged as follows. In Section 2 we present the model of n-player absorbing games and the special case of absorbing team games. In Section 3 we define perturbed equilibria. In Section 4 we provide three sufficient conditions for an n-player absorbing game to have a perturbed equilibrium, and finally, in Section 5, we prove that in any absorbing team game at least one of the sufficient conditions hold.

2. THE MODEL

DEFINITION 2.1. An *n-player absorbing game* G is given by $((A^i, h^i, u^i)_{i=1}^n, w)$ where

- A^i is a finite set of actions available for player i. Denote $A = \times_{i=1}^n A^i$.
- $r^i: A \to \mathbf{R}$ for $i = 1, \ldots, n$. For every $a \in A$, $r^i(a)$ is the daily payoff for player i.
- $w: A \to [0, 1]$. For every $a \in A$, $w(a)$ is the probability the game is absorbed if the action combination a was played by the players.
- $u^i: A \to \mathbf{R}$ for $i = 1, \ldots, n$. Given that the game was absorbed by action combination $a \in A$, $u^i(a)$ is the constant payoff player i receives at every future stage.

We assume w.l.o.g. that $\| r \|, \| u \| \leq 1$, and denote $X = \times_{i=1}^n \Delta(A^i)$ the set of mixed-action combinations. Let r_t^i be the payoff of player i at stage t.

DEFINITION 2.2. An $n+m$-player absorbing game is called an *absorbing team game* if $r^i = r^j$ and $u^i = u^j$ whenever $1 \leq i, j \leq n$ or $n+1 \leq i, j \leq n+m$.

Most of the results we prove are valid for general n-player absorbing games. Only in Theorem 3.5 do we use the special structure of absorbing team games.

Let G be an n-player absorbing game. Let $N = \{1, \ldots, n\}$ be the set of players, and let c^i be the min–max value of player i in the game. By Neyman (1988) c^i exists for every $i \in N$. Moreover, players $N \setminus \{i\}$ have an ϵ-min–max profile against player i, that is, a profile τ_ϵ^{-i} such that for every strategy σ^i of player i

$$\mathrm{E}_{\tau_\epsilon^{-i}, \sigma^i}\left(\frac{r_1^i + \ldots + r_t^i}{t}\right) \leq c^i + \epsilon$$

and for every $t > t_0$ (where t_0 is independent of σ^i)

$$\mathrm{E}_{\tau_\epsilon^{-i}, \sigma^i}\left(\limsup_{t\to\infty} \frac{r_1^i + \ldots + r_t^i}{t}\right) \leq c^i + \epsilon.$$

Note that in a team game, though players of the same team have equal payoff functions, their min–max value may be different.

Each vector $x^i \in \Delta(A^i)$ is a probability distribution over A^i. Therefore, every $x \in X$ is a probability distribution over A. For any $x \in X$ and $a \in A$ denote $x_a = \Pi_{i\in N} x_{a^i}^i$.

We define the multilinear extension of r^i and w by

$$r^i(x) = \sum_{a\in A} x_a r^i(a)$$

and

$$w(x) = \sum_{a\in A} x_a w(a)$$

for every $x \in X$.

A mixed-action combination $x \in X$ is *absorbing* if $w(x) > 0$ and *nonabsorbing* if $w(x) = 0$.

For every absorbing mixed-action $x \in X$ we define

$$u^i(x) = \sum_{a\in A} \frac{x_a w(a)}{w(x)} u^i(a).$$

$u^i(x)$ is the expected absorption payoff for player i if the players play the stationary profile x. Note that $u^i(\cdot)$ is continuous whenever it is defined.

