TZANKO DONCHEV[1] and ELZA FARKHI

Approximations of One-Sided Lipschitz Differential Inclusions with Discontinuous Right-Hand Sides

1. Introduction.

Denote by I the interval [0,1], $X = \mathbf{R}^n$, and by $\mathcal{K}$ (respectively $\mathcal{KK}$) the set of all nonempty compact (respectively convex compact) subsets of X. Consider the differential inclusion:

$$\dot{x}(t) \in F(t, x(t)) \quad \text{for a.e. } t \in I, \quad x(0) = x_0, \tag{1}$$

where $x(\cdot)$ is an absolutely continuous (AC) function from I to X, while $F : I \times X \to \mathcal{K}$ is a measurable multifunction.

We study the Euler discrete approximation of (1) by polygonal functions $y(\cdot)$, satisfying

$$\begin{cases} y(t) = y(t_i) + (t - t_i) f_i \quad \text{for all} \quad t \in [t_i, t_{i+1}], \quad y(0) = x_0, \\ \quad \text{where} \quad f_i \in F(t_i, y(t_i)), \quad i = 0, 1, \ldots, N-1 \ , \end{cases} \tag{2}$$

where the mesh points on I are $0 = t_0 < t_1 < t_2 <, \ldots, < t_N = 1$.

The aim of this paper is to investigate the accuracy of Euler method i.e. the approximation of the solution set S_1 of (1) by the solution set S_2 of (2), when the right-hand side $F(t, \cdot)$ is one-sided Lipschitz (OSL) and upper semicontinuous (USC).

Our analysis is based on a modification of a fundamental theorem estimating the distance between a given quasitrajectory and the set of solutions of the dynamical system (1), established by Filippov [14] and Pliś [23] under different conditions. It has been proved by Pliś [23] for convex-valued right-hand side, measurable in the time and satisfying a Kamke-type Lipschitz condition in the state variable. Filippov [14] proved the estimate for continuous right-hand side, Lipschitz continuous in the state variable and non-necessarily convex-valued (for a more general formulation see [3]). This theorem implies Lipschitz continuity of the solution set with respect to perturbations in the initial condition and the right-hand side. It has extensive applications in the qualitative theory of the differential inclusions (see e.g. [2, 3]), as well as in the numerical analysis (cp. [13, 21]).

An extension of this theorem for one-sided Lipschitz $F(\cdot)$ is published in [10]. We prove here a further modification of the theorem with an additional perturbation

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of the state argument x of F and obtain Hölder continuity of order one half of the solution set.

The one-sided Lipschitz condition (or its modifications) replace successfully the Lipschitz one in different cases: existence of solutions [11], relaxation theorems [7], approximation of the solution set [10], stability under regular and singular perturbations ([8], [26]). Veliov [26] first applied a modified one-sided Lipschitz condition with a negative constant to singularly perturbed differential inclusions. In this case the classical Lipschitz condition is not helpful and the advantage of the one-sided Lipscitz condition with negative constant is obvious, since it replaces some hardly verifiable conditions ([8], [26]).

Let us note that the OSL condition with a nonpositive (negative) constant provides weak stability (asymptotic weak stability) of the dynamical system (1) (see Section 3) and better approximation estimates.

The Euler approximation of differential inclusions is studied in a large number of papers (see for instance the surveys [13, 21]). Approximation estimates are obtained also in [1], where a uniform continuity of $F(\cdot)$ and a stability condition for the reachable set are presumed. As is proven in [10], the latter kind of stability holds for measurable in the time one-sided Lipschitz systems.

In [12] the estimates are given by means of an averaged modulus of continuity of the set-valued mapping $F(t,x)$, called there τ-modulus and defined for continuous in the state variable and measurable in the time right-hand side. This modulus, defined in the spirit of Sendov's averaged moduli of smoothness ([24]), is exploited also in [10], to get estimates for Euler scheme when F is continuous in x and one-sided Lipschitz. Thus the estimates of [10] extend the global estimate of [1] for one-sided Lipschitz systems continuous only in the state. The τ-modulus, however, is not defined for discontinuous in x right-hand side, and in this sense, it does not provide error estimates for discontinuous in the state variable systems.

Introducing a strengthened one-sided Lipschitz condition (SOSL) ([20]) and its modification ([21]), Lempio and Veliov obtained $\mathcal{O}(h)$ convergence of the solution set of (2) even for discontinuous right-hand side. The reader may find in [21] a direct proof of the convergence order $\mathcal{O}(h^{\frac{1}{2}})$ of the Euler method for autonomous OSL systems. So far there is no definite answer to the question whether this order is sharp for OSL systems.

The main contributions of the present paper are:

1) Extending the range of Filippov theorem to systems with perturbations in the state argument of the right-hand side in the case when F is one-sided Lipschitz and upper-semicontinuous. The lack of Lipschitz continuity implies an estimate of order one half in the state perturbation.

2) Introducing a new averaged modulus of continuity, ρ-modulus, of the nonautonomous set-valued map $F(t,x)$, suited to discontinuous in x mapping better than

the τ-modulus ([12], [10]) and solving the measurability problem for the local continuity modulus. The mentioned theorem and the ρ-modulus enable to get a general error estimate of Euler scheme for differential inclusions (or equations) with discontinuous OSL right-hand sides that implies $\mathcal{O}(h^{\frac{1}{2}})$ convergence for a right-hand side of bounded 2-variation.

The paper is organized as follows: In the next section some basic definitions and notations are given. The basic stability theorems are proved in Section 3. The new modulus of continuity of $F(t,x)$ is introduced and applied to error analysis of the Euler approximation in section 4. Some examples, comments and open problems are given in the last section.

