Three-player absorbing games
Solan, Eilon
Mathematics of Operations Research; Aug 1999; 24, 3; ABI/INFORM Global
pg. 669

MATHEMATICS OF OPERATIONS RESEARCH
Vol. 24, No. 3, August 1999
Printed in U.S.A.

THREE-PLAYER ABSORBING GAMES

EILON SOLAN

An n-player absorbing game is an n-player stochastic game where all the states but one are absorbing (a state is absorbing if once it is reached, the probability to leave it is zero, whatever the players play).

We prove that every three-player absorbing game has an undiscounted equilibrium payoff.

1. Introduction. An n-player absorbing game can be described as follows. A fixed one-shot game is played over and over again. However, for every action combination, there is a probability that once it is played the game is terminated (or is absorbed), and an absorbing payoff vector which the players receive at every future stage if the game is absorbed by this combination.

A payoff vector $g \in \mathbf{R}^n$ is an *ϵ-equilibrium payoff* if there exists a strategy profile $\sigma_\epsilon = (\sigma^i_\epsilon)$ such that if the players follow σ_ϵ then the expected payoff for each player i in every sufficiently long game, as well as the expected liminf of his average payoffs in the infinite game, is at least $g^i - \epsilon$, while if any player i deviates from σ^i_ϵ, then his expected payoff in every sufficiently long game, as well as the expected limsup of his average payoffs in the infinite game, is at most $g^i + \epsilon$. The profile σ_ϵ is an *ϵ-equilibrium profile* for g.

The payoff vector g is an *equilibrium payoff* if it is an ϵ-equilibrium payoff for every $\epsilon > 0$.

It is well known (see, e.g., Blackwell and Ferguson 1968) that 0-equilibrium profiles need not exist, and that stationary and Markov profiles are not sufficient to construct ϵ-equilibrium profiles.

Kohlberg (1974) proved that every two-player zero-sum absorbing game has an equilibrium payoff, which is the value of the game. Vrieze and Thuijsman (1989) proved that every two-player non zero-sum absorbing game has an equilibrium payoff. Moreover, the ϵ-equilibrium profiles can be chosen to be 'almost' stationary (i.e., they are given by a stationary profile plus threat strategies). Recently Flesch et al. (1997) gave an example of a three-player absorbing game where no 'almost' stationary equilibrium payoff exists, and all equilibrium profiles have a cyclic nature.

In the present paper we generalize the result of Vrieze and Thuijsman for three-player games, i.e., we prove that every three-player absorbing game has an equilibrium payoff. By Neyman and Sorin (1998), our result implies that every three-player repeated game with incomplete information and symmetric signaling has an equilibrium payoff.

The basic idea of the proof is, as in Vrieze and Thuijsman (1989), to consider a sequence of discounted equilibria that converges to a limit as the discount factor goes to 1, and to construct different types of ϵ-equilibrium profiles according to various properties of this sequence. Denote by $x = (x^i)$ the limit of the discounted stationary equilibrium profiles, and by $g = (g^i)$ the limit of the corresponding discounted equilibrium payoff vectors. The action combination x will be viewed also as a stationary profile.

When there are two players, Vrieze and Thuijsman prove that three cases can occur. (i) x

Received February 5, 1997; revised March 13, 1998 and February 17, 1999.
AMS 1991 subject classification. Primary: 90D20; Secondary: 90D15.
OR/MS subject classification. Primary: Games/Stochastic.
Key words. Absorbing games, repeated games with absorbing states, stochastic games, undiscounted equilibrium payoff.

0364-765X/99/2403/0669/$05.00
Copyright © 1999, Institute for Operations Research and the Management Sciences

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

is absorbing (that is, if the players play the stationary strategy x, absorption occurs eventually with probability 1), and then, by adding threat strategies to the stationary profile x, one can construct an ϵ-equilibrium profile in the undiscounted game. (ii) x is nonabsorbing, but the expected nonabsorbing payoff for the players if they follow the stationary profile x is at least g. Then, by adding threat strategies to the stationary strategy x, one can construct an ϵ-equilibrium profile in the undiscounted game. (iii) x is nonabsorbing, but the nonabsorbing payoff for one of the players, say player 1, if the players follow the stationary profile x, is strictly less then g^1. In this case Vrieze and Thuijsman prove that player 2 has an action a^2 (or a *perturbation*) such that the stationary profile (x^1, a^2) is absorbing, and it yields both players a payoff which is at least g. Using this perturbation and threat strategies, one can construct an ϵ-equilibrium profile in the undiscounted game.

When trying to generalize this approach for more than two players, it turns out that if the first two cases do not hold, then there does not necessarily exist a perturbation (of any subset of the players), nor a convex combination of perturbations, which yields every player i a payoff which is at least g^i.

To overcome this difficulty, we define an auxiliary game, where the nonabsorbing payoffs are bounded by the min-max value of the players in the original game, and the absorbing payoff remains the same. We consider a sequence of discounted equilibria in the auxiliary game, and denote by x the limit of the discounted stationary equilibrium profiles and by g the limit of the corresponding discounted equilibrium payoffs. It turns out that the first two cases of Vrieze and Thuijsman yield an 'almost' stationary ϵ-equilibrium profile in the original game as above, and, if they do not hold, then there exists a *convex combination* of some absorbing perturbations that yields each player i a payoff which is at least g^i. Since the absorbing payoff in the auxiliary game is equal to the absorbing payoff in the original game, this convex combination yields each player i a payoff which is at least g^i in the original game as well. When there are three players, one can construct, using this convex combination, an ϵ-equilibrium profile in the original game for every $\epsilon > 0$. Moreover, this profile has a cyclic nature, but the length of the cycle may depend on ϵ.

The existence of a "good" convex combination of absorbing perturbations turns out to be useful in deriving various results for absorbing games, such as (i) existence of an equilibrium payoff in games where the players are divided into two teams, and the players in each team have the same payoff function (Solan 1997), and (ii) existence of an extensive-form correlated equilibrium (for a more general result see Solan and Vieille 1998a).

In all the ϵ-equilibrium profiles that we construct, the players play mainly the limit of the discounted stationary equilibrium profiles, and perturb to other actions with a very small probability, while checking statistically whether the other players do not deviate. Once a deviation is detected, the deviator is punished with an ϵ-min-max profile forever.

Unfortunately, our approach cannot be generalized for more than three players. In §9 we give an example of a four-player absorbing game where there exists a sequence of discounted equilibrium profiles that converges to a limit, but one cannot construct an ϵ-equilibrium profile where the players play mainly the limit mixed-action. It is currently not known whether equilibrium payoff exists in every n-player absorbing game, for $n \geq 4$.

The paper is arranged as follows. In §2 we give three examples of three-player absorbing games, show some of the equilibrium payoffs of these games, and illustrate the construction of the ϵ-equilibrium profiles for the general case. In §3 we define the model of absorbing games and state the main result, and in §4 we state two additional theorems, that have their own importance, and are crucial to the proof of the main result. In §5 we give four sets of sufficient conditions for existence of an equilibrium payoff in absorbing games. Sections 6–8 are devoted to the proof of the first theorem, and §10 is devoted to the proof of the second theorem.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

2. Examples. In this section we illustrate different types of ϵ-equilibrium profiles that can occur in absorbing games, by giving three examples.

EXAMPLE 1.

	W: L	W: R	E: L	E: R
T	0, 1, 0*	0, 1, 2*	1, 0, 0	0, 1, 0
B	1, 0, 0	0, 0, 0	0, 0, 1	1, 1, 1

In this game player 1 chooses a row, player 2 chooses a column and player 3 chooses a matrix. An asterisked cell means that once this cell is chosen, the game is absorbed with probability 1 and the absorbing payoff is as indicated by the cell. A nonasterisked cell means that whenever this cell is chosen the game is not absorbed, and the nonabsorbing payoff is as indicated by the cell. Note that the min-max value of each player is 0.

One equilibrium payoff in the game is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Consider the nonabsorbing stationary profile $(\frac{1}{2}T + \frac{1}{2}B, \frac{1}{2}L + \frac{1}{2}R, E)$. Player 1 is indifferent between his actions, but player 2 can profit by playing always R. Hence players 1 and 3 must conduct a statistical test, check statistically whether player 2 plays each action approximately half the time, and punish him with a min-max strategy if a deviation is detected. In order to prevent player 3 from playing always W (and receiving an expected payoff 1), players 1 and 2 threaten player 3 with a min-max strategy if he deviates.

Another equilibrium payoff is (0, 1, 1), and an equilibrium profile is to play the stationary absorbing profile $(T, \frac{1}{2}L + \frac{1}{2}R, W)$, while checking for a deviation of players 1 and 3 and punish a deviator with a min-max strategy profile. Note that deviations of player 2 cannot be detected, since if the other two follow this profile, then whatever he plays, the game is absorbed at the first stage. However, player 2 is indifferent between his actions, since he receives 1 whatever he plays; hence the fact that his deviations cannot be detected does not affect this equilibrium. □

These two types of equilibria were used by Vrieze and Thuijsman (1989) for two-player games.

EXAMPLE 2.

	W: L	W: C	W: R	E: L	E: C	E: R
T	0, 0, 0	0, 0, 0	0, 1, 0*	0, 0, 0	0, 0, 0	
M	0, 0, 0	1, 4, 1*		3, 0, 1*		
B	1, −2, 3*					

The empty cells may be arbitrary (both payoff and whether they are absorbing or not). Let R be the maximal payoff (in absolute values). For the rest of the section we denote $\delta = \epsilon/R$. Note that the min-max value of players 1 and 2 is at most 1, and the min-max value of player 3 is at most 0.

One equilibrium payoff is (1, 4, 1) and an ϵ-equilibrium profile for it is to play at every

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

stage the mixed-action combination $((1-\delta)T+\delta M, (1-\delta)L+\delta C, W)$. Clearly if the players follow this profile they receive the desired payoff, and neither player 1 nor player 2 can profit more than $\delta R=\epsilon$ by deviating. To deter player 3 from deviating, once he deviates he is punished with his min-max value.

Absorption occurs at every stage with probability δ^2, while perturbations of players 1 and 2 occur with probability δ. Therefore, if δ is sufficiently small, the players can conduct statistical tests to check whether player 1 plays the action M in frequency δ, and whether player 2 plays the action L with frequency δ. Though these tests are not necessary, since no player can profit by deviating, they can still be employed.

Another equilibrium payoff is $(2, 2, 1)=\frac{1}{2}(1, 4, 1)+\frac{1}{2}(3, 0, 1)$, and an ϵ-equilibrium profile for it is:

- At odd stages play $((1-\delta)T+\delta M, (1-\delta)L+\delta C, W)$.
- At even stages play $((1-\delta)T+\delta M, L, (1-\delta)W+\delta E)$.

If the players follow this profile then their expected payoff is approximately $(2, 2, 1)$. Since player 1 prefers absorption by (M, L, E), while player 2 prefers absorption by (M, C, W), the players should check statistically whether each of them follows this profile, and punish a deviator with an ϵ-min-max profile.

Yet a third equilibrium payoff is $(1, 1, 2)=\frac{1}{2}(1, 4, 1)+\frac{1}{2}(1, -2, 3)$, and an ϵ-equilibrium profile for it is:

- At odd stages play $((1-\delta)T+\delta M, (1-\delta)L+\delta C, W)$.
- At even stages play $((1-\delta^2)T+\delta^2 B, L, W)$.

If the players follow this profile then their expected payoff is approximately $(1, 1, 2)$. In this case the players cannot check whether player 1 perturbs at even stages in the pre-specified probability, since once he plays B the game is absorbed with probability of at least $1-\delta$, and the probability that absorption occurs before he ever plays B is approximately $\frac{1}{2}$. However, player 1 has no incentive to deviate, and therefore such a check is not needed. Nevertheless the players do need to check whether player 2 perturbs at odd stages as he should.

Actually, every convex combination (g^1, g^2, g^3) of the four absorbing cells $(1, -2, 3)$, $(1, 4, 1)$, $(0, 1, 0)$ and $(3, 0, 1)$ in which $g^1, g^2 \geq 1$, $g^3 \geq 0$ that satisfies:

- if $(1, -2, 3)$ has a positive weight in this combination then $g^1=1$;
- if $(0, 1, 0)$ has a positive weight in this combination then $g^2=1$; is an equilibrium payoff. □

EXAMPLE 3.

	W: L	W: R	E: L	E: R
T	0, 0, 0	0, 1, 3*	3, 0, 1*	1, 1, 0*
B	1, 3, 0*	1, 0, 1*	0, 1, 1*	0, 0, 0*

This game was studied by Flesch et al. (1997). The game is symmetric in the sense that for every player i and action combination $a=(a^1, a^2, a^3)\neq(T, L, W)$ we have

$$u^i(a^1, a^2, a^3)=u^{i+1 \bmod 3}(a^3, a^1, a^2),$$

where $u^i(a)$ is the absorbing payoff to player i if the action combination a is played, and we identify $T=L=W$ and $B=R=E$.

Flesch et al. (1997) prove that the game possesses no stationary ϵ-equilibrium. However,

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

they prove that (1, 2, 1) is an equilibrium payoff. The equilibrium profile that Flesch et al. suggest is the following.

- At the first stage, the players play $(\frac{1}{2}T + \frac{1}{2}B, L, W)$.
- At the second stage, the players play $(T, \frac{1}{2}L + \frac{1}{2}R, W)$.
- At the third stage, the players play $(T, L, \frac{1}{2}W + \frac{1}{2}E)$.
- Afterwards, the players play cyclically those three mixed-action combinations, until absorption occurs.

If the players follow this profile then their expected payoff is (1, 2, 1), and it can easily be checked that no player has a profitable deviation. Note that this profile guarantees that the game will eventually be absorbed.

Let $\epsilon > 0$, $\delta = \epsilon/3$ and $n \in \mathbf{N}$ satisfy that $(1 - \delta)^n = \frac{1}{2}$. A more robust ϵ-equilibrium profile for this game, that does not depend on the payoffs of the cells (T, R, E), (B, R, W) and (B, L, E) is the following:

- The players play $((1 - \delta)T + \delta B, L, W)$ for n stages (thus, the overall probability to be absorbed by the action combination (B, L, W) is $\frac{1}{2}$).
- Then the players play $(T, (1 - \delta)L + \delta R, W)$ for n stages.
- Then the players play $(T, L, (1 - \delta)W + \delta E)$ for n stages.
- Afterwards, the players play cyclically those three phases, until absorption occurs.