DEFINITION 2.3. The vector $v = (v^i)_{i\in N}$ is a *(uniform) equilibrium payoff* if for every $\epsilon > 0$ there exists $t_0 \in \mathbf{N}$ and a strategy profile $\sigma_\epsilon = (\sigma_\epsilon^i)_{i=1}^n$ such that for every player $i \in N$, every strategy τ^i of player i and every $t > t_0$

$$\mathrm{E}_{\sigma_\epsilon}\left(\frac{r_1^i + \ldots + r_t^i}{t}\right) \geq v^i - \epsilon,$$

$$\mathrm{E}_{\sigma_\epsilon}\left(\liminf_{t\to\infty} \frac{r_1^i + \ldots + r_t^i}{t}\right) \geq v^i - \epsilon,$$

$$\mathrm{E}_{\sigma_\epsilon^{-i}, \tau^i}\left(\frac{r_1^i + \ldots + r_t^i}{t}\right) \leq v^i + \epsilon \quad \text{and}$$

$$\mathrm{E}_{\sigma_\epsilon^{-i}, \tau^i}\left(\limsup_{t\to\infty} \frac{r_1^i + \ldots + r_t^i}{t}\right) \leq v^i + \epsilon.$$

The profile σ_ϵ is an *ϵ-equilibrium profile* for v.

3. ON PERTURBED EQUILIBRIA

Let $H = \cup_{t\in\mathbf{N}} A^t$ be the space of all finite histories. Define a partial order on H by $(a_1, a_2, \ldots, a_t) \geq (a'_1, a'_2, \ldots, a'_{t'})$ if and only if $t \geq t'$ and $a_i = a'_i$ for every $i = 1, \ldots, t'$, i.e., $(a'_1, a'_2, \ldots, a'_{t'})$ is a beginning of $(a_1, a_2, \ldots, a_t)$.

DEFINITION 3.1. A function $f: H \to \{0, 1\}$ is *monotonic* if $h \geq h'$ implies $f(h) \geq f(h')$.

f is monotonic if $f(h) = 0$ implies that $f(h') = 0$ for every beginning h' of h. A monotonic function f can represent a statistical test—as long as $f(h) = 0$ no "deviation" is detected, but as soon as $f(h) = 1$ a "deviation" is announced.

DEFINITION 3.2. Let $f = (f^i)_{i \in N}$ be a vector of monotonic functions. A history $h \in H$ is a 0-*history* if $f^i(h) = 0$ for every $i \in N$ and 1-*history* otherwise. It is a *minimal 1-history with index* i_0 (w.r.t. f) if

- Any $h' < h$ is a 0-history.
- $f^i(h) = 0$ for every $i < i_0$.
- $f^{i_0}(h) = 1$.

If f represents a vector of statistical tests, i.e., f^i is a statistical test that checks player i, then a minimal 1-history with index i_0 is a history in which a "deviation" is announced for the first time, and player i_0 is the first player whose "deviation" is detected.

For any two histories $h = (a_1, a_2, \ldots, a_t)$ and $h' = (a_1, a_2, \ldots, a_{t'})$ such that $h' < h$ we define the subtraction $h \setminus h' = (a_{t'+1}, a_{t'+2}, \ldots, a_t) \in H^{t-t'}$.

DEFINITION 3.3. Let $x \in X$ and $\epsilon > 0$. A profile σ is (x, ϵ)-*perturbed* if there exist

- a vector $f_\epsilon = (f_\epsilon^i)_{i \in N}$ of monotonic functions
- and, for every $i \in N$, an ϵ-min–max profile τ_ϵ^{-i} against player i,

such that for every 1-history $h \in H$, $\sigma^{-i_0}(h) = \tau_\epsilon^{-i_0}(h \setminus h')$ where h' is a beginning of h which is a 1-minimal history with index i_0 (w.r.t. f_ϵ) and for every 0-history $h \in H$, $\| \sigma(h) - x \| < \epsilon$.

In other words, the players play an ϵ-perturbation of the stationary profile x. Meanwhile, the actions of each player are screened by some statistical test. The first player who fails the test is punished with an ϵ-min–max profile forever.

DEFINITION 3.4. Let $x \in X$ and $v \in \mathbf{R}^n$. v is an *x-perturbed equilibrium payoff* if it is an equilibrium payoff, and for every $\epsilon > 0$ there exists an (x, ϵ)-perturbed profile σ_ϵ that is an ϵ-equilibrium profile for v. x is the *base* of the perturbed equilibrium.