2. Definitions and Notations.

Let us note first that all the concepts nondiscussed in this paper can be found: differential inclusions in [2, 3, 6, 25]; measurable multifunctions [2, 4]; averaged moduli of smoothness in [24, 9, 12]; theory of ordinary differential equations [19]. We will consider everywhere, except in theorem 3.3 and some examples, convex-valued multifunctions.

The notation $\langle\cdot,\cdot\rangle$ is used for the scalar product and $|\cdot|$ for the Euclidean norm in $X=\mathbf{R}^n$. Let $S=\{x\in X : |x|=1\}$ be the unit sphere and U be the open unit ball in X centered in the origin. We denote for the sets $A,B\in\mathcal{K}$ by

$$D_H(A,B)=\max\{\sup_{a\in A} dist(a,B), \sup_{b\in B} dist(b,A)\}$$

the Hausdorff distance between A and B, where $dist(a,B)=\inf_{b\in B}|a-b|$. We will use the same notations also when A and B are compact subsets of a metric space (then the norm of the difference is replaced by the distance function). We denote for $A\in\mathcal{K}$ $|A|=D_H(A,\{0\})$ and the **support function** of A is denoted by $\sigma(x,A)=\max_{a\in A}\langle x,a\rangle$. If $A,B\in\mathcal{KK}$, then

$$D_H(A,B)=\max_{l\in S}|\sigma(l,A)-\sigma(l,B)|.$$

For a nonempty set $A\subset X$ we denote by $\bar{A}$ (resp. $co\ A$) the closed (resp. convex) hull of A.

Let Y be a normed space. A multifunction $F:Y\to\mathcal{K}$ is said to be: **upper semi-continuous (USC)** (respectively **lower semi-continuous (LSC)**; or **continuous**) at $x_0\in Y$ when for every $\varepsilon>0$ there exists $\delta>0$ such that $F(x_0+\delta U)\subset F(x_0)+\varepsilon U$ (respectively $F(x_0)\subset F(x)+\varepsilon U$; or $D_H(F(x_0),F(x))\le\varepsilon$) for all $x\in x_0+\delta U$. We note only that in our case F is LSC iff for every $l_0\in F(x_0)$ and every sequence $x_i\to x_0$ there exist $l_i\in F(x_i)$ such that $l_i\to l_0$. Recall that the multifunction $F(t,\cdot)$ is called almost USC (resp. almost LSC; or almost continuous) when for given $\varepsilon>0$

there exists $I_\varepsilon \subset I$ with $meas I_\varepsilon > 1 - \varepsilon$ such that F is USC (resp. LSC; continuous) on $I_\varepsilon \times X$.

Remember also that the multifunction $F(t, \cdot)$ is called **Lipschitz continuous** on the set $A \subset X$ with a constant $L \geq 0$ iff $D_H(F(t,x), F(t,y)) \leq L|x-y|$ for every $x, y \in A,\ t \in I$.

Definition 2.1 The multifunction $F : I \times X \to \mathcal{K}$ is called **one-sided Lipschitz (OSL)** with a constant L (not necessarily positive) iff for every $x, y \in X$, $f_x \in F(t,x)$ and $t \in I$ there exists $f_y \in F(t,y)$ satisfying

$$\langle x - y, f_x - f_y \rangle \leq L|x-y|^2.$$

This is equivalently expressed by the support function:

$$\sigma(x-y, F(t,x)) - \sigma(x-y, F(t,y)) \leq L|x-y|^2$$

for every $x, y \in X$ and $t \in I$.

Remark 2.1 Obviously OSL property is a generalization of Lipschitz continuity for set-valued maps. Further, F is OSL iff $\overline{co}\ F$ is OSL. Let us note again that the constant L may be negative and, as we will see later, this is an important case from stability and approximation point of view. As it is shown in [10], there exist discontinuous maps which are OSL.

Remark 2.2 We like to stress that our OSL condition is weaker than the one existing in the literature. The following stronger condition, used in [15, 16], [18] and [13], will be called here **uniform one-sided Lipschitz (UOSL)** condition:

F is called uniformly one-sided Lipschitz iff there exists a constant L such that

$$\langle x - y, f_x - f_y \rangle \leq L|x-y|^2$$

for every $t \in I$, $x, y \in X$ and every $f_x \in F(t,x)$, $f_y \in F(t,y)$.

UOSL mappings with a constant $L = 0$ are called **dissipative**.

A more detailed comparison between OSL and UOSL property and examples can be found in [10]. Here we give only one example showing that the extension is essential:

Example 2.1 Let $g : R \times X \times R^m$ be continuous in the last argument, $g(\cdot, \cdot, u)$ is OSL for fixed u with a constant L not depending on u, $U \subset R^m$ is compact. Then $\overline{co}\ g(t, \cdot, U)$ is OSL in the sense of Definition 2.1, but generally does not satisfy the uniform one-sided Lipschitz condition.

3. Stability Results.

In this section we prove two versions of the basic theorem of Filippov-Pliś.

In the following we suppose:

B1. $F : I \times X \to \mathcal{K}$ is OSL with a constant L and bounded on bounded sets.

B2.F is convex-valued, $F(\cdot, x)$ is measurable, while $F(t, \cdot)$ is USC.

Consider the absolutely continuous (AC) solutions of

$$\dot{x}(t) \in F(t_i, x(t_i)) \text{ on } [t_i, t_{i+1}],\ i = 0, 1, \ldots, N-1, \quad x(0) = x_0 \tag{3}$$

Clearly every solution of (2) is also a solution of (3) and since F is convex-valued, if $x(\cdot)$ is a solution of (3), then there exist a solution $y(\cdot)$ of (2) such that $x(t_i) = y(t_i)$. This observation enable us to approximate the solution set S_1 by the solution set S_3 of (3) instead of S_2.