The exact conditions for existence of such an ϵ-equilibrium profile appear in §5. □

3. The model and the main result.

An *absorbing game* is a 5-tuple $G = (N, (A^i, h^i, u^i)_{i\in N}, w)$ where:

- N is a finite set of players.
- For every player $i \in N$, A^i is a finite set of pure actions available to player i. Denote $A = \times_{i\in N} A^i$.
- For every player i, $h^i : A \to \mathbf{R}$ is a function that assigns to each action combination $a \in A$ a nonabsorbing payoff for player i.
- For every action combination $a \in A$, $w(a)$ is the probability that the game is absorbed if this action combination is played. If the game is absorbed, $u^i(a)$ is the payoff that player i receives, at each future stage.

We denote $X^i = \Delta(A^i)$, the set of all probability distributions over A^i, $X = \times_{i\in N} X^i$, $X^L = \times_{i\in L} X^i$ for each $L \subset N$ and $X^{-i} = \times_{j\neq i} X^j$. Let $R \geq 1$ be a bound on $|h^i|$ and $|u^i|$ for every i, and $H = \bigcup_{t\in\mathbf{N}} A^t$ be the space of all finite histories.

DEFINITION 3.1. A (*behavioral*) *strategy* for player i is a function $\sigma^i : H \to X^i$. A vector $\sigma = (\sigma^i)_{i\in N}$ where each σ^i is a strategy for player i is a *strategy profile* (or simply a *profile*).

Sometimes we view a mixed action combination $x \in X$ as a stationary strategy profile.

Every profile σ induces a probability distribution over $A^{\mathbf{N}}$, the space of infinite histories (equipped with the σ-algebra that is spanned by all the finite cylinders). We denote expectation according to this probability distribution by $\mathbf{E}_\sigma$.

Let r_t^i be the payoff that player i receives at stage t.

DEFINITION 3.2. A vector $g \in \mathbf{R}^N$ is an *equilibrium payoff* if for every $\epsilon > 0$ there exists $t_\epsilon \in \mathbf{N}$ and a strategy profile σ_ϵ such that

- For every player $i \in N$,

$$\mathbf{E}_{\sigma_\epsilon}\left(\liminf_{t\to\infty} \frac{r_1^i + r_2^i + \cdots + r_t^i}{t}\right) \geq g^i - \epsilon,$$

and for every $t > t_\epsilon$,

$$\mathbf{E}_{\sigma_\epsilon}\left(\frac{r_1^i + r_2^i + \cdots + r_t^i}{t}\right) \geq g^i - \epsilon.$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

- For every player i and every strategy τ^i of player i,

$$\mathbf{E}_{\sigma_\epsilon^{-i},\tau^i}\left(\limsup_{t\to\infty} \frac{r_1^i + r_2^i + \cdots + r_t^i}{t}\right) \leq g^i + \epsilon,$$

and for every $t > t_\epsilon$,

$$\mathbf{E}_{\sigma_\epsilon^{-i},\tau^i}\left(\frac{r_1^i + r_2^i + \cdots + r_t^i}{t}\right) \leq g^i + \epsilon.$$

The strategy profile σ_ϵ is an *ϵ-equilibrium profile* for the payoff g.

The main result of the paper is:

THEOREM 3.3. *Every three-player absorbing game has an equilibrium payoff.*

4. An outline of the proof. The proof is divided into three main steps:

(i) We give four sufficient conditions for existence of an equilibrium payoff in absorbing games.

(ii) We prove that in any absorbing game there exists a mixed action combination that satisfies one of three sets of properties. The first (resp. second) set of properties implies that the first (resp. second) sufficient condition holds.

(iii) Only the third set of properties poses a difficulty. We then prove a geometric result which implies that if the third set of properties is satisfied and $|N| = 3$, then one of the last two sufficient conditions hold.

In this section we present two theorems that correspond to steps (ii) and (iii). The sufficient conditions are given in the next section.

To prove the second step, we define an auxiliary absorbing game where the nonabsorbing payoff of each player is bounded by his min-max value in the original game, and the absorbing payoff remains the same. Every limit of discounted equilibria in the auxiliary game defines a mixed action combination that satisfies (ii). An important step in the proof is to prove that the discounted min-max values in the auxiliary game converge to the min-max value of the original game.

We begin with some preliminary definitions. We denote the multilinear extensions of w and h over X by w and h. We define an extension of u to X by

$$u(x) = \frac{\sum_{a\in A} (\prod_{i\in N} x^i_{a^i}) w(a) u(a)}{w(x)} \tag{1}$$

whenever $w(x) > 0$ and 0 otherwise. Note that u is continuous at every point x such that $w(x) > 0$, and that wu is multilinear over X. We identify each action $a^i \in A^i$ with the probability distribution over A^i that gives probability 1 to a^i.

DEFINITION 4.1. The real number $c^i \in \mathbf{R}$ is the *min-max value* of player i if for every $\epsilon > 0$ there exist $t_\epsilon \in \mathbf{N}$ and a profile σ_ϵ^{-i} of players $N\backslash\{i\}$ such that

- For every strategy σ^i of player i,

$$\mathbf{E}_{\sigma_\epsilon^{-i},\sigma^i}\left(\limsup_{t\to\infty} \frac{r_1^i + r_2^i + \cdots + r_t^i}{t}\right) \leq c^i + \epsilon,$$

and for every $t > t_\epsilon$,

$$\mathbf{E}_{\sigma_\epsilon^{-i},\sigma^i}\left(\frac{r_1^i + r_2^i + \cdots + r_t^i}{t}\right) \leq c^i + \epsilon.$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

- For every strategy profile σ^{-i} of players $N\setminus\{i\}$ there exists a strategy σ^i of player i such that

$$\mathbf{E}_\sigma\left(\liminf_{t\to\infty}\frac{r_1^i + r_2^i + \cdots + r_t^i}{t}\right) \ge c^i - \epsilon,$$

and for every $t > t_\epsilon$,

$$\mathbf{E}_\sigma\left(\frac{r_1^i + r_2^i + \cdots + r_t^i}{t}\right) \ge c^i - \epsilon.$$

The profile σ_ϵ^{-i} is an *ϵ-min-max profile against player i.*

LEMMA 4.2. *For every player i, the min-max value c^i exists. Moreover, $c^i = \lim_{\beta\to 1} c^i(\beta)$, where $c^i(\beta)$ is the min-max value of player i in the β-discounted game.*

This result was proved by Mertens and Neyman (1981) for two-player stochastic games, and an unpublished proof of Neyman (1996) that follows similar lines proves the result for n-player stochastic games.

For every player i define the function $e^i : X \to \mathbf{R}$ by

$$e^i(x) = w(x)u^i(x) + (1 - w(x))c^i.$$

$e^i(x)$ is the maximal payoff that player i can receive if at the current stage the players play the mixed-action combination x, and from tomorrow on player i is punished with an ϵ-min-max profile (with an arbitrary small ϵ).

DEFINITION 4.3. A mixed action combination $x \in X$ is *absorbing* if $w(x) > 0$ and *nonabsorbing* otherwise.

Let $a, b \in \mathbf{R}^d$. We say that $a \ge b$ if $a_i \ge b_i$ for every $i = 1, \dots, d$, and that $a > b$ if $a \ge b$ and $a \ne b$.

As seen in Example 2, a basic notion is that of absorbing neighbors:

DEFINITION 4.4. Let $x \in X$ be a nonabsorbing mixed-action combination and $L \subseteq N$. An action combination $b^L \in \times_{i\in L} A^i$ is *an absorbing neighbor of x by L* if

- $w(x^{-L}, b^L) > 0$.
- $w(x^{-L'}, b^{L'}) = 0$ for every strict subset L' of L.

If $L = \{i\}$ the absorbing neighbor is called a *single absorbing neighbor* of player i.

We denote by $\mathcal{B}(x)$ the set of all absorbing neighbors of x, and by $\mathcal{B}_i(x)$ the set of all single absorbing neighbors of player i. Note that $\mathcal{B}(x)$ is never empty (as long as there is an absorbing action combination), but $\mathcal{B}_i(x)$ may be empty.

The following theorem turns out to be very useful for absorbing games. In addition to its crucial part in the proof of Theorem 3.3, it is used to prove the existence of an equilibrium payoff in absorbing team games (the players are divided into two teams, and the players in each team have the same payoff functions; see Solan 1997) and can be used to prove the existence of an extensive-form correlated equilibrium in n-player absorbing games (for a more general result, see Solan and Vieille 1998a).

THEOREM 4.5. *There exist $x \in X$, $g \in \mathbf{R}^N$, $\mu \in \Delta(\mathcal{B}(x))$ and $d : \bigcup_i A^i \to [0, \infty)$ that satisfy:*

(a) $g^i \ge e^i(x^{-i}, a^i)$ *for every player i and action $a^i \in A^i$.*

(b) *At least one of the following holds:*

  (i) *x is nonabsorbing and $h(x) \ge g$.*

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

(ii) *x is absorbing*, $u(x) = g$ *and* $u^i(x^{-i}, a^i) = g^i$ *for every player i and every action* $a^i \in \text{supp}(x^i)$ *such that* $w(x^{-i}, a^i) > 0$.

(iii) (1) *x is nonabsorbing.*

(2) **Average payoff condition:** *For every player* $i \in N$, $\sum_{b^L \in \mathcal{B}(x)} \mu(b^L) u^i(x^{-L}, b^L) \geq g^i$.

(3) **Vanishing action condition:** *For every player i and every action* $a^i \notin \text{supp}(x^i)$

$$\sum_{b^L \mid i \in L, b^i = a^i} \mu(b^L) u^i(x^{-L}, b^L) = \left(\sum_{b^L \mid i \in L, b^i = a^i} \mu(b^L) \right) g^i.$$

(4) **Degree condition:** $d(a^i) = 0$ *for every player i and every* $a^i \in \text{supp}(x^i)$. *Moreover, if there is a strict inequality in* (b)(iii)(2) *for at least one player, then* $\sum_{i \in L} d(b^i) = 1$ *for every* $b^L \in \text{supp}(\mu)$.

Note that the second condition in (b)(ii) follows from the third condition in (b)(ii). Condition (a) states that g is a "good" payoff vector; that is, no player can profit by deviating and being punished from the second stage. Condition (a) and either (b)(i) or (b)(ii) imply that there exists an equilibrium payoff of the types presented in Example 1.

If there is an equality in (b)(iii)(2) for all the players, we say that the average payoff condition holds with equality. Otherwise we say that it holds with strict inequality. The names of the last two conditions are taken from the proof of the theorem: we will consider a sequence of discounted equilibria in the auxiliary game that converges to a limit. The vanishing action condition will relate to actions that have positive probability in any discounted equilibrium in this sequence, but vanish in the limit. The function d in the degree condition will be defined as the degree of some power series.

For $|N| = 2$, Theorem 4.5 can be deduced by the results of Vrieze and Thuijsman (1989). In this case, μ is supported by single absorbing neighbors.

One should now consider the case that condition (b)(iii) holds. In that case, the following theorem states that if there are only three players then there exists an equilibrium payoff of the types presented in Examples 2 and 3.

To state the theorem, we need to make precise the conditions in which we have a cyclic equilibrium payoff. It turns out that the following definition plays an important role:

DEFINITION 4.6. Let $a, b, c \in \mathbf{R}^3$. The three vectors (a, b, c) are *left-cyclic* if $b_1 > a_1 > c_1$, $c_2 > b_2 > a_2$ and $a_3 > c_3 > b_3$, and *right-cyclic* if $b_1 < a_1 < c_1$, $c_2 < b_2 < a_2$ and $a_3 < c_3 < b_3$. They are *cyclic* if they are either left-cyclic or right-cyclic. They are *positive cyclic* if they are cyclic and

$$\det\begin{pmatrix} 0 & b_1 - a_1 & c_1 - a_1 \\ a_2 - b_2 & 0 & c_2 - b_2 \\ a_3 - c_3 & b_3 - c_3 & 0 \end{pmatrix} > 0.$$

Whenever we say that three vectors are cyclic, it should be understood that the first vector serves as the a in the above definition, the second vector serves as the b, and the third serves as the c.

Note that if (a, b, c) are positive left-cyclic, then $((a_1, a_3, a_2), (c_1, c_3, c_2), (b_1, b_3, b_2))$ are positive right-cyclic.

THEOREM 4.7. *If* 4.5(b)(iii) *holds and* $|N| = 3$ *then at least one of the following holds*:

(a) *There exists* $\nu \in \Delta(\mathcal{B}(x))$ *such that* $\text{supp}(\nu) \subseteq \text{supp}(\mu)$, *for every player i*,

$$\sum_{b^L \in \text{supp}(\nu)} \nu(b^L) u^i(x^{-L}, b^L) \geq g^i \tag{2}$$

and for every player i such that $\mathcal{B}_i(x) \cap \text{supp}(\nu) \neq \emptyset$ *there is an equality in* (2).

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

(b) *For every player i there exists* $y^i \in \Delta(\mathcal{B}_i(x))$ *such that* $\mathrm{supp}(y^i) \subseteq \mathrm{supp}(\mu)$ *and* $(u(x^{-1}, y^1), u(x^{-2}, y^2), u(x^{-3}, y^3))$ *are positive cyclic vectors.*

As we will see, if (a) (resp. (b)) holds then there exists an equilibrium payoff of the type presented in Example 2 (resp. 3). The function d of the degree condition is essential to prove this theorem.

Theorem 3.3 is an immediate consequence of Theorems 4.5, 4.7 and the four sufficient conditions for existence of an equilibrium payoff that we provide in the next section.

5. Four types of equilibria.

In this section we present four sets of sufficient conditions for the existence of an equilibrium payoff in absorbing games. Three of the four sets are for games with an arbitrary number of players, while the fourth is given only for three-player games. With every set of sufficient conditions we give the corresponding ϵ-equilibrium profiles. The profiles use various constants, which are chosen in such a way to insure that no player can profit too much by deviating, and that a false detection of deviation occurs with a small probability. The way to choose these constants appears in Appendix 1.