The main result of the paper is:

THEOREM 3.5. *Any absorbing team game admits a perturbed equilibrium payoff.*

4. SUFFICIENT CONDITIONS FOR PERTURBED EQUILIBRIA

In this section we consider n-player absorbing games. We provide three sufficient conditions for a mixed-action combination x to be a base of a perturbed equilibrium. The first two conditions were used by Vrieze and Thuijsman (1989) for the case $n = 2$, but the third condition is new. Since the proofs given by Vrieze and Thuijsman hold for the general case as well, we provide for the first two sufficient conditions only an intuitive proof. A more general form of the third condition was used in Solan (1999) for 3-player absorbing games. The 3-player game studied by Flesch *et al*. (1997) does not satisfy any of these three conditions.

For any player i define $e^i: X \to \mathbf{R}$ by

$$e^i(x) = w(x)u^i(x) + (1 - w(x))c^i.$$

If $w(x) = 0$ we define $w(x)u^i(x) = 0$. $e^i(x)$ is the maximal payoff that player i can guarantee if at the current stage the players play the mixed-action combination x and from tomorrow on player i is punished with an ϵ-min–max profile, with an arbitrarily small ϵ. Note that $w(x)u^i(x) = \sum_{a\in A} x_a w(a)u^i(a)$, and therefore e^i is a continuous function.

DEFINITION 4.1. Let $v \in \mathbf{R}^n$ be a payoff vector. A mixed-action combination x is *individually rational* for v if for every player $i \in N \cup M$ and every action $a^i \in A$

$$v^i \geq e^i(x^{-i}, a^i).$$

In particular, if the players play today a mixed-action combination that is ϵ-close to x and their expected payoff is ϵ-close to v, then none of them can profit more than 2ϵ by a unilateral deviation that will be followed by a punishment.

For every mixed-action combination $x \in X$ let $v^i(x)$ be the expected undiscounted payoffs for player i if the players play the stationary profile x; i.e.,

$$v^i(x) = \begin{cases} r^i(x) & w(x) = 0 \\ u^i(x) & w(x) > 0. \end{cases}$$

LEMMA 4.2. *Let $x \in X$ be a mixed-action combination that satisfies two conditions*:

1. *x is individually rational for $v(x)$.*

2. *If x is absorbing, then $u^i(x) = u^i(x^{-i}, a^i)$ for every player $i \in N$ and every action $a^i \in \operatorname{supp}(x^i)$ such that $w(x^{-i}, a^i) > 0$.*

Then $v(x)$ is an x-perturbed equilibrium payoff.

Proof. Let $\epsilon > 0$ be fixed. Define the following (x, ϵ)-perturbed profile σ:

1. As long as no deviation is detected, each player i plays the mixed action combination x^i.

2. Each player is checked for the following:

(a) that his or her realized action is in $\text{supp}(x^i)$, and

(b) that the distribution of his or her realized actions is ϵ-close to x^i.

Test (a) is employed from the first stage, while test (b) is employed only from stage t_1, where t_1 is chosen sufficiently large such that the probability that a player who plays the stationary strategy x^i will fail it is smaller than ϵ.

It is clear that if the players follow σ then their expected payoff in every sufficiently long game, as well as in the infinite game, is close to $v(x)$.

By the first condition, deviations outside $\text{supp}(x^i)$ are not profitable (that is, the deviator cannot profit more than 2ϵ). Moreover, it follows from the first condition that if x is nonabsorbing then deviations in $\text{supp}(x^i)$ are not profitable as well. By both conditions, the same conclusion holds if x is absorbing. ■

LEMMA 4.3. *Let x be a nonabsorbing mixed-action combination. If there exist player i_0 and an action $b^{i_0} \in A^{i_0}$ such that $w(x^{-i_0}, b^{i_0}) > 0$ and x is individually rational for $u(x^{-i_0}, b^{i_0})$, then $u(x^{-i_0}, b^{i_0})$ is an x-perturbed equilibrium payoff.*

Proof. Let $\epsilon > 0$ be fixed and $\delta \in (0, \epsilon)$ be sufficiently small. Define the following (x, ϵ)-perturbed profile σ:

1. As long as no deviation is detected, each player $i \neq i_0$ plays the mixed action x^i, while player i_0 plays the mixed action $(1-\delta)x^{i_0} + \delta b^{i_0}$.