We prove next two technical lemmas ensuring boundedness of the trajectory set of the perturbed inclusion.

Lemma 3.1 *If $F : I \times X \to \mathcal{KK}$ is OSL with constant L, then the mapping $G : I \times X \to \mathcal{KK}$, $G(t, x) = F(I, x + U)$ is also OSL with a constant L.*

Proof. Fix $\varepsilon > 0$. Then for every $x, y \in X$ there exist $\tilde{t} \in I$ and $l \in U$ such that

$$\begin{aligned}
&\sigma(x-y, F(I, x+U)) - \sigma(x-y, F(I, y+U)) \leq \\
&\sigma(x-y, F(\tilde{t}, x+l)) - \sigma(x-y, F(I, y+U)) + \varepsilon \leq \\
&\sigma(x-y, F(\tilde{t}, x+l)) - \sigma(x-y, F(\tilde{t}, y+l)) + \varepsilon \leq L|x-y|^2 + \varepsilon.
\end{aligned}$$

Consequently

$$\sigma(x-y, F(I, x+U)) - \sigma(x-y, F(I, y+U)) \leq L|x-y|^2$$

since ε is arbitrary. QED.

Next we prove that under B1, B2 the solution sets S_1 and S_3 are uniformly bounded.

Lemma 3.2 *Under the assumptions B1, B2 there exist constants M, K and N_o such that $|x(t)| \leq M - 1$ and $|F(t, x(t))| \leq K - 1$ for every $x \in S_1 \bigcup S_3$ and for $N > N_o$.*

Proof. Obviously every solution $x(\cdot)$ of (1) is also a solution of

$$\dot{x}(t) \in \overline{co}\, F(I, x(t) + U), \quad x(0) = x_0 \tag{4}$$

Let $x(\cdot)$ be a solution of the last differential inclusion. Then by the previous lemma

$\langle x(t), \dot{x}(t) \rangle \leq \sigma(x(t), F(I, x(t) + U)) \leq L|x(t)|^2 + \sigma(x(t), F(I, U))$. Hence
$\langle x(t), \dot{x}(t) \rangle \leq L|x(t)|^2 + Q|x(t)|$, where $Q = |F(I, U)|$

Since $x(\cdot)$ is AC, then $|x(\cdot)|$ is also AC, hence a.e. differentiable. Therefore

$$\frac{1}{2}\frac{d}{dt}(|x|^2) = |x(t)|\frac{d}{dt}|x(t)| \leq L|x(t)|^2 + Q|x(t)| \quad \text{for a.e.} \quad t \in I.$$

Denote $m(t) = |x(t)|$. Let $T = \{t \in I : m(t) = 0\}$ and let T' be the set of the points of density of T. It is well known that $meas(T') = meas(T)$. So if $\dot{m}(t)$ exists for $t \in T'$, then $\dot{m}(t) = 0$. Thus $\dot{m}(t) \leq Lm(t) + Q$ for a.e. $t \in I$. Therefore

$$m(t) \leq \max_{t\in I}\left\{\exp\left(Lt\right)\left(|x_0| + Q\int_0^t \exp\left(-L\tau\right)d\tau\right)\right\} = M - 1.$$

By B1 there exists K such that $|F(I, MU)| \leq K - 1$. Choose $N_o = K$. Then $Kh = \dfrac{K}{N} \leq 1$. Now we can show by induction that every $x(\cdot) \in S_3$ is also a solution of (4) and therefore the estimates found hold also for $x(\cdot)$. Indeed, let $x(\cdot)$ be a solution of (3). Assume it solves (4) on $[0, t_i]$, then $|\dot{x}(t)| \leq K$ for $t \in [t_i, t_{i+1}]$. Since $\dfrac{K}{N} \leq 1$, one has that $x(t_i) \in x(t) + U$ for $x \in [t_i, t_{i+1}]$, i.e. $x(\cdot)$ satisfies (4) on $[0, t_{i+1}]$. QED.

Remark 3.1 We have seen that if $x(\cdot)$ is a solution of (4), then $|x(t)| \leq M - 1$ and $|F(t, x(t))| \leq K - 1$. If $\dot{x}(t) \in F(I, x(t) + U) + U, x(0) = x_0$ then from Filippov theorem (cp. e.g. [3]) follows that there are constants M and K such that for every solution $x(\cdot)$ of the last inclusion $|x(t)| \leq M$ and $|F(t, x(t))| \leq K$.

The first modification of Filippov - Pliś theorem follows:

Theorem 3.1 *Let B1 and B2 hold and $g(t), h(t)$ be measurable functions such that $|h(t)| \leq 1,\ 0 \leq g(t) \leq 1$ for a.e. $t \in I$. If $y(\cdot)$ is a solution of*

$$\dot{y}(t) \in F(t, y(t) + h(t)) + g(t)U, \qquad y(0) = y_0$$

then there exists a solution $x(\cdot)$ of (1) such that

$$\begin{aligned} |x(t) - y(t)| \ \leq\ & e^{Lt}\Big\{|x_0 - y_0| + \int_0^t e^{-Ls}g(s)\,ds \\ & + [\int_0^t e^{-2Ls}|h(s)|(4K + 8|L|M + 2L_+)\,ds]^{\frac{1}{2}}\Big\}, \end{aligned}$$

where M and K are defined in Lemma 3.2 and $L_+ = \max\{0, L\}$.

Proof. First we will show that there exists a solution $z(\cdot)$ of the inclusion

$$\dot{z}(t) \in F(t, z(t) + h(t)), \quad z(0) = x_0 \tag{5}$$

such that $|z(t) - y(t)| \leq e^{Lt}\{|x_0 - y_0| + \int_0^t e^{-Ls}g(s)\,ds\}$.