5.1. An 'almost' stationary nonabsorbing equilibrium.

LEMMA 5.1. *Let* $x \in X$ *be a nonabsorbing mixed action combination such that*

$$h^i(x) \geq e^i(x^{-i}, a^i) \qquad \forall\, i \in N,\ a^i \in A^i.$$

Then $h(x)$ *is an equilibrium payoff.*

Note that by the assumption it follows that $h^i(x) \geq c^i$ for every i.

PROOF. Let $\epsilon > 0$ be fixed. Consider the following strategy profile σ: At every stage the players play the mixed action combination x. In addition, the players conduct the following statistical tests:

(1) Each player is checked whether his past actions are compatible with σ.

(2) At every stage $t \geq t_1$ (where t_1 is defined below), each player i is checked whether the distribution of his realized actions is ϵ-close to x^i.

The first player that fails one of these statistical tests is punished with an ϵ-min-max profile forever. If more than one player fails one of these tests at the same stage, then the player with the minimal index is punished.

The constant $t_1 \geq t_c$ is chosen sufficiently large such that the probability of false detection of deviation is bounded by ϵ; that is, for every $i \in N$,

$$\Pr(\|\bar{X}^i_t - x^i\| < \epsilon \ \ \forall\, t > t_1) > 1 - \epsilon/|N|,$$

where $\bar{X}^i_t = (1/t)\sum_{j=1}^{t} X^i_j$ and $\{X^i_j\}$ are i.i.d. r.v. with distribution x^i.

It is easy to verify that by the condition no player can profit more than $2\epsilon R$ by deviating in the infinite game, as well as in any t-stage game, for $t \geq 2Rt_1/\epsilon$, and therefore σ is a $2\epsilon R$-equilibrium profile for $h(x)$. For a more detailed analysis of this profile the reader may refer to Vrieze and Thuijsman (1989). □

5.2. An 'almost' stationary absorbing equilibrium.

LEMMA 5.2. *Let* $x \in X$ *be an absorbing mixed action combination that satisfies the following two conditions*:

(1) $u^i(x) \geq e^i(x^{-i}, a^i)$ $\forall i \in N,\ a^i \in A^i$.

(2) $u^i(x) = u^i(x^{-i}, a^i)$ *for every* $i \in N$ *and* $a^i \in \mathrm{supp}(x^i)$ *such that* $w(x^{-i}, a^i) > 0$.

Then $u(x)$ *is an equilibrium payoff.*

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

PROOF. Let $\epsilon > 0$ be fixed. Consider the strategy profile σ that was defined in the proof of Lemma 5.1, where t_1 is sufficiently large to satisfy $(1 - w(x))^{t_1} < \epsilon$: absorption occurs in the first t_1 stages with probability greater than $1 - \epsilon$.

If the players follow σ then their expected payoff in the infinite game, as well as in every t-stage game, for $t \geq t_1$, is $2\epsilon R$-close to $u(x)$. By the first condition no player i can profit more than $3\epsilon R$ by playing an action outside $\text{supp}(x^i)$, and by the second condition neither can he profit more than $2\epsilon R$ by altering the probabilities in which he plays actions in $\text{supp}(x^i)$.

Therefore σ is a $3\epsilon R$-equilibrium profile for $u(x)$. For a more detailed analysis of this profile the reader may refer to Vrieze and Thuijsman (1989). □

5.3. An average of absorbing cells.

LEMMA 5.3. *Let $x \in X$ be a nonabsorbing mixed action combination. Let $\mu \in \Delta(\mathcal{B}(x))$ and denote $g = \sum_{b^L \in \mathcal{B}(x)} \mu(b^L) u(x^{-L}, b^L)$. Assume the following conditions hold*:

(1) $g^i \geq e^i(x^{-i}, a^i)$ $\forall i \in N$, $a^i \in A^i$.

(2) $u^i(x^{-i}, a^i) = g^i$ *for every player i and every action $a^i \in \mathcal{B}_i(x) \cap \text{supp}(\mu)$.*

Then g is an equilibrium payoff.

PROOF. Let $\epsilon, \eta > 0$ be sufficiently small, $T \in \mathbf{N}$ sufficiently large, and $f: [1, \ldots, T] \to \text{supp}(\mu)$ such that

$$\left| \frac{\#\{j | f(j) = b^L\}}{T} - \mu(b^L) \right| < \epsilon/3 \qquad \forall\, b^L \in \text{supp}(\mu). \tag{3}$$

In words, f is a discrete approximation of μ. Extend the domain of f to $\mathbf{N}$ by $f(t) = f(t \bmod T)$ for every $t > T$. Let $L(t)$ be the set of players for which $f(t)$ is an absorbing neighbor of x.

In the sequel, $\delta \in (0, \epsilon)$ is sufficiently small, such that

$$(1 - \delta)^T > 1 - \epsilon/3 \tag{4}$$

and $t_1, t_2 \geq 1/\epsilon$ are sufficiently large. For every $b^L \in \text{supp}(\mu)$ let $\delta(b^L) = (\delta / w(x^{-L}, b^L))^{1/|L|}$. Define a profile σ as follows:

- At stage t the players play the mixed action combination $(1 - \delta(f(t)))x + \delta(f(t))(x^{-L(t)}, f(t))$.

If the players follow σ then the probability of absorption at each stage t is $\delta(f(t))^{|L(t)|} w(x^{-L(t)}, f(t)) = \delta$. Fix consecutive T stages. By (3) and (4), the probability that the game is absorbed by a neighbor $b^L \in \text{supp}(\mu)$, given absorption occurs in these T stages, is ϵ-close to $\mu(b^L)$. Hence, for every stage t, if absorption has not occurred before stage t then the expected payoff for the players is ϵR close to g.

We add the following statistical tests to σ. Each player i is checked for the following:

(1) Whether his realized actions are compatible with σ.

(2) For every $b^L \in \text{supp}(\mu)$ such that $i \notin L$, whether the distribution of his realized actions, restricted to stages j such that $f(j) = b^L$, is η-close to x^i. This check is done only after stage $t_1 T$.

(3) For every $b^L \in \text{supp}(\mu)$ such that $i \in L$, whether player i plays the action b^i during stages j such that $f(j) = b^L$ with probability $\delta(b^L)$ (that is, the realized probability p should satisfy $1 - \eta/|N| < p/\delta(b^L) < 1 + \eta/|N|$). This check is done only after stage $t_2 T$.

(4) If $\text{supp}(\mu) \subseteq \mathcal{B}_i(x)$, whether the game was absorbed before stage t_0 (where t_0 is chosen below).

By the first condition no player can profit more than ϵR by playing an action which is not

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

compatible with his strategy, while by the second condition no player i can profit more than ϵR by altering the probabilities in which he plays at every stage any single absorbing neighbor a^i of x. The second and third statistical tests verify that the players indeed follow, modulo statistical errors, the prespecified profile. The last test deals with the case that *all* the absorbing neighbors in $\text{supp}(\mu)$ are single absorbing neighbors of some player i. If such a case arises, it may be of interest to player i *never* to be absorbed, and to receive the nonabsorbing payoff $h^i(x)$.

As we prove in Appendix 1, since absorption occurs at every single stage with probability δ while for every $b^L \in \text{supp}(\mu)$ with $|L| \geq 2$ the probability to play the action b^i is $\delta(b^L)$, and $\delta = o(\delta(b^L))$, the constants δ, t_1 and t_2 can be chosen in such a way to ensure that no player can profit more than $2\epsilon R$ by any deviation, and that the probability of false detection of deviation is at most ϵ. □

5.4. A cyclic equilibrium. The last sufficient condition is given only for three-player absorbing games, hence we assume that $N = \{1, 2, 3\}$.

LEMMA 5.4. *If (a, b, c) are positive right-cyclic vectors then the system of equations*

$$a_1 = \frac{\beta b_1 + (1-\beta)\gamma c_1}{\beta + (1-\beta)\gamma}, \tag{5}$$

$$b_2 = \frac{\gamma c_2 + (1-\gamma)\alpha a_2}{\gamma + (1-\gamma)\alpha}, \tag{6}$$

$$c_3 = \frac{\alpha a_3 + (1-\alpha)\beta b_3}{\alpha + (1-\alpha)\beta}, \tag{7}$$

has a unique solution. Moreover, this solution satisfies $\alpha, \beta, \gamma \in (0, 1)$.

PROOF. Assume w.l.o.g. that $a_1 = b_2 = c_3 = 0$. Since the vectors are cyclic, it follows that $a_2, a_3, b_1, b_3, c_1, c_2 \neq 0$. Note that every solution (α, β, γ) satisfies that $\alpha, \beta, \gamma \neq 0, 1$.

We are now going to calculate β. Similar calculations can be done for α and γ. By (6) and (7) we have

$$\alpha = \frac{-\gamma c_2}{(1-\gamma)a_2} = \frac{\beta b_3}{\beta b_3 - a_3}, \tag{8}$$

and by (5) we have

$$\gamma = \frac{-\beta b_1}{(1-\beta)c_1}. \tag{9}$$

Substituting (9) in (8) and dividing by β yields

$$\frac{b_1 c_2}{c_1 - \beta c_1 + \beta b_1} = \frac{b_3 a_2}{\beta b_3 - a_3}.$$

Hence

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

$$\beta = \frac{a_2b_3c_1 + a_3b_1c_2}{b_3b_1c_2 + a_2b_3c_1 - a_2b_3b_1}$$

is uniquely determined. Since (a, b, c) are right-cyclic it follows that the denominator is positive, while since they are positive cyclic, the numerator is also positive. Hence $\beta > 0$. To prove that $\beta < 1$ it is sufficient to prove that $b_3b_1c_2 - a_2b_3b_1 - a_3b_1c_2 > 0$, which holds since (a, b, c) are right-cyclic. □

LEMMA 5.5. *Let $x \in X$ be a nonabsorbing mixed action combination and for every $i \in N$ let $y^i \in X^i$ such that*

(1) for each $i = 1, 2, 3$, $w(x^{-i}, y^i) > 0$ and $u^i(x^{-i}, y^i) \geq e^i(x^{-i}, a^i)$ for every $a^i \in A^i$

(2) $(u(x^{-1}, y^1), u(x^{-2}, y^2), u(x^{-3}, y^3))$ are positive cyclic vectors.

(3) For every player i and action $a^i \in \operatorname{supp}(y^i)$, $w(x^{-i}, a^i) > 0$ and $u^i(x^{-i}, a^i) = u^i(x^{-i}, y^i)$.

Then there exists an equilibrium payoff.

PROOF. Assume w.l.o.g. that $(u(x^{-1}, y^1), u(x^{-2}, y^2), u(x^{-3}, y^3))$ are right-cyclic (otherwise, change the names of players 2 and 3, and recall the remark after Definition 4.6). By condition (2) and Lemma 5.4 there exist $\alpha, \beta, \gamma \in (0, 1)$ such that

$$u^1(x^{-1}, y^1) = \frac{\beta u^1(x^{-2}, y^2) + (1-\beta)\gamma u^1(x^{-3}, y^3)}{\beta + (1-\beta)\gamma}, \tag{10}$$

$$u^2(x^{-2}, y^2) = \frac{\gamma u^2(x^{-3}, y^3) + (1-\gamma)\alpha u^2(x^{-1}, y^1)}{\gamma + (1-\gamma)\alpha} \tag{11}$$

and

$$u^3(x^{-3}, y^3) = \frac{\alpha u^3(x^{-1}, y^1) + (1-\alpha)\beta u^3(x^{-2}, y^2)}{\alpha + (1-\alpha)\beta}. \tag{12}$$

Let $\epsilon, \eta > 0$ be fixed. Let $\delta_1, \delta_2, \delta_3 \in (0, \epsilon)$ be sufficiently small and $n_1, n_2, n_3 \in \mathbf{N}$ satisfy the following

$$\begin{aligned} &(1 - \delta_1 w(x^{-1}, y^1))^{n_1} = 1 - \alpha, \\ &(1 - \delta_2 w(x^{-2}, y^2))^{n_2} = 1 - \beta, \quad \text{and} \\ &(1 - \delta_3 w(x^{-3}, y^3))^{n_3} = 1 - \gamma. \end{aligned} \tag{13}$$

Later we shall impose more conditions on $\{n_i\}_{i=1}^3$ and $\{\delta_i\}_{i=1}^3$.

Define a profile σ as follows:

- Phase 1: The players play the mixed action combination $(1 - \delta_1)x + \delta_1(x^{-1}, y^1)$ for n_1 stages.
- Phase 2: The players play the mixed action combination $(1 - \delta_2)x + \delta_2(x^{-2}, y^2)$ for n_2 stages.
- Phase 3: The players play the mixed action combination $(1 - \delta_3)x + \delta_3(x^{-3}, y^3)$ for n_3 stages.
- The players repeat cyclically these three phases until absorption occurs.

If the players follow σ then the probability that the game is absorbed during the first phase

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

is $1 - (1 - \delta_1 w(x^{-1}, y^1))^{n_1} = \alpha$. Similarly, the probability that the game is absorbed during the second and third phases is β and γ respectively.

Hence by (10) the expected payoff for player 1 is

$$\frac{\alpha u^1(x^{-1}, y^1) + (1 - \alpha)\beta u^1(x^{-2}, y^2) + (1 - \alpha)(1 - \beta)\gamma u^1(x^{-3}, y^3)}{1 - (1 - \alpha)(1 - \beta)(1 - \gamma)} = u^1(x^{-1}, y^1).$$

Moreover, for every $j \leq n_1$ his expected payoff given absorption has not occurred in the first j stages is $u^1(x^{-1}, y^1)$.

Similarly, the expected payoff of player 2, given absorption has not occurred during the first n_1 stages, is $u^2(x^{-2}, y^2)$. By condition 2, $u^2(x^{-1}, y^1) > u^2(x^{-2}, y^2)$, and the expected payoff of player 2 is

$$\alpha u^2(x^{-1}, y^1) + (1 - \alpha)u^2(x^{-2}, y^2) \geq u^2(x^{-2}, y^2).$$

Moreover, for every $j \leq n_1$ the expected payoff for player 2, given absorption has not occurred in the first j stages, is at least $u^2(x^{-2}, y^2)$.