2. Each player $i \neq i_0$ is checked as in the profile constructed in the proof of Lemma 4.2.

3. Player i_0, in addition to the two checks done in the proof of Lemma 4.2, is checked for

(c) whether the stage of the game is smaller than t_2, where t_2 is sufficiently large such that if no player fails one of the previous tests, then absorption occurs before stage t_2 with probability greater than $1 - \epsilon$.

Test (c) is required, since if $r^{i_0}(x) > u^{i_0}(x^{-i_0}, b^{i_0})$ then player i_0 receives more by *never* playing b^{i_0}.

The constant δ is chosen sufficiently small such that the probability that player i_0 plays b^{i_0} before test (b) in the proof of Lemma 4.2 is employed is smaller than ϵ; that is, $(1-\delta)^{t_1} > 1 - \epsilon$.

It is clear that if the players follow σ and no deviation is detected then the game will be eventually absorbed, and their expected payoff in every sufficiently long game, as well as in the infinite game, is close to $u(x^{-i_0}, b^{i_0})$. As in Lemma 4.2, no player $i \neq i_0$ can profit more than 3ϵ by deviating, and the three tests for player i_0 imply that he or she cannot profit more than 3ϵ as well. ■

DEFINITION 4.4. Let $x \in X$ be a mixed-action combination and let $\emptyset \neq S \subseteq N$. An action combination $b^S \in A^S$ is a *neighbor* of x if

- $w(x^{-S}, b^S) > 0$.
- $w(x^{-T}, b^T) = 0$ for every proper subset T of S.

In particular, if x has a neighbor then for $T = \emptyset$ we get $w(x) = 0$, which means that x is nonabsorbing. It follows that an absorbing mixed-action combination does not have neighbors.

If $S = \{i_0\}$ contains a single player, $b^S = b^{\{i_0\}}$ is a *neighbor of distance 1*. In such a case we write b^{i_0} instead of $b^{\{i_0\}}$.

LEMMA 4.5. *If there exists a nonabsorbing mixed-action combination $x \in X$, a set $\mathcal{B}$ of neighbors of x such that $|S| \geq 2$ for every $b^S \in \mathcal{B}$, and a probability distribution $\mu \in \Delta(\mathcal{B})$ such that x is individually rational for $\sum_{b^S \in \mathcal{B}} \mu(b^S) u(x^{-S}, b^S)$, then $\sum_{b^S \in \mathcal{B}} \mu(b^S) u(x^{-S}, b^S)$ is an x-perturbed equilibrium payoff.*

Proof. Denote $\mathcal{B} = \{b_1^{S_1}, \ldots, b_K^{S_K}\}$, and let $\epsilon > 0$ be given. We are going to construct an (x, ϵ)-perturbed equilibrium profile, where the equilibrium path has a cycle of length K. At stage t, the players try to be absorbed with a small probability δ by the neighbor $b_k^{S_k}$, where $k = t \bmod K$. Thus, each player $i \notin S_k$ plays x^i, whereas each player $i \in S_k$ plays $(1 - \delta_k)x^i + \delta_k b_k^i$, where δ_k is appropriately chosen. Since players need not be indifferent between the various neighbors in $\mathcal{B}$, suitable statistical tests are needed to make deviation nonprofitable, and they can be performed effectively since $|S| \geq 2$ for every $b^S \in \mathcal{B}$.

We define here a profile that depends on various constants and prove that if the constants satisfy several properties then the profile is an ϵ-equilibrium. The exact way to choose the constants appears in the Appendix.

Let $\delta \in (0, \epsilon)$ be sufficiently small. Assume w.l.o.g. that μ has a full support. Denote $\alpha_k = \mu(b_k^{S_k})$, and

$$\delta_k = (\delta \alpha_k / w(x^{-S_k}, b_k^{S_k}))^{1/|S_k|}.$$

Note that

$$w(x^{-S_k}, b_k^{S_k}) \delta_k^{|S_k|} = \delta \alpha_k. \tag{1}$$

Define the following (x, ϵ)-perturbed profile σ:

1. Denote $k = t \bmod K$, where t is the current stage. If no deviation was detected before stage t, then each player i plays as follows. If $i \notin S_k$ player i plays the mixed-action x^i, while if $i \in S_k$ player i plays the mixed-action $(1-\delta_k)x^i + \delta_k b^i_k$.