Consider the mapping

$$H(t,u) = \{v \in F(t, u + h(t)) : \langle y(t) - u, \dot{y}(t) - v \rangle \leq |y(t) - u|(L|y(t) - u| + g(t))\}.$$

We prove first that $H(t,u)$ is nonempty for every $t \in I, u \in X$.

Let $\alpha \in F(t, y(t) + h(t))$ be such that $|\dot{y}(t) - \alpha| = dist(\dot{y}(t), F(t, y(t) + h(t))) \leq g(t)$. Then from OSL condition there exists $w \in F(t, u + h(t))$ such that

$$\langle y(t) - u, \alpha - w \rangle \leq L|y(t) - u|^2.$$

Therefore

$$\langle y(t) - u, \dot{y}(t) - w \rangle \leq \langle y(t) - u, \alpha - w \rangle + |\dot{y}(t) - \alpha||y(t) - u| \leq L|y(t) - u|^2 + g(t)|y(t) - u|,$$

i.e. $H(t,u) \neq \emptyset$. Obviously $H(\cdot,\cdot)$ is convex-valued, since F is convex-valued.

Let us show that $H(t,\cdot)$ has a closed graph for fixed $t \in I$. Let the sequences $\{u_i\}_{i=1}^{\infty}, \{v_i\}_{i=1}^{\infty}$, be such that $u_i \to u$, $v_i \to v$, $v_i \in H(t, u_i)$. Then

$\langle y(t) - u_i, \dot{y}(t) - v_i \rangle \to \langle y(t) - u, \dot{y}(t) - v \rangle$ and $|y(t) - u_i| \to |y(t) - u|$. Thus
$\langle y(t) - u, \dot{y}(t) - v \rangle \leq L|y(t) - u|^2 + g(t)|y(t) - u|$.

Hence $H(t,\cdot)$ is closed valued USC.

It is not hard to show that $H(\cdot, u)$ is measurable since F is almost continuous. Furthermore $H(t,u) \subset F(t, u + h(t))$. Therefore there exists a solution $z(\cdot)$ of the following differential inclusion:

$$\dot{z}(t) \in H(t, z(t)), \quad z(0) = x_0 \tag{6}$$

Let us note that the established boundedness of the solution set of the inclusion (4) replaces the growth condition in the corresponding existence theorem (cp. [6], Theorem 5.2). From the definition of $H(\cdot,\cdot)$ follows

$$\langle y(t) - z(t), \dot{y}(t) - \dot{z}(t) \rangle \leq L|y(t) - z(t)|^2 + g(t)|y(t) - z(t)|.$$

Denoting $m(t) = |y(t) - z(t)|$ as in the proof of lemma 3.2 one obtains for a.e. $t \in I$ $\dot{m}(t) \leq Lm(t) + g(t)$, $m(0) = |y_0 - x_0|$. Hence

$$|z(t) - y(t)| \leq e^{Lt}\{|x_0 - y_0| + \int_0^t e^{-Ls} g(s)\, ds\}.$$

Now define the mapping

$$G(t,u) = \{v \in F(t,u) : \langle z(t) + h(t) - u, \dot{z}(t) - v \rangle \leq L|z(t) + h(t) - u|^2\}.$$

It follows from the OSL condition that $G(t,u) \neq \emptyset$. As above we establish that it is closed and convex valued, $G(\cdot,u)$ is measurable, $G(t,\cdot)$ is USC. Let

$$\dot{x}(t) \in G(t,x(t)), \quad x(0) = x_0 \tag{7}$$

Therefore one derives

$$\begin{aligned}
&\langle x(t) - z(t), \dot{x}(t) - \dot{z}(t)\rangle = \\
&\langle x(t) - z(t) - h(t), \dot{x}(t) - \dot{z}(t)\rangle + \langle h(t), \dot{x}(t) - \dot{z}(t)\rangle \\
&\leq L|x(t) - z(t) - h(t)|^2 + 2K|h(t)| \\
&\leq L|x(t) - z(t)|^2 + 2|L||h(t)||x(t) - z(t)| + 2K|h(t)| + L|h(t)|^2.
\end{aligned}$$

Denote $p(t) = |x(t) - z(t)|^2$. The last inequality gives

$$\dot{p}(t) \leq 2Lp(t) + 2(2K + 4M|L|)|h(t)| + 2L|h(t)|^2,$$

since $|x(t) - z(t)| \leq 2M$. Thus, taking into account that $|h(t)| \leq 1$, we obtain

$$|x(t) - z(t)| = \sqrt{p(t)} \leq e^{Lt}\{\int_0^t e^{-2Ls}[(4K + 8M|L| + 2L_+)|h(s)|]\, ds\}^{1/2}.$$

Finally, by the triangle inequality $|x(t) - y(t)| \leq |x(t) - z(t)| + |z(t) - y(t)|$ we get the desired result. QED.

Remark 3.2 For $h(\cdot) \equiv 0$ we get the known generalization of the Filippov theorem [10]. The presence of a nonzero state perturbation $h(\cdot)$ causes to decrease the order of convergence to $1/2$ in the L_2-norm of $h(\cdot)$ for an OSL system. We conjecture that under a suitably defined SOSL condition one may obtain first order convergence in $h(\cdot)$ in the upper theorem.