In a similar way, the expected payoff of player 3, given absorption has not occurred during the first $n_1 + n_2 + n_3$ stages, is $u^3(x^{-3}, y^3)$. Since the profile is cyclic, it follows that his expected payoff at the beginning of the game is $u^3(x^{-3}, y^3)$. By condition 2, $u^3(x^{-1}, y^1) < u^3(x^{-3}, y^3)$, and therefore for every $j \leq n_1$ his expected payoff, given absorption has not occurred during the first j stages, is at least $u^3(x^{-3}, y^3)$.

By adding the following test to σ,

• if at some stage any player plays an action which is not compatible with σ, then this player is punished with an ϵ-min-max profile forever. If some players deviate at the same stage, then the player with the minimal index is punished; it follows by the first condition that no player can profit more than ϵ by playing an action which is not compatible with σ.

• By the third condition it follows that player 1 cannot profit by changing the probabilities in which he perturbs to actions in $\text{supp}(y^1)$.

• In order to prevent deviations within $\text{supp}(x^i)$, the players employ a similar statistical test as in the proof of Lemma 5.1. That is, during each phase the players check whether the distribution of the realized actions of each player i, when restricted to $\text{supp}(x^i)$, is η-close to x^i.

• Since the profile is cyclic, no player can gain more than $2\epsilon R$ by any deviation, and, if η is sufficiently small, false detection of deviation occurs before absorption with probability smaller than ϵ. □

6. On the discounted game. For every $\beta \in (0, 1)$ let $v_\beta(x)$ be the expected β-discounted payoff for the players if they follow the stationary profile x. The function $v_\beta(x)$ satisfies the recursion formula:

$$v_\beta(x) = (1 - \beta)h(x) + \beta w(x)u(x) + \beta(1 - w(x))v_\beta(x). \tag{14}$$

Solving (14) we get

$$v_\beta(x) = \frac{(1 - \beta)h(x) + \beta w(x)u(x)}{1 - \beta(1 - w(x))}. \tag{15}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Since h, w and uw are multilinear, it follows that v_β is continuous and semi-algebraic. Moreover, it is quasi-concave. Indeed, if $v^i_\beta(x)$, $v^i_\beta(y) \geq c$ for some $c \in \mathbf{R}$ then

$$(1-\beta)h(x) + \beta w(x)u(x) \geq c(1-\beta+\beta w(x)) \quad \text{and}$$

$$(1-\beta)h(y) + \beta w(y)u(y) \geq c(1-\beta+\beta w(y)).$$

Therefore, by (15), for any $\lambda \in [0, 1]$,

$$v_\beta(\lambda x + (1-\lambda)y)$$
$$= \frac{\lambda((1-\beta)h(x) + \beta w(x)u(x)) + (1-\lambda)((1-\beta)h(y) + \beta w(y)u(y))}{\lambda(1-\beta+\beta w(x)) + (1-\lambda)(1-\beta+\beta w(y))} \geq c$$

as desired. Let

$$\alpha_\beta(x) = \frac{1-\beta}{1-\beta(1-w(x))} \tag{16}$$

then

$$v_\beta(x) = \alpha_\beta(x)h(x) + (1-\alpha_\beta(x))u(x). \tag{17}$$

Vrieze and Thuijsman (1989) already derived this formula. $\alpha_\beta(x)$ is defined only for $\beta \in (0, 1)$, but whenever $w(x) > 0$ we can define it continuously for $\beta = 1$.

If $w(x) = 0$ then $\alpha_\beta(x) = 1$ for any β, and therefore $v_\beta(x) = h(x)$, while if $w(x) > 0$ then $\lim_{\beta\to 1} \alpha_\beta(x) = 0$, which means that if x is absorbing then $\lim_{\beta\to 1} v_\beta(x) = u(x)$.

We now derive several properties of the limit of stationary equilibria in the discounted game as the discount factor tends to 1. As we will see later, the *rate* at which the probability to play different actions tends to 0 must be taken into account. Therefore we consider Puiseux functions of the form $\beta \to x(\beta)$, where $x(\beta)$ is a stationary equilibrium in the β-discounted game, rather than just a sequence as is done in the literature. For a quick reminder on Puiseux functions the reader is referred to Appendix 2. In the sequel, Puiseux functions are always denoted with a hat.

DEFINITION 6.1. Let I be a finite set. A vector $\hat{f} = (\hat{f}_i)_{i\in I}$ of Puiseux functions is a *Puiseux probability distribution* if $\hat{f}_i \geq 0$ for every $i \in I$ and $\sum_{i\in I} \hat{f}_i = 1$.

For every Puiseux function $\hat{f}$ we define $\hat{f}(1) = \lim_{\theta\to 1} \hat{f}(\theta)$. Clearly we have:

LEMMA 6.2. *Let I be a finite set and $\hat{f}$ a Puiseux probability distribution over I. Then $\hat{f}(1)$ is a probability distribution over I.*

DEFINITION 6.3. A vector of Puiseux functions $\hat{x} = (\hat{x}^i_{a^i})_{i\in N, a^i\in A^i}$ is a *Puiseux stationary profile* if $\hat{x}^i_{a^i}$ is a Puiseux function for every $i \in N$ and $a^i \in A^i$, and $\hat{x}(\theta) = (\hat{x}^i_{a^i}(\theta))_{i\in N, a^i\in A^i}$ is a stationary profile for every $\theta \in (0, 1)$.

For the rest of the section we fix a Puiseux stationary profile $\hat{x}$ and we study the properties of the vector of Puiseux functions $\hat{g} = (\hat{g}^i)$ that are defined by $\hat{g}(\theta) = v_\theta(\hat{x}(\theta))$; that is, the θ-discounted payoff that the stationary profile $\hat{x}(\theta)$ induces.

By (17), $\hat{g}(1) = \lim_{\theta\to 1} v_\theta(\hat{x}(\theta))$ is a convex combination of a nonabsorbing part and an absorbing part. Lemma 6.4 below connects the weight of each part to the rate of absorption induced by the Puiseux stationary profile $\hat{x}$. Lemma 6.5 below gives us a condition when the

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

weight of the absorbing part is 1, and Lemma 6.6 below provides an explicit expression of the absorbing part of the payoff.

Since the set of Puiseux functions is a field, the function $\theta \to \alpha_\theta(\hat{x}(\theta))$ is a Puiseux function. If $w(\hat{x}(1)) > 0$ then $\lim_{\theta\to 1} \alpha_\theta(\hat{x}(\theta)) = 1$, and therefore in this case $\lim_{\theta\to 1} v_\theta(\hat{x}(\theta)) = u(\hat{x}(1))$.

The following lemma, which follows from Lemma 12.2, connects the degree of the probability of absorption to Eq. (17).

LEMMA 6.4. $\deg(w(\hat{x})) = 1$ *if and only if* $\lim_{\theta\to 1} \alpha_\theta(\hat{x}(\theta)) \in (0, 1)$. $\deg(w(\hat{x})) < 1$ *if and only if* $\lim_{\theta\to 1} \alpha_\theta(\hat{x}(\theta)) = 0$. $\deg(w(\hat{x})) > 1$ *if and only if* $\lim_{\theta\to 1} \alpha_\theta(\hat{x}(\theta)) = 1$.

We can now derive the following.

LEMMA 6.5. *Assume that* $\lim_{\theta\to 1} \alpha_\theta(\hat{x}(\theta)) < 1$. *Let* $i \in N$ *and* $a^i \in A^i$ *such that* $\deg(w(\hat{x}^{-i}, a^i)) < \deg(w(\hat{x}))$. *Then* $\lim_{\theta\to 1} \alpha_\theta(\hat{x}^{-i}(\theta), a^i) = 0$.

PROOF. By the assumption and Lemma 6.4 it follows that $\deg(w(\hat{x}^{-i}, a^i)) < \deg(w(\hat{x})) \leq 1$. By a second use of Lemma 6.4 the result follows. □

For every $L \subseteq N$ and $b^L \in A^L$ we define the Puiseux function $\hat{x}_{b^L}$ by:

$$\hat{x}_{b^L} = \prod_{i\in L} \hat{x}^i_{b^i}.$$

A Puiseux stationary profile $\hat{x}$ is *absorbing* if for every θ sufficiently close to 1, $\hat{x}(\theta)$ is an absorbing stationary profile.

Define for every $L \subset N$,

$$\hat{x}_*^{-L}(\theta) = \sum_{a^{-L}\in \operatorname{supp}(\hat{x}^{-L}(1))} \hat{x}_{a^{-L}}(\theta).$$

$\hat{x}_*^{-L}(\theta)$ is the probability that the realized action of every player $i \in N\backslash L$ is in $\operatorname{supp}(\hat{x}^i(1))$, given they play the mixed action combination $\hat{x}^{-L}(\theta)$. Note that $\hat{x}_*^{-L}(1) = 1$ for every $L \subset N$.

Denote

$$\mathcal{B}^0 \stackrel{\text{def}}{=} \{b^L \in A^L | L \subseteq N,\ w(x^{-L}, b^L) > 0\} \supseteq \mathcal{B}(\hat{x}(1)).$$

For any θ, $\hat{x}(\theta)$ induces a natural probability distribution $\nu_{\hat{x}(\theta)}$ over $\mathcal{B}^0$:

$$\nu_{\hat{x}(\theta)}(b^L) = \frac{\hat{x}_*^{-L}(\theta)\hat{x}_{b^L}(\theta) w(\hat{x}^{-L}(1), b^L)}{w(\hat{x}(\theta))}.$$

Clearly

$$u(\hat{x}(\theta)) = \sum_{b^L\in\mathcal{B}^0} \nu_{\hat{x}(\theta)}(b^L) u(\hat{x}^{-L}(1), b^L).$$

Define the probability distribution $\mu_{\hat{x}}$ as the limit of $(\nu_{\hat{x}(\theta)})$:

$$\mu_{\hat{x}}(b^L) = \lim_{\theta\to 1} \nu_{\hat{x}(\theta)}(b^L).$$

By the definition of $\mathcal{B}(x)$, if $b^L \in \mathcal{B}^0\backslash\mathcal{B}(\hat{x}(1))$ then there exists a strict subset L' of L such

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

that $b^{L'} \in \mathcal{B}(\hat{x}(1))$. In particular, $\lim_{\theta\to 1} \hat{x}_{b^L}(\theta)/\hat{x}_{b^{L'}}(\theta) = 0$, and therefore $\operatorname{supp}(\mu_{\hat{x}}) \subseteq \mathcal{B}(\hat{x}(1))$. Hence one has:

LEMMA 6.6. *If $\hat{x}$ is absorbing but $w(\hat{x}(1)) = 0$ then*

$$\lim_{\theta\to 1} u(\hat{x}(\theta)) = \sum_{b^L\in\mathcal{B}(\hat{x}(1))} \mu_{\hat{x}}(b^L)u(\hat{x}^{-L}(1), b^L).$$

7. Proof of Theorem 4.5. Define the function $\tilde{h} : X \to \mathbf{R}^N$ by

$$\tilde{h}^i(x) = \min\{h^i(x), c^i\}.$$

Note that $\tilde{h}$ is not necessarily multilinear, but it is quasi-concave and semi-algebraic. We now define an auxiliary absorbing game $\tilde{G}$, which is played as the original game, but has a different nonabsorbing payoff function. Whereas in the original game the nonabsorbing payoff at each stage t is $h(a_t)$, where a_t is the action combination that the players play at that stage, the nonabsorbing payoff in $\tilde{G}$ is $\tilde{h}(x_t)$, where x_t is the *mixed action combination* that the profile that the players follow indicates them to play.

Formally, the expected β-discounted payoff for the players in $\tilde{G}$, if the players follow the profile σ, is:

$$(18)\qquad \tilde{v}_\beta(\sigma) = (1-\beta)\mathbf{E}_\sigma\left(\sum_{t=1}^{\infty} \beta^{t-1}(\mathbf{1}_{t_*\geq t}\tilde{h}(x_t) + \mathbf{1}_{t_*<t}u(x_{t_*}))\right)$$

where t_* is the stage in which absorption occurs.

EXAMPLE 4. Consider the following two-player zero-sum absorbing game:

	y	$1-y$
x	$1, -1$	$0, 0^*$
$1-x$	$-1, 1$	$0, 0^*$

The min-max value of both players in this game is 0. Any stationary profile (x, y) where $x \in [\frac{1}{2}, 1]$ and $y = 0$ is an equilibrium profile.

Note that the equilibrium profiles mentioned above are equilibrium profiles in $\tilde{G}$ too, for every discount factor.

The stationary profile $(x = \frac{1}{2}, y = \frac{1}{2})$ is also an equilibrium in $\tilde{G}$, since if player 1 plays $x = 1$ then his expected β-discounted payoff in $\tilde{G}$ is 0. Note, however, that if player 1 plays $x = 0$ then his expected β-discounted payoff in $\tilde{G}$ is strictly less than 0, though $x = 0$ is in the support of his β-discounted stationary equilibrium strategy. □

The equivalent of Eq. (17) for the game $\tilde{G}$ is:

$$(19)\qquad \tilde{v}_\beta(x) = \alpha_\beta(x)\tilde{h}(x) + (1-\alpha_\beta(x))u(x)$$

where α_β is given in (16). Since $\tilde{h}$ is quasi-concave it follows as before that $\tilde{v}_\beta$ is continuous and quasi-concave for every fixed β.

Since the absorbing payoff in the auxiliary game is as in the original game, if $\hat{x}$ is a Puiseux stationary profile such that $\hat{x}(1)$ is absorbing, then for every player i,

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

$$(20) \qquad \lim_{\theta \to 1} \tilde{v}_\theta(\hat{x}(\theta)) = u(\hat{x}(1)) = \frac{\sum_{a^i \in \operatorname{supp}(\hat{x}^i(1))} \hat{x}^i_{a^i}(1) w(\hat{x}^{-i}(1), a^i) u(\hat{x}^{-i}(1), a^i)}{w(\hat{x}(1))}.$$

PROOF OF THEOREM 4.5. The idea is to consider a sequence of β-discounted stationary equilibria in the auxiliary game $\tilde{G}$ that converges to a limit. The values of x, g, μ and d are derived from this sequence, and conditions (a) and (b) are proven using various inequalities that β-discounted equilibria satisfy, and by taking the limit $\beta \to 1$.