2. For the statistical test, the players consider at stage t only stages $j < t$ such that $j = t \bmod K$. All other stages are ignored.

(a) Each player i is checked whether his or her realized action is compatible with this profile; that is, if it is in $\mathrm{supp}(x^i)$ and if $i \in S_k$, it may be also b^i_k.

(b) Each player i is checked whether the distribution of his or her realized actions, when restricted to $\mathrm{supp}(x^i)$, is ϵ-close to x^i.

(c) Each player $i \in S_k$ is checked whether he or she plays the action b^i_k with frequency δ_k. Formally, the realized probability p that player i plays b^i_k at stages $j < t$ such that $j = t \bmod K$ should satisfy $|p/(\delta_k t/K) - 1| < \epsilon$.

The first two tests are used in the previous sufficient conditions as well and, if δ is sufficiently small, can be employed effectively. The third statistical test is employed only from stage $t_3 K$, where t_3 is chosen such that (i) no player can profit more than ϵ by changing the frequency in which he or she plays b^i_k before the test is employed (t_3 is not too large), and (ii) the probability of false detection of deviation is smaller than ϵ (t_3 should be large enough).

It is clear that if the players follow σ then the game will be eventually absorbed, and their expected payoff in every sufficiently long game, as well as in the infinite game, is close to $\sum_{b^S \in \mathscr{B}} \mu(b^S) u(x^{-S}, b^S)$. By the statistical tests, no player can profit more than ϵ by deviating.

The only question that arises is whether there exist $t_3 \in \mathbf{N}$ and $\delta > 0$ such that test (c) can be employed effectively. This question is answered affirmatively in the Appendix. Intuitively such constants exist since absorption occurs at every stage with probability $O(\delta)$, while player i perturbs at stage k to b^i_k with probability $O(\delta^{1/|S_k|}) \geq O(\delta^{1/2})$. Hence, until absorption occurs, player i should perturb at least $O(\delta^{-1/2})$ times, which is sufficient for statistical tests. ■

Remark. If the game is a team game, then we can assume w.l.o.g. that $|\mathscr{B}| = 2$. Indeed, consider the convex polyhedron

$$Q \stackrel{\text{def}}{=} \left\{ \sum_{k=1}^{K} \alpha_k u(x^{-S_k}, b_k^{S_k}) \right\},$$

where $\alpha = (\alpha_k)$ ranges on all probability distributions over $\{1, \ldots, K\}$. Since $q = \sum_{k=1}^{K} \mu(b_k^{S_k}) u(x^{-S_k}, b_k^{S_k}) \in Q$ and both q and Q are essentially two-dimensional, there exists a probability distribution $\nu \in \Delta(\mathcal{B})$ such that its support includes at most two elements, and $\sum_{k=1}^{K} \nu(b_k^{S_k}) u(x^{-S_k}, b_k^{S_k}) \geq \sum_{k=1}^{K} \mu(b_k^{S_k}) u(x^{-S_k}, b_k^{S_k}) = q$. Hence x is individually rational w.r.t. $\sum_{k=1}^{K} \nu(b_k^{S_k}) u(x^{-S_k}, b_k^{S_k})$, and the conditions of Lemma 4.5 are satisfied with ν replacing μ.

Note that in a general n-player absorbing game, a similar argument shows that if the conditions of Lemma 4.5 are satisfied and the dimension of Q is d, then there exists a perturbed equilibrium profile where the equilibrium path has a cycle of length d.

5. EXISTENCE OF AN EQUILIBRIUM PAYOFF

In this section we prove that in any absorbing team game at least one of the sufficient conditions presented in Section 4 holds.

Let G be an absorbing team game. We denote by N and M the two sets of players in each team, and let $n = |N|$ and $m = |M|$.