Using the last theorem, it is not difficult to prove the following corollary:

Corollary 3.1 *Consider the system (1) and the system*

$$\dot{y}(t) \in G(t,y(t)), \quad y(0) = y_0.$$

Denote the solution set of the last differential inclusion by S_G. Under the conditions of the previous theorem for F and G, if $D_H(Graph G(t,\cdot), Graph F(t,\cdot)) \leq h(t) \leq 1$, then

$$\begin{aligned}
D_H(S_F, S_G) \ \leq\ & e^{Lt}\Big\{|x_0 - y_0| + \int_0^t e^{-Ls} h(s)\, ds \\
& + [\int_0^t |h(s)|(4K + 2|L|(4M+1))\, ds]^{\frac{1}{2}}\Big\}.
\end{aligned}$$

Let $I = [0, \infty)$. The bondedness assumption B1 assures extendability of each solution on every finite interval and thus the solution set of the system (1) on $[0, \infty)$ is nonempty. Let us denote by $S_1(x_0)$ the set of all solutions of (1) starting from the point x_0. The solution set is equipped with the uniform metrics for continuous functions.

Remark 3.3 Remind that the system (1) is **weakly stable** [16] on $I = [0, \infty)$ iff for every $\varepsilon > 0$ and for each trajectory $x(\cdot)$ of (1) there is a $\delta > 0$ such that whenever $|y_0 - x_0| < \delta$, there is a trajectory $y(\cdot)$, starting from y_0 such that $|y(t) - x(t)| < \varepsilon$ for $0 \leq t < \infty$, or equivalently

$$D_H(S_1(y_0), S_1(x_0)) < \varepsilon.$$

Weak asymptotic stability is defined similarly with the additional condition $|y(t) - x(t)| \to 0$ as $t \to \infty$.

Corollary 3.2 *If the system (1) satisfies the conditions B1 and B2 with a non-positive L, then it is weakly stable, and if $L < 0$, then it is weakly asymptotically stable.*

Proof. Follows directly from Theorem 3.1 with $g \equiv 0$, $h \equiv 0$, since $e^{Lt} \leq 1$ for $L \leq 0$ and $\lim_{t\to\infty} e^{Lt} = 0$ for $L < 0$. QED.

The following norm in the continuous functions space provides a stronger result even for positive OSL constant L. Define for a given scalar α

$$\|x\|_\alpha = \sup_{t \in I} \frac{e^{-\alpha L t}}{2} |x(t)|.$$

Denote by $D_{H,\alpha}(A, B)$ the respective Hausdorff distance between two sets A and B of continuous functions, defined on $I = [0, \infty)$.

Corollary 3.3 *Let $I = [0, \infty)$ and $\alpha = sign\ L$. Under the conditions of the upper theorem*

$$D_{H,\alpha}(S_1(x_0), S_1(y_0)) \leq \frac{e^{(1-\alpha)Lt}}{2} |x_0 - y_0| \leq \frac{1}{2} |x_0 - y_0|,$$

i.e. the set-valued map $S_1(\cdot)$ is not only Lipschitz continuous, but also contractive with respect to $\|\cdot\|_\alpha$.

Note that the last corollaries hold also in an infinite time interval $I = [0, \infty)$. The last corollary generalizes the result of Constantin [5] for one-sided Lipschitz F. In the case when $F(\cdot, x)$ is measurable, $F(t, \cdot)$ Lipschitz and X is a separable Banach space, Kannai and Tallos [17] have proven that the solution set of (1) depends in a Lipschitz manner on the initial condition with respect to essentially stronger topology.

Denote by $\mathcal{L} \otimes \mathcal{B}$ the product measurable space on $I \times X$, where $\mathcal{L}$ is the Lebesgue measure on I, while $\mathcal{B}$ is the Borel measure on X. Lojasiewicz has proved the following result

Theorem 3.2 *([22]) Let $F(\cdot,\cdot)$ be $\mathcal{L} \otimes \mathcal{B}-$ measurable compact valued multifunction. $F(t,\cdot)$ is USC at the points x where $F(t,x)$ is convex valued. Let $F(t,\cdot)$ be LSC on some neighborhoods of the points z, for which $F(t,z)$ is not convex-valued. If F is integrably bounded on the bounded sets, then the differential inclusion $\dot{z}(t) \in F(t, z(t))$, $z(0) = x_0$ admits a local solution.*

Now we prove the following more general version of Filippov- Pliś theorem.

Theorem 3.3 *Let F satisfy all the assumptions of theorem 3.2 and the condition B1. If $g(\cdot)$, $h(\cdot)$ and $y(\cdot)$ are as in theorem 3.1, then for every $\varepsilon > 0$ there exists a solution $x(\cdot)$ of (1) such that*

$$\begin{aligned} |x(t) - y(t)| \leq \quad & e^{Lt}\Big\{|x_0 - y_0| + \int_0^t e^{-Ls} g(s)\, ds \\ & + \Big[\int_0^t e^{-2Ls}|h(s)|(4K + 8|L|M + 2L_+)\, ds\Big]^{1/2}\Big\} + \varepsilon. \end{aligned}$$

Proof. We use similar method of proof as in theorem 3.1. Fix $\delta > 0$ and consider the multifunction

$$H_\delta(t,u) = cl\{v \in F(t, u + h(t)) : \langle y(t) - u, \dot{y}(t) - v\rangle < L|y(t) - u|^2 + (g(t) + \delta)|y(t) - u|\}$$

As in the proof of theorem 3.1 one can verify that $H_\delta(\cdot,\cdot)$ is nonempty. Moreover, it is convex and compact valued, measurable on the first and USC on the second variable, if $F(t,u)$ is convex-valued. Using Lusin's theorem, it is not difficult to prove that this map is $\mathcal{L} \otimes \mathcal{B}-$measurable. Let now $F(t,u)$ be LSC at the point u. Then there are $v \in H_\delta(t,u)$ and $\mu > 0$, such that $\langle y(t) - u, \dot{y}(t) - v\rangle = L|y(t) - u|^2 + (g(t) + \delta - \mu)|y(t) - u|$. From the lower semi-continuity of $F(t,u)$ it follows that for a given sequence $\{u_i\}_{i=1}^\infty$, if $u_i \to u$, then there exist $v_i \in F(t, u_i + h(t))$, such that $v_i \to v$. Then for sufficiently large i $v_i \in H_\delta(t, u_i)$, i.e. $H_\delta(t,\cdot)$ is LSC in u.