STEP 1: PROPERTIES OF $\tilde{G}$. Since $\tilde{v}_\beta$ is continuous it follows that $\tilde{c}^i(\beta)$, the min-max value of player i in $\tilde{G}$, exists. Since $\tilde{v}^i_\beta$ is continuous and quasi-concave, for every β there exists a stationary β-discounted equilibrium in $\tilde{G}$. It is clear that the corresponding β-discounted equilibrium payoff is at least $\tilde{c}(\beta)$.

STEP 2: $c^i = \lim_{\beta \to 1} \tilde{c}^i(\beta)$. The proof of this result is long and independent of the other results, hence it is postponed to the next section.

STEP 3: DEFINITION OF A PUISEUX STATIONARY PROFILE $\hat{x}$. Consider the set

$$\mathscr{E} = \{(\beta, x) \in (0, 1) \times X \,|\, \tilde{v}^i_\beta(x) \geq \tilde{v}^i_\beta(x^{-i}, y^i) \ \ \forall\, i \in N, y^i \in X^i\}.$$

$(\beta, x) \in \mathscr{E}$ if and only if x is a β-discounted stationary equilibrium profile in $\tilde{G}$. First we note that $\mathscr{E}$ is a semi-algebraic set. Indeed, the functions w, $\tilde{h}$ and u are semi-algebraic. By (19) the function $\tilde{v} : (0, 1) \times X \to \mathbf{R}^N$ is semi-algebraic. It follows that the function $f(x)$: $(0, 1) \times X \to \mathbf{R}^N$ that is defined by

$$f^i(\beta, x) = \sup_{y^i \in X^i} \tilde{v}^i_\beta(x^{-i}, y^i)$$

is semi-algebraic. Finally, $\mathscr{E} = \{(\beta, x) | \tilde{v}_\beta(x) = f(\beta, x)\}$ is semi-algebraic.

By Step 1 the projection of $\mathscr{E}$ over the first coordinate is the interval $(0, 1)$. By Theorem 12.4 there exists a Puiseux stationary profile $\hat{x}$ such that $(\beta, \hat{x}(\beta)) \in \mathscr{E}$ for every $\beta \in (0, 1)$.

STEP 4: DEFINITION OF x, g, μ, d. Define $x = \hat{x}(1)$, $g = \lim_{\beta \to 1} \tilde{v}_\beta(\hat{x}(\beta))$, $\mu = \mu_{\hat{x}}$ and $d(a^i) = \deg(\hat{x}^i_{a^i})$ for every player i and every $a^i \in \cup_i A^i$.

Since $\tilde{v}^i_\beta(\hat{x}(\beta))$ is a Puiseux function in β, the limit g is well defined. By Lemma 4.2 and Step 2,

$$(21) \qquad g = \lim_{\beta \to 1} \tilde{v}_\beta(\hat{x}(\beta)) \geq \lim_{\beta \to 1} \tilde{c}(\beta) = c.$$

STEP 5: ASSERTION 4.5(a) HOLDS. Fix a player i and an action $a^i \in A^i$. If $w(x^{-i}, a^i) = 0$ then 4.5(a) follows from (21). If, on the other hand, $w(x^{-i}, a^i) > 0$ then $\lim_{\beta \to 1} \alpha_\beta(\hat{x}^{-i}(\beta), a^i) = 0$. Since $\hat{x}^i(\beta)$ is a best reply against $\hat{x}^{-i}(\beta)$, by (19) and the continuity of u at absorbing mixed action combinations, it follows that

$$(22) \quad g^i = \lim_{\beta \to 1} \tilde{v}^i_\beta(\hat{x}(\beta)) \geq \lim_{\beta \to 1} \tilde{v}^i_\beta(\hat{x}^{-i}(\beta), a^i) = \lim_{\beta \to 1} u^i(\hat{x}^{-i}(\beta), a^i) = u^i(x^{-i}, a^i).$$

Assertion 4.5(a) follows from (21), (22) and the definition of e^i.

STEP 6: THE DISTINCTION BETWEEN THE POSSIBILITIES IN 4.5(b). Denote $q = \lim_{\beta \to 1} \alpha_\beta(\hat{x}(\beta))$. Substituting $\hat{x}(\beta)$ instead of x in (19) and taking the limit as $\beta \to 1$ yields, by the continuity of $\tilde{h}$ and (21),

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

$$c \leq g = \lim_{\beta \to 1} \tilde{v}_\beta(\hat{x}(\beta)) = q\tilde{h}(x) + (1-q) \lim_{\beta \to 1} u(\hat{x}(\beta)). \tag{23}$$

If $q = 1$ then $h(x) \geq \tilde{h}(x) = g$ and $w(x) = 0$. Therefore 4.5(b)(i) holds.

If $w(x) > 0$ then $q = 0$ and the second claim in 4.5(b)(ii) holds. In Step 7 we prove that in this case 4.5(b)(ii) holds.

Otherwise $q < 1$ and $w(x) = 0$. Therefore $\hat{x}$ is absorbing. Since $\tilde{h}(x) \leq c$, it follows by (23) that $\lim_{\beta \to 1} u(\hat{x}(\beta)) \geq g$, and by Lemma 6.6, 4.5(b)(iii)(2) holds. We shall later see that in this case 4.5(b)(iii) holds.

STEP 7: IF $w(x) > 0$ THEN ASSERTION 4.5$(b)(ii)$ HOLDS. Let $i \in N$ and $a^i \in \operatorname{supp}(x^i)$ such that $w(x^{-i}, a^i) > 0$. Then $\lim_{\beta \to 1} \alpha_\beta(\hat{x}^{-i}(\beta), a^i) = 1$ and, by the optimality of $\hat{x}^i(\beta)$,

$$u^i(x^{-i}, a^i) = \lim_{\beta \to 1} \tilde{v}^i_\beta(\hat{x}^{-i}(\beta), a^i) \leq \lim_{\beta \to 1} \tilde{v}^i_\beta(\hat{x}(\beta)) = g^i. \tag{24}$$

By (20) and the definition of g there is an equality in (24), as desired.

From now on we assume that $q < 1$ and $w(x) = 0$.

STEP 8: THE VANISHING ACTION CONDITION HOLDS. Let $i \in N$ be fixed. For every $a^i \in A^i$, denote $\mathcal{B}(\mu, a^i) = \{b^L \in \operatorname{supp}(\mu) | i \in L, b^i = a^i\}$.

By Lemmas 6.5 and 6.6, for every $a^i \in \operatorname{supp}(\hat{x}^i(\beta))$ such that $\sum_{b^L \in \mathcal{B}(\mu, a^i)} \mu(b^L) > 0$ we have

$$g^i = \lim \tilde{v}^i_\beta(\hat{x}^{-i}(\beta), a^i) = \lim u^i(\hat{x}^{-i}(\beta), a^i)$$

$$= \sum_{b^L \in \mathcal{B}(\mu, a^i)} \mu(b^L) u^i(x^{-L}, b^L) \Big/ \sum_{b^L \in \mathcal{B}(\mu, a^i)} \mu(b^L).$$

STEP 9: THE DEGREE CONDITION HOLDS. If the average payoff condition holds with equality then $q \in (0, 1)$. Hence, by Lemma 6.4, $\deg(w(\hat{x}(\beta))) = 1$. Therefore, for every $b^L \in \operatorname{supp}(\mu)$,

$$\sum_{i \in L} d(b^i) = \sum_{i \in L} \deg(\hat{x}^i_{b^i}) = \deg(w(\hat{x}(\beta))) = 1. \quad \square$$

8. Proof of Step 2 of Theorem 4.5. In this section we prove the second step in the proof of Theorem 4.5. Throughout the section we fix a player $i \in N$. Our goal is to show that for every mixed action x^{-i} there exists a reply x^i of player i such that $\tilde{v}^i_\beta(x) \geq c^i - \epsilon$ for all $\beta \geq \beta(\epsilon)$, where $\beta(\epsilon)$ is independent of x^{-i}. Since x^{-i} is arbitrary, the result will follow.

For every $\epsilon > 0$, define

$$\beta(\epsilon) = \inf\{\beta' \in [0, 1) | |c^i(\beta) - c^i| \leq \epsilon \qquad \forall\, \beta \in (\beta', 1)\}.$$

By Lemma 4.2, $\beta(\epsilon) < 1$ for every $\epsilon > 0$.

For every $\epsilon > 0$, every action $a^i \in A^i$ and every stationary profile $x^{-i} \in X^{-i}$, let

$$\Gamma_\epsilon(x^{-i}, a^i) = \{\beta \in (\beta(\epsilon), 1) | v^i_\beta(x^{-i}, a^i) \geq c^i - \epsilon\};$$

that is, the set of all discount factors β, such that a^i is an ϵ-good reply of player i against x^{-i} in the β-discounted game.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

By the definition of $\beta(\epsilon)$,

$$\bigcup_{a^i \in A^i} \Gamma_\epsilon(x^{-i}, a^i) = (\beta(\epsilon), 1). \tag{25}$$

By Eq. (17), if $\Gamma_\epsilon(x^{-i}, a^i) \neq \emptyset$ then at least one of the following inequalities holds:

$$h^i(x^{-i}, a^i) \geq c^i - \epsilon \quad \text{or} \tag{26}$$

$$u^i(x^{-i}, a^i) \geq c^i - \epsilon. \tag{27}$$

Clearly if for a given pair (x^{-i}, a^i) both (26) and (27) hold, then $\Gamma_\epsilon(x^{-i}, a^i) = (\beta(\epsilon), 1)$. If $\Gamma_\epsilon(x^{-i}, a^i) \neq \emptyset$, $w(x^{-i}, a^i) > 0$ and $h^i(x^{-i}, a^i) < c^i - \epsilon$ then $u^i(x^{-i}, a^i) \geq c^i - \epsilon$ and vice versa; if $u^i(x^{-i}, a^i) < c^i - \epsilon$ then $h^i(x^{-i}, a^i) > c^i - \epsilon$.

By the continuity of $v_\beta(x)$, as a function of β for every fixed x, $\Gamma_\epsilon(x^{-i}, a^i)$ is relatively closed in $(\beta(\epsilon), 1)$.

Lemma 8.1. *Let $\epsilon > 0$, $x^{-i} \in X^{-i}$, and $a^i \in A^i$. The set $\Gamma_\epsilon(x^{-i}, a^i)$ is either empty, or has the form $(\beta(\epsilon), \beta_1]$, $[\beta_2, 1)$ or $(\beta(\epsilon), 1)$.*

Proof. For every fixed $x \in X$, the function

$$\alpha_\beta(x) = 1 - \frac{\beta w(x)}{1 - \beta(1 - w(x))}$$

is monotonic decreasing in β.

Assume that $\beta' \in \Gamma_\epsilon(x^{-i}, a^i)$. Since $\alpha_\beta(x^{-i}, a^i)$ is monotonic decreasing in β, if equation (26) holds, then every $\beta \in (\beta(\epsilon), \beta')$ is also in $\Gamma_\epsilon(x^{-i}, a^i)$. Symmetrically, if Eq. (27) holds, then every $\beta \in (\beta', 1)$ is also in $\Gamma_\epsilon(x^{-i}, a^i)$. Since $\Gamma_\epsilon(x^{-i}, a^i)$ is relatively closed in $(\beta(\epsilon), 1)$ the result follows. □

Since $\lim_{\beta \to 1} \alpha_\beta(x) = 0$ for every absorbing stationary profile x, and by the continuity of w and u^i we get:

Lemma 8.2. *Let $\epsilon > 0$, $x^{-i} \in X^{-i}$, and $a^i \in A^i$ such that $w(x^{-i}, a^i) > 0$. If there exists $\beta' \in (\beta(\epsilon), 1)$ such that $\Gamma_\epsilon(x^{-i}, a^i) = [\beta', 1)$ then $u^i(x^{-i}, a^i) > c^i - \epsilon$, whereas if $\Gamma_\epsilon(x^{-i}, a^i) = (\beta(\epsilon), \beta']$ for some $\beta' < 1$ then $u^i(x^{-i}, a^i) < c^i - \epsilon$.*

Lemma 8.3. *Let $\epsilon > 0$, and $x^{-i} \in X^{-i}$. Assume there is no action a^i of player i such that $\Gamma_\epsilon(x^{-i}, a^i) = (\beta(\epsilon), 1)$. Then there exists $y^i \in X^i$ such that $h^i(x^{-i}, y^i) = c^i - \epsilon$ and $u^i(x^{-i}, y^i) \geq c^i - \epsilon$.*

Proof. Let $x^{-i} \in X^{-i}$ satisfy the assumption. Since A^i is finite, by (25) and Lemma 8.1 there exist actions a^i and b^i of player i such that $\Gamma_\epsilon(x^{-i}, a^i) = (\beta(\epsilon), \beta_1]$, $\Gamma_\epsilon(x^{-i}, b^i) = [\beta_2, 1)$ *and* $\beta_1 \geq \beta_2$.

Indeed, let a^i be an action that maximizes β_1 among all actions $a' \in A^i$ such that $\Gamma_\epsilon(x^{-i}, a') = (\beta(\epsilon), \beta_1]$, and let b^i be an action that minimizes β_2 among all actions $a' \in A^i$ such that $\Gamma_\epsilon(x^{-i}, a') = [\beta_2, 1)$. It follows from Lemma 8.1 and (25) that $\beta_1 \geq \beta_2$, as desired.

Note that $w(x^{-i}, a^i) > 0$, otherwise $h^i(x^{-i}, a^i) \geq c^i - \epsilon$, which implies that $\Gamma_\epsilon(x^{-i}, a^i) = (\beta(\epsilon), 1)$. Similarly, $w(x^{-i}, b^i) > 0$.

By Lemma 8.2, $u^i(x^{-i}, b^i) > c^i - \epsilon$ and $u^i(x^{-i}, a^i) < c^i - \epsilon$. By (17) and the assumption it follows that $h^i(x^{-i}, b^i) < c^i - \epsilon$ and $h^i(x^{-i}, a^i) > c^i - \epsilon$.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Let $q \in (0, 1)$ solve the equation

$$qh^i(x^{-i}, a^i) + (1 - q)h^i(x^{-i}, b^i) = c^i - \epsilon.$$

Let y^i be the mixed action of player i where he plays the action a^i with probability q and b^i with probability $1 - q$.