Let

$$d^i = \begin{cases} \max_{j \in N} c^j & i \in N \\ \max_{j \in M} c^j & i \in M \end{cases}.$$

It is clear that for any equilibrium payoff $v = (v^i)_{i \in N \cup M}$, $v^i = v^j$ whenever $i, j \in N$ or $i, j \in M$. Moreover, $v^i \geq c^i$ for every $i \in N \cup M$. Therefore $v^i \geq d^i$ for each $i \in N \cup M$.

The following lemma is crucial for the proof of the main result. The lemma is proved in Solan (1999). Since its proof is involved, we provide here only a sketch of the proof.

LEMMA 5.1. *There exists a mixed-action combination x, a set $\mathcal{B}$ of neighbors of x, and a vector $g = (g^i)_{i \in N \cup M} \in \mathbf{R}^{n+m}$ such that the following four conditions are satisfied for some probability distribution $\mu \in \Delta(\mathcal{B})$:*

1. *At least one of the following holds,*

$$v^i(x) \geq g^i \text{ for every } i \in N \cup M \tag{2}$$

or

$$\sum_{b^S \in \mathcal{B}} \mu(b^S) u^i(x^{-S}, b^S) \geq g^i \text{ for every } i \in N \cup M. \tag{3}$$

2.

$$g^i \geq d^i \text{ for every } i \in N \cup M. \tag{4}$$

3. *For every player i and every action a^i of player i such that* $w(x^{-i}, a^i) > 0$

$$g^i \geq u^i(x^{-i}, a^i). \tag{5}$$

4. *For every player i and every neighbor $a^i \in \mathcal{B}$ of x of distance* 1

$$g^i = u^i(x^{-i}, a^i). \tag{6}$$

Sketch of the Proof. Define a function $\tilde{r}: X \to \mathbf{R}^N$ by

$$\tilde{r}^i(x) = \min\{r^i(x), d^i\}.$$

Define an auxiliary game $\tilde{G}$, which is played like the original game G, but the daily nonabsorbing payoff that player i receives at stage t is $\tilde{r}^i(x_t)$, where x_t is the mixed-action combination that the players play at that stage. Formally, for every discount factor $\beta \in (0, 1)$ and every profile σ, the expected β-discounted payoff for the players if they follow σ is

$$\mathrm{E}_\sigma\left((1-\beta)\sum_{t=1}^{\infty}\beta^t(1_{t\leq t_\star}\tilde{r}(x_t) + 1_{t>t_\star}u(x_{t_\star}))\right),$$

where $t_\star$ is the stage of absorption.

It turns out that the discounted min–max value of each player in the auxiliary game exists and converges, as β tends to 1, to his or her min–max value in the original game.

For every β we choose a stationary β-discounted equilibrium profile x_β in $\tilde{G}$. We define x to be the limit of x_β, g to be the limit of the corresponding β-discounted equilibrium payoff vectors, and μ to be the limit of the probability distribution over the neighbors of x induced by x_β. We assume that the limits exist by taking a subsequence.

Using the fact that x_β is a discounted equilibrium, (4) hold. If in addition we recall that u is continuous whenever it is defined, (5) and (6) hold as well. Since $\tilde{r}^i(x) \leq d^i \leq g^i$, it follows that either (2) or (3) holds as well. ■

We are now ready to prove our main result.

Proof of Theorem 3.5. Let $(x, \mathcal{B}, g, \mu)$ satisfy the conclusion of Lemma 5.1. We prove that x is a base of a perturbed equilibrium.

We shall have 4 cases:

1. x is absorbing.
2. x is nonabsorbing and $v^i(x) \geq g^i$ for every $i \in N \cup M$.
3. x is nonabsorbing, and there exists a neighbor of x $b^{i_0} \in \mathcal{B}$ of distance 1 such that $u^i(x^{-i_0}, b^{i_0}) \geq g^i$ for each $i \in N \cup M$.

4. None of the above holds.

We prove that in Cases 1 and 2 the conditions of Lemma 4.2 hold, and in Case 3 the conditions of Lemma 4.3 hold. If Case 4 holds, we prove that the conditions of Lemma 4.5 hold.