From theorem 3.2 we infer that the differential inclusion

$$\dot{z}(t) \in H_\delta(t, z(t)), \quad z(0) = x_0 \tag{8}$$

has a solution $z_\delta(\cdot)$. As in the previous proof one can show that

$$\langle y(t) - z_\delta(t), \dot{y}(t) - \dot{z}_\delta(t)\rangle \leq L|y(t) - z_\delta(t)|^2 + (g(t) + \delta)|y(t) - z_\delta(t)|,$$

hence

$$|z_\delta(t) - y(t)| \leq e^{Lt}\{|x_0 - y_0| + \int_0^t e^{-Ls}[g(s) + \delta]\, ds.$$

Now consider the mapping $G_\delta(\cdot,\cdot)$,

$$\begin{aligned} G_\delta(t,u) &= cl\{v \in F(t,u) : \langle z_\delta(t)+h(t)-u, \dot{z}_\delta(t)-v\rangle \\ &< [L|z_\delta(t)+h(t)-u|+2\delta]|z_\delta(t)+h(t)-u|. \end{aligned}$$

By the same arguments as above one can show that G_δ satisfies all the assumptions of theorem 3.2. If $x_\delta(\cdot)$ is a solution of

$$\dot{x}_\delta(t) \in G_\delta(t, x_\delta(t)), \quad x_\delta(0) = x_0,$$

then, as in the previous proof we get

$$|x_\delta(t) - z_\delta(t)| \le e^{Lt}\{\int_0^t e^{-2Ls}[(4K+8M(|L|+\delta))|h(s)|+2(L_+ +\delta)|h(s)|]\,ds\}^{1/2}.$$

Finally

$$\begin{aligned} &|x_\delta(t)-y(t)| \le e^{Lt}\Big\{|x_0-y_0| + \int_0^t e^{-Ls}(g(s)+\delta)\,ds \\ &+[\int_0^t e^{-2Ls}|h(s)|(4K+8M(|L|+\delta)+2L_+ +2\delta)\,ds]^{1/2}\Big\}. \end{aligned}$$

Obviously it is possible to choose $\delta > 0$ such that the conclusion is fulfilled. QED.

Remark 3.4 We presented the last proof in a concise way. This theorem will not be used essentially further in the paper. Theorem 3.1, however, is very important itself and has a number of different applications. Some of them will be demonstrated in the last section.

4. The Euler Method.

In this section we assume that B1 and B2 are satisfied, i.e. all the continuous and discrete trajectories are contained in a sufficiently large compact set A (see Remark 3.1), $F(\cdot,x)$ is measurable for every $x \in A$ and F is bounded on $I \times A$.

Let us first define the averaged ρ-modulus of continuity of the right-hand side F.

Let $\Delta = \{t_0, t_1, \cdots, t_N\}$ be a partition of I consisting of $N+1$ points, $0 = t_0 \le t_1 \le \cdots \le t_N = 1$. Denote $I_k = [t_{k-1}, t_k], k = 1, \cdots, N$. If the vector $\mathbf{y} = (y_0, y_1, \cdots, y_{N-1}) \in \mathbf{R}^{N\cdot n}$ is such that $y_i \in A$ for $i = 0, \cdots, N-1$, we write simply $\mathbf{y} \in A$.

For given partition Δ, $h \in (0,1)$ and fixed $x \in X$ we define the local modulus of continuity of F at the point $t \in I_k$

$$\omega(F,\Delta,x,h,t) = \sup\{D_H(F(s,x),F(r,x)) : s,r \in [t-\frac{h}{2}, t+\frac{h}{2}] \cap I_k\}.$$

Then for the given $1 \leq p < \infty$, h, the partition Δ and $\mathbf{y} = (y_0, y_1, \cdots, y_{N-1})$ denote

$$\rho(F, \Delta, \mathbf{y}, h)_p = \Big\{ \sum_{k=1}^{N} \int_{I_k} \omega(F, \Delta, y_{k-1}, h, t)^p \, dt \Big\}^{\frac{1}{p}}.$$

Finally, taking supremum over all partitions Δ,we define the global L_p-averaged modulus of continuity of F with step h :

$$\rho(F, h)_p = \sup_{\Delta} \sup_{\mathbf{y} \in A} \rho(F, \Delta, \mathbf{y}, h)_p.$$

The last modulus is smaller than the τ-modulus (cp. [12], [10]), defined as $\tau(F, h)_p = \left(\int_0^1 (\tau(F, t, h))^p dt \right)^{\frac{1}{p}}$, where the integrand is

$$\tau(F, t, h) = \sup \Big\{ \sup \Big(haus(F(s, x), F(r, x)) : s, r \in [t - \frac{h}{2}, t + \frac{h}{2}] \cap I \Big) : x \in A \Big\} .$$

Indeed, one can see that in ρ-modulus definition one integrates for fixed values of the state variable x and the supremum over x is taken after the integration, while for the τ- modulus the supremum over $x \in A$ is inside the integral. Moreover, for the ρ-modulus there is no problem with the measurability of the integrand even for discontinuous in the state variable right-hand side. This problem stays open for τ-modulus when F is not continuous in the state variable.