Clearly

$$h^i(x^{-i}, y^i) = qh^i(x^{-i}, a^i) + (1 - q)h^i(x^{-i}, b^i) = c^i - \epsilon \tag{28}$$

and

$$w(x^{-i}, y^i) = qw(x^{-i}, a^i) + (1 - q)w(x^{-i}, b^i) > 0. \tag{29}$$

Since both $v^i_\beta(x^{-i}, a^i) \geq c^i - \epsilon$ and $v^i_\beta(x^{-i}, b^i) \geq c^i - \epsilon$ we get by the quasi-concavity of v^i_β that $v^i_\beta(x^{-i}, y^i) \geq c^i - \epsilon$. By (17), (28) and (29) we get that $u^i(x^{-i}, y^i) \geq c^i - \epsilon$, as desired. □

Proof of Step 2 of Theorem 4.5. It is clear that $\tilde{c}^i(\beta) \leq c^i(\beta)$ for every $\beta \in (0, 1)$, hence $\limsup_{\beta\to 1} \tilde{c}^i(\beta) \leq \lim_{\beta\to 1} c^i(\beta) = c^i$.

For the opposite inequality, let $x^{-i} \in X^{-i}$ and let $\epsilon > 0$. First we show that there exists a mixed action $y^i \in X^i$ of player i such that

$$\tilde{v}^i_\beta(x^{-i}, y^i) \geq c^i - \epsilon \qquad \forall\, \beta \in (\beta(\epsilon), 1). \tag{30}$$

Case 1. There is no action $a^i \in A^i$ of player i such that $\Gamma_\epsilon(x^{-i}, a^i) = (\beta(\epsilon), 1)$.

By Lemma 8.3 there exists $y^i \in X^i$ such that $h^i(x^{-i}, y^i) = c^i - \epsilon$ and $u^i(x^{-i}, y^i) \geq c^i - \epsilon$. In particular by (19), $\tilde{v}^i_\beta(x^{-i}, y^i) \geq c^i - \epsilon$ for every $\beta \in (\beta(\epsilon), 1)$.

Case 2. There exists $a^i \in A^i$ such that $\Gamma_\epsilon(x^{-i}, a^i) = (\beta(\epsilon), 1)$.

By the definition of Γ_ϵ, $v^i_\beta(x^{-i}, a^i) \geq c^i - \epsilon$ for every $\beta \in (\beta(\epsilon), 1)$. Therefore, by (17), for every $\beta \in (\beta(\epsilon), 1)$,

$$c^i - \epsilon \leq v^i_\beta(x^{-i}, a^i) = \alpha_\beta(x^{-i}, a^i)h^i(x^{-i}, a^i) + (1 - \alpha_\beta(x^{-i}, a^i))u^i(x^{-i}, a^i).$$

If $w(x^{-i}, a^i) = 0$ then $\alpha_\beta(x^{-i}, a^i) = 1$ for every $\beta \in (\beta(\epsilon), 1)$, and therefore $h^i(x^{-i}, a^i) \geq c^i - \epsilon$. In particular $\tilde{v}^i_\beta(x^{-i}, a^i) = \tilde{h}^i(x^{-i}, a^i) \geq c^i - \epsilon$.

If, on the other hand, $w(x^{-i}, a^i) > 0$ then $\lim_{\beta\to 1} \alpha_\beta(x^{-i}, a^i) = 0$ and therefore $u^i(x^{-i}, a^i) \geq c^i - \epsilon$. It follows by (19) and the definition of $\tilde{h}$ that $\tilde{v}^i_\beta(x^{-i}, a^i) \geq c^i - \epsilon$, as desired.

Since for every $x^{-i} \in X^{-i}$ there exists $y^i \in X^i$ such that (30) holds, it follows that $\liminf_{\beta\to 1} \tilde{c}^i(\beta) \geq c^i - \epsilon$. Since ϵ is arbitrary, the result follows. □

9. More than three players.

Unfortunately, our approach cannot be generalized for more than three players. In our proof we construct for every $\epsilon > 0$ and every sequence of discounted equilibria in the auxiliary game $\tilde{G}$ that converges to a mixed action x, an ϵ-equilibrium profile, where the players play mainly x and perturb to other actions with a small probability. Such a construction does not need to be possible for games with more than three players, as can be seen by the following four-player game:

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

4

		2			2	
	1	0, 0, 0, 0	4, 1, 0, 0*		0, 0, 4, 1*	0, 1, 0, 1*
		1, 4, 0, 0*	1, 1, 0, 0*		1, 0, 0, 1*	1, 1, 0, 1*
3						
	1	0, 0, 1, 4*	0, 1, 1, 0*		0, 0, 1, 1*	0, 1, 1, 1*
		1, 0, 1, 0*	1, 1, 1, 0*		1, 0, 1, 1*	1, 1, 1, 1*

In this game player 1 chooses a row, player 2 chooses a column, player 3 chooses either the top two matrices or the bottom two matrices, and player 4 chooses either the left two matrices or the right two matrices.

Note that the auxiliary game is essentially the same as the original game, and that there are the following symmetries in the payoff function: for every 4-tuple of actions (a, b, c, d) we have:

$$v^1(a, b, c, d) = v^2(b, a, d, c),$$

$$v^3(a, b, c, d) = v^4(b, a, d, c) \quad \text{and}$$

$$v^1(a, b, c, d) = v^3(c, d, a, b).$$

For every $\lambda \in [0, 1]$, let $T^i_\beta(\lambda)$ be the set of all the best replies of player i in the β-discounted game if the other three players play the stationary strategy $(\lambda, 1 - \lambda)$. By the symmetries of the game it follows that $T^i_\beta(\lambda) = T^j_\beta(\lambda)$ for each pair of players i and j.

Every fixed-point λ of the correspondence T^i_β is a stationary equilibrium for the β-discounted game, where all the players play the same mixed action $(\lambda, 1 - \lambda)$ at every stage.

Note that for every $\beta \in (0, 1)$, $\lambda = 0$ is a fixed point of T^i_β, and $\lambda = 1$ is not a fixed point. We shall see that there is a fixed point $\lambda(\beta) \in (0, 1)$ such that $\lim_{\beta\to1} \lambda(\beta) = 1$.

Fix $\beta \in (0, 1)$, and assume that players 2, 3 and 4 play $(\lambda, 1 - \lambda)$, where $\lambda \in (0, 1)$. If player 1 plays the bottom row then his expected payoff is 1, while if he plays the top row then his expected payoff is $g^1 = 4\lambda^2(1 - \lambda) + \beta\lambda^3 g^1$. Player 1 is indifferent between his two actions if $g^1 = 1$, hence

$$f_\beta(\lambda) \stackrel{\text{def}}{=} (4 - \beta)\lambda^3 - 4\lambda^2 + 1 = 0. \tag{31}$$

We claim that (31) has a solution $\lambda(\beta) \in (0, 1)$ such that $\lim_{\beta\to1} \lambda(\beta) = 1$. Indeed, $f_\beta(1) > 0$ for every $\beta \in (0, 1)$, and $f_\beta(\beta^2) < 0$ for β sufficiently close to 1. The claim follows since f_β is continuous and $\lim_{\beta\to1} \beta^2 = 1$.

However, the conditions of Lemmas 5.1, 5.2, 5.3 and 5.5 are not satisfied for the nonabsorbing cell. It is clear that the condition of Lemma 5.1 is not satisfied for the nonabsorbing cell, and Lemma 5.2 is irrelevant for this cell.

It is easily checked that the only convex combinations z of the four vectors $a_1 = (1, 4, 0, 0)$, $a_2 = (4, 1, 0, 0)$, $a_3 = (0, 0, 1, 4)$ and $a_4 = (0, 0, 4, 1)$ such that $z^i = 1$ if a_i has a positive weight in the combination, are the four combinations that include only a single vector a_i. Therefore the conditions of Lemma 5.3 are not satisfied for the nonabsorbing cell.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

We shall now see that the conditions of Lemma 5.5 w.r.t. the nonabsorbing cell do not hold. Note that essentially there can be a cyclic ϵ-equilibrium profile where a *subset* of the players perturb in each phase (rather than a single player). However, the argument that ruled out the possibility of existence of an equilibrium payoff of the third type implies also that only a single player can perturb in each phase of a cyclic equilibrium profile.

Assume to the contrary that $g = (g^1, g^2, g^3, g^4) \geq (1, 1, 1, 1)$ is a cyclic equilibrium payoff, that player 1 is the first player to perturb, $\gamma > 0$ is the overall probability in which player 1 perturbs in the first phase, and $f = (f^1, f^2, f^3, f^4)$ is the expected payoff for the players given player 1 did not perturb in the first phase.

Clearly $g = \gamma(1, 4, 0, 0) + (1 - \gamma)f$; hence $f^3, f^4 > 1, f^1 = 1$ and $\gamma < 1$. Therefore the only player that can perturb after player 1 has finished his perturbations is player 2. Let $y = (y^1, y^2, y^3, y^4)$ be the expected payoff for the players after player 2 has finished his phase of perturbations. Note that if this phase is endless then $g^3 = g^4 = 0$—a contradiction. Since $f^1 = 1$ it follows that $y^1 < 1$, a contradiction to the individual rationality of the strategy.

It is a little more technical to show that there is no general ϵ-equilibrium profile where the players play mainly the nonabsorbing cell at every stage. For more details see Solan and Vieille (1998b).

10. Proof of Theorem 4.7. In this section we prove Theorem 4.7. For the rest of the section we fix $x \in X$ and $g \in \mathbf{R}^N$ for which there exist $\mu \in \Delta(\mathcal{B}(x))$ and $d : \bigcup A^i \to [0, \infty)$ such that 4.5(b)(iii) holds for (x, g, μ, d).

Assume that 4.7(b) does not hold. Our objective is to construct, using μ, a new probability distribution ν that satisfies 4.7(a). The construction depends on the number of players that have single absorbing neighbors in the support of μ. Denote by r the number of those players.

It is clear that if $r = 0$ then one can take $\nu = \mu$. It turns out that if $r = 2$ or $r = 3$ then it is sufficient to change the relative weights of the single absorbing neighbors to construct ν. The difficult case is when $r = 1$. In that case one needs to change the relative weights of *all* absorbing neighbors in supp(μ) to construct ν. The weights of the different absorbing neighbors is determined by the function d. This is the only place where the function d plays any role.

The proof is arranged as follows. After some preliminaries we provide two lemmas that show how modifying the relative weights of single absorbing neighbors can turn an inequality in the average payoff condition into an equality. We then use these two lemmas to solve the cases $r = 2$ and $r = 3$. Finally we solve the case $r = 1$.

By adding a constant to the payoff functions we assume w.l.o.g. that $g = (0, 0, 0)$. For every vector $0 < \lambda \in \mathbf{R}^3_+$, the *normalization* of λ, denoted by $\bar{\lambda}$, is defined by:

$$\bar{\lambda}_i = \frac{\lambda_i}{\sum_{j=1}^3 \lambda_j}.$$

Let $z \in \mathbf{R}^3$. The *signed form* of z is the vector of the signs of the values of (z^1, z^2, z^3). If $z^i = 0$ then the ith coordinate of the sign vector is 0.

Let $\mathcal{MD} = \{(\mu, d) | \mu \in \mathcal{B}(x),\ d : \bigcup_i A^i \to [0, \infty)$ and 4.5(b)(iii) holds for $(x, g, \mu, d)\}$. We denote by $\mathcal{M}$ the projection of $\mathcal{MD}$ over the first coordinate. By the assumption, $\mathcal{MD} \neq \emptyset$. From now on we fix a pair $(\mu, d) \in \mathcal{MD}$ such that μ has a minimal support among the elements in $\mathcal{M}$.

Let $R = \{i \in N | \mathcal{B}_i(x) \cap \text{supp}(\mu) \neq \emptyset\}$. R is the set of players that have single absorbing neighbors in supp(μ). We denote $r = |R|$.

If $r = 0$ or the average payoff condition holds with equality, then 4.7(a) holds with $\nu = \mu$. From now on we assume that $\sum_{i \in L} d(b^i) = 1$ for every $b^L \in \text{supp}(\mu)$. In particular, $d(a^i) = 1$ for every player $i \in N$ and every single absorbing neighbor $a^i \in \mathcal{B}_i(x) \cap \text{supp}(\mu)$.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

For every $i \in R$ we define $y^i \in X^i$ as follows:

$$y^i_{a^i} = \begin{cases} 0 & d(a^i) \neq 1, \\ \dfrac{\mu(a^i)}{\sum_{b^i \in \mathcal{B}_i(x)} \mu(b^i)} & d(a^i) = 1. \end{cases}$$

y^i is the mixed action induced by μ over $\mathcal{B}_i(x)$, the set of single absorbing neighbors of player i. Note that $w(x^{-i}, a^i) > 0$ for every $a^i \in \text{supp}(y^i)$, and by the vanishing action condition it follows that $u^i(x^{-i}, a^i) = g^i$.

For every function $\rho : \mathcal{B}(x) \to [0, \infty)$ and player $j \in N$ we define

$$\langle \rho, \mu^j \rangle = \sum_{b^L \in \mathcal{B}(x)} \rho(b^L) u^j(x^{-L}, b^L).$$

If $1 \in R$ and the signed form of $u(x^{-1}, y^1)$ is $(0, +, +)$ then y^1, viewed as a probability distribution over $\mathcal{B}(x)$, satisfies 4.7(a). By the minimality of μ, it follows that $u(x^{-1}, y^1)$ does not have the signed form $(0, -, -)$. Both observations remain true if some of the "+" signs (resp. "−" signs) in the first case (resp. second case) are replaced by "0." Hence we assume that if $1 \in R$ then $u(x^{-1}, y^1)$ has the signed form $(0, +, -)$ or $(0, -, +)$. Symmetric assumptions are made for players 2 and 3.

The basic idea of the proof is to gradually increase or decrease the weights of the single absorbing neighbors of the players, until there is an equality in the average payoff condition for one more player.

(i) Suppose that the signed form of $u(x^{-2}, y^2)$ is $(-, 0, +)$. By gradually increasing the weights of the single absorbing neighbors of player 2, we keep the average payoff of players 2 and 3 positive and we decrease the average payoff of player 1. By increasing the relative weights properly, the average payoff of player 1 becomes 0.