Case 1. x is absorbing.

In this case $\mathcal{B} = \emptyset$ (since there are no neighbors of x) and therefore (2) holds. By (2) and (4) for each $i \in N \cup M$

$$u^i(x) = v^i(x) \geq g^i \geq d^i \geq c^i. \tag{7}$$

By (5), (7), and the definition of e^i, x is individually rational for $v(x)$. Since x is absorbing we have by (7) and (5) for each $i \in N \cup M$

$$g^i \leq v^i(x) = u^i(x) = \frac{\sum_{a^i \in \operatorname{supp}(x^i)} x^i_{a^i} w(x^{-i}, a^i) u^i(x^{-i}, a^i)}{\sum_{a^i \in \operatorname{supp}(x^i)} x^i_{a^i} w(x^{-i}, a^i)} \leq g^i. \tag{8}$$

Equation (8) implies that $u^i(x^{-i}, a^i) = g^i$ for each player i and $a^i \in \operatorname{supp}(x^i)$, and therefore the second condition of Lemma 4.2 holds, as desired.

Case 2. x is nonabsorbing and $v^i(x) \geq g^i$ for each $i \in N \cup M$.

We prove that the conditions of Lemma 4.2 hold. Since x is nonabsorbing, it is sufficient to show that x is individually rational for $v(x)$.

By the assumption and (4)

$$v^i(x) \geq g^i \geq d^i \geq c^i \qquad \text{for every } i \in N \cup M. \tag{9}$$

By (5), (9), and the definition of e^i it follows that x is indeed individually rational for $v(x)$.

Case 3. There exists $b^{i_0} \in \mathcal{B}$ such that

$$u^i(x^{-i_0}, b^{i_0}) \geq g^i \qquad \text{for every } i \in N \cup M. \tag{10}$$

In this case the conditions of Lemma 4.3 hold. Indeed, by (10) and (4) $u^i(x^{-i_0}, b^{i_0}) \geq g^i \geq d^i \geq c^i$ for each $i \in N \cup M$. Therefore, by (5) and the definition of e^i, x is individually rational for $u(x^{-i_0}, b^{i_0})$, as desired.

Case 4. None of the above holds.

We prove that the conditions of Lemma 4.5 hold.

Since Case 1 does not hold, x is nonabsorbing. Since Case 2 does not hold, Eq. (3) holds.

Let $b^{i_0} \in \mathcal{B}$ be a neighbor of distance 1 of x. By (6), $u^{i_0}(x^{-i_0}, b^{i_0}) = g^{i_0}$, and since the game is a team game, $u^i(x^{-i_0}, b^{i_0}) = g^i$ for every player i of the same team as i_0. Since Case 3 does not occur, there exists a player i of the opposing team such that $u^i(x^{-i_0}, b^{i_0}) < g^i$, and therefore $u^i(x^{-i_0}, b^{i_0}) < g^i$ for every player i in the opposing team. Therefore

$$\sum_{b^S \in \mathcal{B} | |S|=1} \mu(b^S) u^i(x^{-S}, b^S) \leq \sum_{b^S \in \mathcal{B} | |S|=1} \mu(b^S) g^i \qquad \text{for every } i \in N \cup M \tag{11}$$

and, if $\sum_{b^S \in \mathcal{B} | |S|=1} \mu(b^S) > 0$,

$$\sum_{b^S \in \mathcal{B} | |S|=1} \mu(b^S) u(x^{-S}, b^S) \neq \sum_{b^S \in \mathcal{B} | |S|=1} \mu(b^S) g. \tag{12}$$

Let $\mathcal{C} = \{b^S \in \mathcal{B} \mid |S| \geq 2\}$. By (3), (11), and (12), $\mathcal{C} \neq \emptyset$ and $\sum_{b^S \in \mathcal{C}} \mu(b^S) > 0$.