Here are some properties of $\rho(F, A, h)_p$ that can be proved by a standard technique [24] (see also [12], [9]):

1) $\rho(F, 2h)_p \leq 2\rho(F, h)_p$.

2) If there exists $\tau(F, h)_p$, then obviously $\rho(F, h)_p \leq \tau(F, h)_p$.

3) If F has bounded p-variation, i.e.

$$W_p(F) = \sup_{\Delta} \sup_{\mathbf{y} \in A} \{ \sum_{i=0}^{N-1} D_H(F(t_{i+1}, y_i), F(t_i, y_i))^p, \quad 0 \leq t_0 \leq t_1 \leq \cdots \leq t_N \leq 1\} < \infty,$$

then $\rho(F, h)_p = \mathcal{O}(h^{\frac{1}{p}})$. Note that the p-variation $W_p(F)$ is not greater than the one proposed in [12] for nonautonomous systems.

In order to estimate the Hausdorff distance $D_H(S_1, S_2)$, we use the triangle inequality: $D_H(S_1, S_2) \leq D_H(S_1, S_3) + D_H(S_2, S_3)$. The Hausdorff distance between S_2 and its superset S_3 is easily estimated. Namely, since every $y(\cdot) \in S_3$ is Lipschitz continuous with a constant K, evidently $D_H(S_2, S_3)) \leq 2Kh$.

Let us estimate the excess of S_1 from S_3.

Lemma 4.1 *If B1 and B2 are satisfied, then there exists a constant C, such that for every solution $x(\cdot)$ of (1) there exists a solution $y(\cdot)$ of (3), such that*

$$|x(t) - y(t)| \leq C(\sqrt{h} + \rho(F, h)_2),$$

where $h = \max_{1 \leq i \leq N} |t_i - t_{i-1}|$, $A = MU$ and M is defined in lemma 3.2.

Proof. Let $y(0) = x_0$. We construct the needed solution $y(\cdot)$ successively on the intervals $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, N-1$. Denote for convenience $y_i = y(t_i)$. Suppose $y(\cdot)$ exists on $[0, t_i]$. Let $w(t) \in F(t, y_i)$ be measurable such that

$$\langle x(t) - y_i, \dot{x}(t) - w(t)\rangle \leq L|x(t) - y_i|^2$$

Such measurable selector exists thanks to B1 and B2. Furthermore, there exists a measurable $v(t) \in F(t_i, y_i)$ such that

$$|v(t) - w(t)| = dist(w(t), F(t_i, y_i)) \leq D_H(F(t, y_i), F(t_i, y_i))$$

Define $y(t) = y_i + \int_{t_i}^t v(s)\, ds$ and denote $p(t) = x(t) - y(t)$. Taking into account lemma 3.2, we get $|\dot{x}(t) - w(t)| \leq 2K$; $|y(t) - y_i| \leq Kh$. Consequently we infer

$$\langle p(t), \dot{p}(t)\rangle = \langle x(t) - y_i, \dot{x}(t) - w(t)\rangle + \langle y_i - y(t), \dot{x}(t) - w(t)\rangle + \langle p(t), w(t) - v(t)\rangle$$
$$\leq L|x(t) - y_i|^2 + 2K^2h + |x(t) - y(t)| D_H(F(t, y_i), F(t_i, y_i))$$

Then

$$L|x(t) - y_i|^2 = Lp^2 + L\langle y(t) - y_i, 2x(t) - y(t) - y_i\rangle$$
$$\leq Lp^2 + 4|L|MKh. \text{ Consequently}$$
$$\frac{d}{dt}p^2(t) \leq (2L+1)p^2(t) + 4K(2M|L| + K)h + D_H^2(F(t, y_i), F(t_i, y_i))$$

Here we have used inequality of the type $2ab \leq a^2 + b^2$. Solving the previous differential inequality we obtain

$$p^2(t) \leq p^2(t_i) + e^{(2L+1)(t-t_i)} \int_{t_i}^t e^{-(2L+1)s}[4K(2M|L| + K)h + D_H^2(F(s, y_i), F(t_i, y_i))]\, ds$$

By induction one can derive

$$\begin{aligned} p^2(t) \quad &\leq \quad e^{(2L+1)t}\{\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} e^{-(2L+1)s} D_H^2(F(s, y_i), F(t_i, y_i))\, ds \\ &\quad + \int_0^t e^{-(2L+1)s} 4K(2M|L| + K)h\, ds\} \end{aligned}$$

Let us denote $\mathbf{y} = (y_0, y_1, \ldots, y_{N-1})$, $c_1 = \max_{0 \leq t \leq 1} \max_{0 \leq s \leq t} e^{(2L+1)(t-s)}$. Then from the last inequality we get $p^2 \leq 4c_1 K(2M|L| + K)h + c_1(\rho(F, \Delta, \mathbf{y}, h)_2)^2$, which implies the desired result for $A = MU$. QED.

The main result of this section follows.

Theorem 4.1 *Under the assumptions B1, B2 there exists a constant C such that*

$$D_H(S_1, S_2) \leq C(h^{\frac{1}{2}} + \rho(F, h)_2).$$

Here $D_H(\cdot, \cdot)$ denotes the Hausdorff distance between sets of continuous vector-functions on I.