On the other hand, if we gradually decrease the weights of the single absorbing neighbors of player 2, we increase the average payoff of players 1 and 2, but decrease the average payoff of player 3. By decreasing the relative weights properly, the average payoff of player 3 becomes 0 (the other possibility is that the weight of the absorbing neighbors of player 2 becomes zero, while the average payoff of player 3 is positive. But this is impossible since μ has a minimal support).

(ii) Suppose now that in addition the signed form of $u(x^{-1}, y^1)$ is $(0, +, -)$ and $\langle \mu, u^3 \rangle = 0$. We can increase the relative weights of the absorbing neighbors of players 1 and 2 such that the average payoff of player 3 *will not be affected* (since $u^3(x^{-2}, y^2) > 0 > u^3(x^{-1}, y^1)$). However, the average payoff of player 2 will remain positive, and that of player 1 will decrease.

(iii) If in addition the signed form of $u(x^{-3}, y^3)$ is, for example, $(-, +, 0)$ then by properly increasing the relative weights of the single absorbing neighbors of players 1 and 2, while decreasing the relative weights of the absorbing neighbors of player 3, the average payoffs of players 2 and 3 remain unchanged. One then has only to check what happens with the average payoff of player 1.

We will now formalize these ideas.

Lemma 10.1. *If $2 \in R$ and $u(x^{-2}, y^2)$ has the signed form $(-, 0, +)$ then there exists $\nu \in \mathcal{M}$ such that* $\text{supp}(\mu) = \text{supp}(\nu)$ *and $\langle \nu, u^3 \rangle = 0$. Furthermore, if $\langle \mu, u^2 \rangle = 0$ then $\langle \nu, u^2 \rangle = 0$.*

Proof. For every $t \in [0, 1]$ define:

$$\nu_t(b^L) = \begin{cases} t\mu(b^L) & b^L \in \mathcal{B}_2(x), \\ \mu(b^L) & \text{otherwise}; \end{cases}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

i.e., as t decreases we decrease the weight of the single absorbing neighbors of player 2. Clearly $\nu_1 = \mu$, for every $t \in [0, 1]$ the vanishing action condition and the degree condition hold w.r.t. $\bar{\nu}_t$, and the average payoff condition holds w.r.t. $\bar{\nu}_t$ for $i = 1, 2$. For every t such that $\langle \bar{\nu}_t, u^3 \rangle \geq 0$ the average payoff condition holds w.r.t. $\bar{\nu}_t$ for $i = 3$ as well. Since μ has a minimal support, there is $t_0 \in (0, 1]$ such that $\langle \bar{\nu}_{t_0}, u^3 \rangle = 0$. If $\langle \mu, u^2 \rangle = 0$ then $\langle \bar{\nu}_t, u^2 \rangle = 0$ for every t; hence $\bar{\nu}_{t_0}$ is the desired vector. □

LEMMA 10.2. *If $2 \in R$ and $u(x^{-2}, y^2)$ has the signed form $(+, 0, -)$ then there exists a vector $\nu \in \mathcal{M}$ such that* $\operatorname{supp}(\mu) = \operatorname{supp}(\nu)$ *and* $\langle \nu, u^3 \rangle = 0$. *Furthermore, if* $\langle \mu, u^2 \rangle = 0$ *then* $\langle \nu, u^2 \rangle = 0$.

PROOF. For every $t \in [1, \infty)$ define:

$$\nu_t(b^L) = \begin{cases} t\mu(b^L) & b^L \in \mathcal{B}_2(x), \\ \mu(b^L) & \text{otherwise;} \end{cases}$$

i.e., as t increases we increase the weight of the single absorbing neighbors of player 2. Clearly $\nu_1 = \mu$, for every $t \in [1, \infty)$ the vanishing action condition and the degree condition hold w.r.t. $\bar{\nu}_t$, and the average payoff condition holds w.r.t. $\bar{\nu}_t$ for $i = 1, 2$. For every t such that $\langle \bar{\nu}_t, u^3 \rangle \geq 0$ condition 4.5(b)(iii)(2) holds w.r.t. $\bar{\nu}_t$ for $i = 3$ too. Since $u(x^{-2}, y^2)$ has the signed form $(+, 0, -)$, for t sufficiently large $\langle \bar{\nu}_t, u^3 \rangle < 0$. Therefore, there is $t_0 \in [1, \infty)$ such that $\langle \bar{\nu}_{t_0}, u^3 \rangle = 0$. If $\langle \mu, u^2 \rangle = 0$ then $\langle \bar{\nu}_t, u^2 \rangle = 0$ for every t; hence $\bar{\nu}_{t_0}$ is the desired vector. □

LEMMA 10.3. *If $r = 2$ then there exists $\nu \in \mathcal{M}$ such that* 4.7(a) *holds.*

PROOF. Assume w.l.o.g. that $R = \{2, 3\}$. By either Lemma 10.1 or Lemma 10.2, according to the signed form of $u(x^{-3}, y^3)$, we can assume that $\langle \mu, u^2 \rangle = 0$. By a second use of either Lemma 10.1 or Lemma 10.2, according to the signed form of $u(x^{-2}, y^2)$, we can assume w.l.o.g. that $\langle \mu, u^i \rangle = 0$ for $i = 2, 3$, as desired. □

LEMMA 10.4. *If $r = 3$ and $(u(x^{-1}, y^1), u(x^{-2}, y^2), u(x^{-3}, y^3))$ are not positive cyclic vectors, then there exists $\nu \in \mathcal{M}$ such that* 4.7(a) *holds.*

PROOF. Up to symmetries we can assume that $u(x^{-1}, y^1)$ has the signed form $(0, +, -)$ and $u(x^{-2}, y^2)$ has the signed form $(-, 0, +)$.

Using Lemma 10.1 twice, once with $u(x^{-1}, y^1)$ and then with $u(x^{-2}, y^2)$, we can assume that $\langle \mu, u^i \rangle = 0$ for $i = 2, 3$. If $\langle \mu, u^1 \rangle = 0$ we are done. Hence assume $\langle \mu, u^1 \rangle > 0$.

Case 1. $u(x^{-3}, y^3)$ has the signed form $(+, -, 0)$.

Let $\alpha, \beta > 0$ solve the equations

$$\begin{aligned} -u^3(x^{-1}, y^1) &= \alpha u^3(x^{-2}, y^2), \\ u^2(x^{-1}, y^1) &= -\beta u^2(x^{-3}, y^3). \end{aligned} \tag{32}$$

Since $(u(x^{-1}, y^1), u(x^{-2}, y^2), u(x^{-3}, y^3))$ are not positive cyclic vectors it follows that $u^2(x^{-1}, y^1)u^3(x^{-2}, y^2)u^1(x^{-3}, y^3) \leq -u^3(x^{-1}, y^1)u^1(x^{-2}, y^2)u^2(x^{-3}, y^3)$. Therefore

$$\beta u^1(x^{-3}, y^3) \leq -\alpha u^1(x^{-2}, y^2). \tag{33}$$

If there is an equality in (33) then μ satisfies 4.7(a). Hence assume that there is strict inequality.

For every $t \geq 0$ define

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

$$\nu_t(b^L) = \begin{cases} \mu(b^L) + t & b^L \in \mathcal{B}_1(x) \cap \operatorname{supp}(\mu), \\ \mu(b^L) + t\alpha & b^L \in \mathcal{B}_2(x) \cap \operatorname{supp}(\mu), \\ \mu(b^L) + t\beta & b^L \in \mathcal{B}_3(x) \cap \operatorname{supp}(\mu), \\ \mu(b^L) & \text{otherwise.} \end{cases}$$

It is clear that $\nu_0 = \mu$, by (32) for every t we have that $\langle \nu_t, u^i \rangle = 0$ for $i = 2, 3$, and since there is strict inequality in (33) it follows that there exists $t_0 > 0$ such that $\langle \bar{\nu}_{t_0}, u^1 \rangle = 0$ as well. Hence $\bar{\nu}_{t_0}$ is the desired probability distribution.

Case 2. $u(x^{-3}, y^3)$ has the signed form $(-, +, 0)$.
Let $\alpha, \beta > 0$ solve the equations

$$\begin{aligned} u^3(x^{-1}, y^1) &= -\alpha u^3(x^{-2}, y^2), \\ u^2(x^{-1}, y^1) &= \beta u^2(x^{-3}, y^3). \end{aligned} \tag{34}$$

For every $t \in \mathbf{R}$ define

$$\nu_t(b^L) = \begin{cases} \mu(b^L) + t & b^L \in \mathcal{B}_1(x) \cap \operatorname{supp}(\mu), \\ \mu(b^L) + t\alpha & b^L \in \mathcal{B}_2(x) \cap \operatorname{supp}(\mu), \\ \mu(b^L) - t\beta & b^L \in \mathcal{B}_3(x) \cap \operatorname{supp}(\mu), \\ \mu(b^L) & \text{otherwise.} \end{cases}$$

Clearly $\nu_0 = \mu$, and for every t we have by (34) and since $u^i(x^{-i}, y^i) = 0$,

$$\langle \nu_t, u^3 \rangle = \langle \mu, u^3 \rangle + t(u^3(x^{-1}, y^1) + \alpha u^3(x^{-2}, y^2)) = 0,$$

$$\langle \nu_t, u^2 \rangle = \langle \mu, u^2 \rangle + t(u^2(x^{-1}, y^1) - \beta u^2(x^{-3}, y^3)) = 0,$$

$$\langle \nu_t, u^1 \rangle = \langle \mu, u^1 \rangle + t(\alpha u^1(x^{-2}, y^2) - \beta u^1(x^{-3}, y^3)).$$

Since μ has minimal support it follows that there exists $t_0 \in \mathbf{R}$ such that $\nu_{t_0}(b^L) > 0$ for every $b^L \in \operatorname{supp}(\mu)$ and $\langle \nu_{t_0}, u^1 \rangle = 0$. Hence $\bar{\nu}_{t_0}$ is the desired probability distribution. □

Recall that $\mathcal{B}(\rho, a^i) = \{b^L \in \mathcal{B}(x) \cap \operatorname{supp}(\rho) | i \in L, b^i = a^i\}$ for every $a^i \in A^i$. Let

$$\langle \rho, u^j | a^i \rangle = \sum_{b^L \in \mathcal{B}(\rho, a^i)} \rho(b^L) u^j(x^{-L}, b^L)$$

(a sum over an empty set of indices is by convention 0). Note that for every $i \in N$, $\langle \rho, u^j \rangle = \sum_{a^i \in A^i} \langle \rho, u^j | a^i \rangle$.

LEMMA 10.5. *If* $r = 1$ *then assertion* 4.7(a) *holds.*

PROOF. Assume that $R = \{1\}$. If $\langle \mu, u^1 \rangle = 0$ then μ itself satisfies 4.7(a). Thus we assume that $\langle \mu, u^1 \rangle > 0$. Clearly if there exists $\nu \in \Delta(\mathcal{B}(x))$ such that $\operatorname{supp}(\nu) \subset \operatorname{supp}(\mu)$, $\operatorname{supp}(\nu) \cap \mathcal{B}_1(x) = \emptyset$ and $\langle \nu, u^j \rangle \geq 0$ for $j = 2, 3$, then the assertion holds. Indeed, if such ν exists then by the minimality of μ it follows that $\langle \nu, u^1 \rangle < 0$, and therefore there exists a convex combination of μ and ν that satisfies 4.7(a).

By the vanishing action condition

$$\langle \mu, u^2 | a^2 \rangle = 0 \qquad \forall\, a^2 \notin \operatorname{supp}(x^2). \tag{35}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

By the above discussion it follows that for every $a^2 \notin \text{supp}(x^2)$ with $\mathcal{B}(\mu, a^2) \neq \emptyset$,

$$\langle \mu, u^3 | a^2 \rangle < 0. \tag{36}$$

Similarly,

$$\langle \mu, u^3 | a^3 \rangle = 0 \qquad \forall\, a^3 \notin \text{supp}(x^3) \tag{37}$$

and if $\mathcal{B}(\mu, a^3) \neq \emptyset$ we have

$$\langle \mu, u^2 | a^3 \rangle < 0. \tag{38}$$

Define for every $b^L \in \mathcal{B}(x)$,

$$\nu(b^L) = \begin{cases} d(b^1)\mu(b^L) & b^L \in \mathcal{B}(x) \text{ and } 1 \in L, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\text{supp}(\mu) \cap \mathcal{B}_1(x) \neq \emptyset$ it follows that $\sum_{b^L \in \mathcal{B}(x)} \nu(b^L) > 0$. By the average payoff condition and the degree condition

$$\begin{aligned} 0 &\leq \sum_{b^L \in \mathcal{B}(x)} \mu(b^L) u^i(x^{-L}, b^L) \\ &= \sum_{b^L \in \mathcal{B}(x)} \mu(b^L) u^i(x^{-L}, b^L) \sum_{j \in L} d(b^j) \\ &= \langle \nu, u^i \rangle + \sum_{a^2 \in A^2} d(a^2) \langle \mu, u^i | a^2 \rangle + \sum_{a^3 \in A^3} d(a^3) \langle \mu, u^i | a^3 \rangle \\ &= \langle \nu, u^i \rangle + \sum_{a^2 \notin \text{supp}(x^2)} d(a^2) \langle \mu, u^i | a^2 \rangle + \sum_{a^3 \notin \text{supp}(x^3)} d(a^3) \langle \mu, u^i | a^3 \rangle \end{aligned} \tag{39}$$

where the last equality holds since $d(a^i) = 0$ whenever $a^i \in \text{supp}(x^i)$. By (35), (36), (37), (38) and (39) it follows that $\langle \nu, u^i \rangle \geq 0$ for $i = 2, 3$, and therefore $\langle \bar{\nu}, u^i \rangle \geq 0$ for $i = 2, 3$ as well.