Let ν be the induced probability distribution of μ over $\mathcal{C}$. By (3), (11), and (4)

$$\sum_{b^S \in \mathcal{C}} \nu(b^S) u^i(x^{-S}, b^S) \geq g^i \geq d^i \geq c^i \qquad \text{for every } i \in N \cup M. \tag{13}$$

By (13), (5), and the definition of e^i, x is individually rational for $\sum_{b^S \in \mathcal{C}} \nu(b^S) u^i(x^{-S}, b^S)$, and therefore the conditions of Lemma 4.5 hold w.r.t. $\mathcal{C}$ and ν. ■

6. APPENDIX: THE CONSTANTS OF LEMMA 4.5

In the Appendix we show how to choose the constants t_3 and δ in the proof of Lemma 4.5.

These constants should satisfy the following inequalities:

1. The probability of false detection of deviation in test (c) is smaller than ϵ,

$$\Pr\left(\left| \frac{\sum_{j=1}^{n} X_j}{n\delta_k} - 1 \right| < \epsilon \qquad \forall n \geq t_3 \right) > 1 - \epsilon, \tag{14}$$

where (X_j) are i.i.d. Bernoulli r.v. with $p(X_j = 1) = \delta_k$.

2. If player i plays b_k^i at *every* stage j with $j = k \bmod K$, then the probability that the game is absorbed *before* stage t_3 is smaller than ϵ,

$$\left(1 - w(x^{-S_k}, b_k^{S_k}) \delta_k^{|S_k - 1|/|S_k|}\right)^{t_3} > 1 - \epsilon. \tag{15}$$

Since t_3 and δ should be chosen simultaneously, we need the following lemma.

LEMMA 6.1. *Let* $p \in (0,1)$ *and* $(X_t)_{t\in\mathbf{N}}$ *be i.i.d. Bernoulli random variables with* $\Pr(X_t = 1) = p$ *and* $\epsilon > 0$. *There exists* $t_\star \in \mathbf{N}$ (*independent of* p) *such that*

$$\Pr\left(\left|\frac{\sum_{j=1}^t X_j}{tp} - 1\right| < \epsilon \qquad \forall t > \frac{t_\star}{p}\right) > 1 - \epsilon. \tag{16}$$

Define

$$C = \max_{k=1,\dots,K}\left(\frac{\alpha_k}{w(x^{-S_k}, b_k^{S_k})}\right)^{\frac{|S_k|-1}{|S_k|}} > 0.$$

Let $\rho_0 > 0$ be sufficiently small such that

$$(1 - C\rho)^{\rho^{-1/2}} = \left((1 - C\rho)^{1/\rho}\right)^{\rho^{1/2}} > 1 - \epsilon \qquad \forall \rho \in (0, \rho_0]. \tag{17}$$

Let $\delta = \min\{\rho_0^2, 1/t_\star^4\}$ and $t_3 = 1/\delta^4$. In particular, $\sqrt{\delta} \le \rho_0$ and $t_3 \ge t_\star$.

Equation (14) holds since $t_3 \ge t_\star$ while Eq. (15) holds since

$$\left(1 - \delta_k^{|S_k - 1|/|S_k|}\right)^{t_3} \ge \left(1 - C\delta^{1/2}\right)^{\delta^{-1/4}} > 1 - \epsilon.$$

Proof of Lemma 6.1. Let $\lambda \in (1, 1+\epsilon)$ and $t_\star = \lambda/\epsilon^3(\lambda - 1)$. By Kolmogorov's inequality (see, e.g., Lamperti (1996), p. 46), for every $k \in \mathbf{N}$

$$\Pr\left(\max_{\lambda^k t_\star/p < t \le \lambda^{k+1} t_\star/p}\left|\sum_{j=1}^t (X_j - p)\right| > \epsilon\lambda^{k+1} t_\star\right) \le \frac{\lambda^{k+1} t_\star p(1-p)}{\epsilon^2 \lambda^{2(k+1)} t_\star^2 p} < \frac{1}{\epsilon^2 \lambda^{k+1} t_\star}. \tag{18}$$

Summing (18) over all $k \ge 0$ yields

$$\Pr\left(\max_{t_\star/p < t}\left|\sum_{j=1}^t (X_j - p)\right| > 2\epsilon t\right) < \frac{\lambda}{\epsilon^2 t_\star(\lambda - 1)} \le \epsilon$$

and the result follows. ■

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