Moreover,

$$\langle \nu, u^1 \rangle = \sum_{a^1 \in A^1} d(a^1) \langle \mu, u^1 | a^1 \rangle.$$

By the vanishing action condition $\langle \mu, u^1 | a^1 \rangle = 0$ whenever $d(a^1) > 0$. Therefore $\langle \bar{\nu}, u^1 \rangle = 0$, and $\bar{\nu}$ is the desired probability distribution. □

11. Appendix 1. In this Appendix we show how to choose the various constants that are used in the ϵ-equilibrium profiles that we constructed in the proofs of Lemmas 5.3 and 5.5.

11.1. The constants of Lemma 5.3. We shall need the following lemma:

LEMMA 11.1. *Let $p = 1/n$ for some $n \in \mathbf{N}$, $(X_t)_{t \in \mathbf{N}}$ be i.i.d. Bernoulli random variables with $P(X_t = 1) = p$ and $\eta > 0$. There exists $t_1 \in \mathbf{N}$, independent of p, such that*

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

$$(40) \qquad P\left(\left| \frac{\sum_{j=1}^{t} X_j}{tp} - 1 \right| < \eta \ \forall\, t > \frac{t_1}{p} \right) > 1 - \eta.$$

PROOF. Let $(X_t)_{t=1}^{\infty}$ be a sequence of i.i.d. Bernoulli random variables with $P(X_t = 1) = p$. The random variable $Y = \sum_{t=1}^{1/p} X_t$ has mean 1 and variance $1 - p < 1$. Let $(Y_k)_{k=1}^{\infty}$ be a sequence of i.i.d. random variables with the same distribution as Y. By Kolmogorov's inequality, there exists $t_1 > 0$, independent of p, such that

$$P\left(\left| \frac{\sum_{i=1}^{k} Y_i}{k} - 1 \right| < \eta \ \forall\, k > t_1 \right) \geq 1 - \eta,$$

and therefore

$$P\left(\left| \frac{\sum_{i=1}^{k/p} X_i}{k} - 1 \right| < \eta \ \forall\, k > t_1 \right) \geq 1 - \eta.$$

Assume t_1 is big enough such that $\eta > 1/t_1$. Let $t > t_1/p$ and $k \in \mathbf{N}$ satisfy

$$k/p < t < (k+1)/p.$$

If $|\sum_{i=1}^{j/p} X_i - j| < j\eta$ for every $j > t_1$ then

$$\sum_{i=1}^{t} X_i \leq \sum_{i=1}^{(k+1)/p} X_i < (k+1)(1+\eta) < tp(1+\eta)^2$$

and

$$\sum_{i=1}^{t} X_i \geq \sum_{i=1}^{k/p} X_i > k(1-\eta) > tp(1-\eta)^2.$$

Therefore

$$P\left(\left| \frac{\sum_{j=1}^{t} X_j}{tp} - 1 \right| < 3\eta \ \forall\, t > \frac{t_1}{p} \right) \geq 1 - \eta > 1 - 3\eta. \quad \square$$

Let $\epsilon > 0$ be fixed and $\eta \in (0, \epsilon/(|N| \cdot |\mathcal{B}(x)|))$. Assume η is sufficiently small such that for every $y \in X$ that satisfies $\|x - y\| < \eta$ we have

$$(41) \qquad w(y^{-L}, b^L) > w(x^{-L}, b^L)/2 \qquad \forall\, b^L \in \mathcal{B}(x),$$

and

$$(42) \qquad \|u(x^{-L}, b^L) - u(y^{-L}, b^L)\| < \epsilon/|\mathcal{B}(x)| \qquad \forall\, b^L \in \mathcal{B}(x).$$

Let $t_1 \in \mathbf{N}$ be sufficiently large such that for every $i \in N$,

$$(43) \qquad \Pr(\|\bar{X}_t^i - x^i\| < \eta \ \forall\, t > t_1) > 1 - \epsilon/|\mathcal{B}(x)|,$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

where $\bar{X}^i_t = (1/t)\sum_{j=1}^{t} X^i_j$ and $\{X^i_j\}$ are i.i.d. r.v. with distribution x^i.

Let ρ_0 be sufficiently small such that

$$(44) \qquad (1-\rho)^{\rho^{-1/2}} = ((1-\rho)^{1/\rho})^{\rho^{1/2}} > 1-\epsilon$$

for every $\rho \in (0, \rho_0)$. Let $t_2 \in \mathbf{N}$ be sufficiently large to satisfy both Lemma 11.1 and the following condition:

$$(45) \qquad \frac{1}{(t_2T)^2} < \rho_0.$$

Let $\delta = 1/(t_2T)^4$. We assume t_2 is sufficiently large such that

$$(46) \qquad (1-\delta)^{t_1T} > 1 - \frac{\epsilon}{|N| \cdot |\mathcal{B}(x)|}.$$

We shall now verify that the profile σ defined in the proof of Lemma 5.3 is a $3\epsilon R$-equilibrium. In view of the conditions, it is sufficient to prove that the following claims hold:

(1) If the players follow σ, then the probability that some player fails the statistical test is smaller than 2ϵ.

(2) If no deviation is detected, and players $N\backslash\{i\}$ follow σ, then the probability that the game is absorbed through some absorbing neighbor $b^L \in \text{supp}(\mu)$ with $|L| \geq 2$ in the first t_1T stages if $i \notin L$, or t_2T stages if $i \in L$, is at most ϵ.

(3) By a nondetectable deviation, a player can profit at most $3\epsilon R$.

(4) By a detectable deviation no player can profit more than $3\epsilon R$.

Indeed, Claim (1) holds by (46) and Lemma 11.1. Claim (2) holds by (46) for $i \notin L$. To see that Claim (2) holds for $i \in L$ as well, note that regardless of the actions of player i at stages j such that $f(j) = b^L$, the probability of nonabsorption through b^L in the first t_2T stages is, by (44), (45) and the definition of δ, at most

$$(1-\delta^{(|L|-1)/|L|})^{t_2T} > (1-\delta^{1/2})^{t_2T} = (1-\delta^{1/2})^{\delta^{-1/4}} > 1-\epsilon,$$

as desired. Claim (3) holds by Claim (2), (42) and the second condition. Finally, Claim (4) holds by the first condition.

Thus σ is a $3\epsilon R$-equilibrium profile for g.

11.2. The constants of Lemma 5.5. Denote $y = (y^i)_{i=1}^3$. Let η be sufficiently small such that for every x', $y' \in X$ with $\|x - x'\| < \eta$ and $\|y - y'\| < \eta$, we have $w(x'^{-i}, y'^i) > w(x^{-i}, y^i)/2$ and

$$(47) \qquad \|u(x'^{-i}, y'^i) - u(x^{-i}, y^i)\| < \epsilon.$$

Let $k \in \mathbf{N}$ be sufficiently large such that

$$(48) \qquad (1-\alpha/2)^k,\ (1-\beta/2)^k,\ (1-\gamma/2)^k < \epsilon.$$

Let t_1 be sufficiently large such that

$$(49) \qquad \Pr(\|\bar{X}^i_t - x^i\| < \eta \ \ \forall\, t > t_1) > 1 - \epsilon/3k,$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

where $\bar{X}_t^i = (1/t)\sum_{j=1}^{t} X_j^i$ and $\{X_j^i\}$ are i.i.d. r.v. with distribution x^i. Let $\delta > 0$ be sufficiently small such that

$$(1-\delta)^{n} > 1-\epsilon. \tag{50}$$

In order to prove that σ is a $3\epsilon R$-equilibrium profile for g, it is sufficient to prove the following claims:

(1) If the players follow σ then the probability that some player fails the statistical test is smaller than 2ϵ.

(2) By a nondetectable deviation, no player can profit more than $3\epsilon R$.

(3) By a detectable deviation, no player can profit more than $3\epsilon R$.

Indeed, Claim (1) holds by (48) and (49), Claim (2) holds by (50), (47) and the third condition, and Claim (3) holds by the first condition.

12. Appendix 2: Puiseux functions and semi-algebraic sets. Denote by $\mathcal{F}$ the collection of all *Puiseux series*, that is, the collection of all the formal sums $\sum_{k=K}^{\infty} a_k(1-\theta)^{k/M}$ where $K \in \mathbf{Z}$, $M \in \mathbf{N}$, $(a_k)_{k=K}^{\infty}$ are real numbers and there exists $\theta_0 \in (0, 1)$ such that $\sum_{k=K}^{\infty} a_k(1-\theta)^{k/M}$ converges for every $\theta \in (\theta_0, 1)$.

We use θ both as an abstract symbol and as a real number. This dual use should not confuse the reader.

It is well known (see, e.g., Walker 1950 or Bewley and Kohlberg 1976) that $\mathcal{F}$ is an ordered field, when addition and multiplication are defined in a similar way to the same operations on power series, and $\sum_{k=K}^{\infty} a_k(1-\theta)^{k/M} > 0$ if and only if $\sum_{k=K}^{\infty} a_k(1-\theta)^{k/M} > 0$ for every θ sufficiently close to 1.

We define the *degree* of any nonzero Puiseux series by:

$$\deg\left(\sum_{k=K}^{\infty} a_k(1-\theta)^{k/M}\right) \overset{\text{def}}{=} \frac{\min\{k \mid a_k > 0\}}{M}$$

and $\deg(0) = \infty$.

DEFINITION 12.1. A function $\hat{f}: [0, 1) \to \mathbf{R}$ is a *Puiseux function* if there exists a Puiseux series $\sum_{k=K}^{\infty} a_k(1-\theta)^{k/M}$ and $\theta_0 \in (0, 1)$ such that $\hat{f}(\theta) = \sum_{k=K}^{\infty} a_k(1-\theta)^{k/M}$ for every $\theta \in (\theta_0, 1)$.

The *degree* of a Puiseux function is the degree of the corresponding Puiseux series, and the order on $\mathcal{F}$ induces an order on Puiseux functions.

Note that $\hat{f}(\theta) = o((1-\theta)^{\deg(\hat{f})-c})$ for every $c > 0$. If $\deg(\hat{f}) \geq 0$ then $\lim_{\theta\to 1} \hat{f}(\theta)$ is finite. In this case define $\hat{f}(1) \overset{\text{def}}{=} \lim_{\theta\to 1} \hat{f}(\theta)$. Note also that

$$\deg(\hat{f}\hat{g}) = \deg(\hat{f}) + \deg(\hat{g}). \tag{51}$$

Clearly we have:

LEMMA 12.2. *Let $\hat{f}$, $\hat{g}$ be two Puiseux functions such that $\hat{g} \neq 0$. $\lim_{\theta\to 1} \hat{f}(\theta)/\hat{g}(\theta) \in (0, \infty)$ if and only if $\deg(\hat{f}) = \deg(\hat{g})$, and $\lim_{\theta\to 1} \hat{f}(\theta)/\hat{g}(\theta) = 0$ if and only if $\deg(\hat{f}) > \deg(\hat{g})$.*

DEFINITION 12.3. Let $d \geq 1$. A subset of $\mathbf{R}^d$ is *semi-algebraic* if it is a finite union of sets of the form

$$\{x \in \mathbf{R}^d \mid p_1(x) = 0, p_2(x) > 0, \ldots, p_m(x) > 0\}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

for polynomial functions $p_1, p_2, \ldots, p_m$.

By Theorem 8.14 in Forster (1981) and Theorem 2.2.1 in Benedetti and Risler (1990) we have:

THEOREM 12.4. *Let $C \subseteq \mathbf{R}^d$ be a semi-algebraic set, whose projection over its first coordinate includes the interval* $(0, 1)$. *Then there exist a vector of Puiseux functions $\hat{f} = (\hat{f}_i)_{i=1}^{d-1} : (0, 1) \to \mathbf{R}^{d-1}$ such that $(\theta, \hat{f}(\theta)) \in C$ for every $\theta \in (0, 1)$.*

Acknowledgments. This paper is part of the Ph.D. thesis of the author done with Professor Abraham Neyman in the Hebrew University of Jerusalem. The author would like to thank Professor Neyman for many discussions and ideas and for the continuous help he offered. The author is also grateful to Janos Flesch for the discussions we had on the subject, and to the comments of Sylvain Sorin and four anonymous referees, that substantially improved the paper. Finally, the author would like to thank the Center for Rationality and Interactive Decision Theory for its hospitality and support.

References

Benedetti, R., J. J. Risler. 1990. *Real Algebraic and Semi-Algebraic Sets*. Hermann, Paris, France.

Bewley, T., E. Kohlberg. 1976. The asymptotic theory of stochastic games. *Math. Oper. Res.* **1** 197–208.

Blackwell, D., T. S. Ferguson. 1968. The big match. *Ann. Math. Statist.* **39** 159–163.

Fink, A. M. 1964. Equilibrium in a stochastic n-person game. *J. Sci. Hiroshima Univ.* **28** 89–93.

Flesch, J., F. Thuijsman, K. Vrieze. 1997. Cyclic Markov equilibria in stochastic games. *Internat. J. Game Theory* **26** 303–314.

Forster, O. 1981. Lectures on Riemann surfaces. *Graduate Texts in Mathematics*. Springer-Verlag, New York.

Kohlberg, E. 1974. Repeated games with absorbing states. *Ann. Statist.* **2** 724–738.

Mertens, J. F., A. Neyman. 1981. Stochastic games. *Internat. J. Game Theory* **10** 53–66.

Neyman, A. 1996. *Lecture Notes of a Course in "Stochastic Games."* The Hebrew University, Jerusalem, Israel.

———, S. Sorin. 1998. Equilibria in repeated games of incomplete information: The general symmetric case. *Internat. J. Game Theory* **27** 201–210.

Solan, E. To appear. Absorbing team games. *Games and Econom. Behavior.*

———, N. Vieille. 1998a. Correlated equilibrium in stochastic games. D.P. #1226, The Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, IL and Cahiers du CEREMADE #9848, Universite Paris 9—Dauphine, France.

———, ———. 1998b. Quitting games. D.P. #1227, The Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, IL. and Cahiers du CEREMADE #9846, Universite Paris 9—Dauphine, France.

Vrieze, O. J., F. Thuijsman. 1989. On equilibria in repeated games with absorbing states. *Internat. J. Game Theory* **18** 293–310.

Walker, R. J. 1950. *Algebraic Curves*. Princeton University Press, Princeton, NJ.

E. Solan: MEDS Department, Kellogg Graduate School of Management, Northwestern University, 2001 Sheridan Road, Evanston, Illinois 60208; e-mail: e-solan@nwu.edu

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.