Proof. Let $x(\cdot)$ be a solution of (3), then denoting $h(t) = x(t_i) - x(t)$ one has $|h(t)| \leq (t - t_i)K \leq Kh$ and

$$\dot{x}(t) \in F(t_i, x(t_i)) \subset F(t, x(t) + h(t)) + D_H(F(t, x(t_i)), F(t_i, x(t_i))U.$$

Thus $\dot{x}(t) \in F(t, x(t)+h(t))+g(t)U$, where $g(t) = D_H(F(t, x_i), F(t_i, x_i))$. From Theorem 3.1 we get $dist(x, S_1) \leq C'(h^{1/2} + \rho(F, h)_1)$. Obviously $\rho(F, h)_1 \leq \rho(F, h)_2$. Let $y(\cdot)$ be a solution of (1). By lemma 4.1 $dist(y, S_3) \leq C(h^{1/2} + \rho(F, h)_2)$. Combining the last two inequalities with $D_H(S_2, S_3) \leq 2Kh$ one obtains the needed estimate. QED.

From property 3 of $\rho(F, A, h)_2$ and the last theorem we obtain

Corollary 4.1 *If F has bounded 2-variation, then under B1, B2* $D_H(S_1, S_2) = \mathcal{O}(\sqrt{h})$.

5. Examples. Concluding Remarks.

Note first that all the results except theorem 3.3 can be proved also in case of Hilbert space X.

In this section we give some examples illustrating our theory and some applications of Filippov-Pliś theorem. Finally some open questions are posed. Now we will present some possible applications of the stability theorems. The first two examples are implementations of Theorem 3.1, while the third is an application of Theorem 3.3.

Example 5.1 (Singularly perturbed system)

For $x \in R^n$ $y \in R^m$ consider the system:

$$\begin{pmatrix} \dot{x}(t) \\ \varepsilon\dot{y}(t) \end{pmatrix} \in F(t, x, y), \quad x(0) = x_0, \quad y(0) = y_0. \tag{9}$$

For $\varepsilon = 0$ we obtain the so-called "degenerate system"

$$\begin{pmatrix} \dot{x}(t) \\ 0 \end{pmatrix} \in F(t, x, y), \quad x(0) = x_0. \tag{10}$$

The solutions of the degenerate system are the pairs of AC function $x(\cdot)$ and integrable $y(\cdot)$ satisfying the system for a.e. $t \in I$. Suppose F satisfies the following modified OSL condition:

There exist positive A, B, D, μ, such that for each $(x_1, y_1), (x_2, y_2) \in R^{n+m}$, if $(f_x, f_y) \in F(t, x_1, y_1)$, then there exists $(g_x, g_y) \in F(t, x_2, y_2)$ such that

$$\langle x_1 - x_2, f_x - g_x \rangle \leq A|x_1 - x_2|^2 + B|y_1 - y_2|^2,$$
$$\langle y_1 - y_2, f_y - g_y \rangle \leq D|x_1 - x_2|^2 - \mu|y_1 - y_2|^2.$$

It is not difficult to prove the following variant of theorem 3.1:

Proposition 5.1 *Let $x(\cdot)$, $y(\cdot)$ be AC functions with*

$$\begin{pmatrix} \dot{x}(t) \\ \varepsilon\dot{y}(t) \end{pmatrix} \in F(t, x + f(t), y + g(t)) + h(t)U, \quad x(0) = x_1, \quad y(0) = y_1.$$

Then under B1, B2 there exists a solution $(u(\cdot), v(\cdot))$ of (9) such that $|x(t) - u(t)|^2 \leq r(t)$, $|y(t) - v(t)|^2 \leq s(t)$, where

$$\begin{aligned} \dot{r}(t) &= Ar(t) + Bs(t) + f(t) + h^2(t), \quad r(0) = |x_0 - x_1|, \\ \varepsilon\dot{s}(t) &= Dr(t) - \mu s(t) + g(t) + h^2(t), \quad s(0) = |y_0 - y_1|. \end{aligned}$$

Now one can show that the solution set of the degenerated system is bounded, i.e. it belongs to $C \times L_\infty$. Furthermore for every L_∞ function $\tilde{y}(\cdot)$ and every $\delta > 0$ there exists Lipschitz continuous function $y(\cdot)$ such that $\int_0^1 |y(t) - \tilde{y}(t)|\, dt \leq \delta$. Therefore under B1, B2 the solution set $Z(\varepsilon)$ is LSC on $C \times L_1$ topology. See for details [8].

Example 5.2 (Regularly perturbed systems)
Let α be parameter in some metric space. Consider

$$\dot{x}(t) \in F(t, x, \alpha), \quad x(0) = x_\alpha$$

Suppose B1, B2 hold uniformly on α and for every compact $A \subset I \times X$

$$\lim_{\alpha \to \beta} Graph F(t, \cdot, \alpha) = Graph F(t, \cdot, \beta).$$

Suppose $x_\alpha \to x_\beta$ and denote the solution set of the last inclusion by $S(\alpha)$. Then due to corollary 3.1 follows $\lim_{\alpha \to \beta} D_H(S(\alpha), S(\beta)) = 0$.

Example 5.3 (Quasitrajectories, relaxation theorem)
Let F satisfy the conditions of theorem 3.3. The AC function $y(\cdot)$ is called quasitrajectory of (1) when there exists a sequence of AC functions $\{y_i(t)\}_{i=1}^{\infty}$ converging to $y(t)$ uniformly on I such that

$$\dot{y}_i(t) \in F(t, y_i(t) + h_i(t)) + g_i(t)U, \quad y_i(0) = x_0$$

whenever $|g_i(t)| \leq 1$, $|h_i(t)| \leq 1$, $h_i(t) \to 0$, $g_i(t) \to 0$ for a.e. $t \in I$. Denote the quasitrajectory set of (1) by Q. From theorem 3.3 immediately follows $Q = \bar{S}_1$. As in the proof of theorem 2.2 of [7] one can show that the solution set of (1) is dense in the solution set of

$$\dot{x}(t) \in H(t, x(t)), \quad x(0) = x_0$$

where $H(t, x) = \overline{co} \bigcap_{\varepsilon > 0} F(t, x + \varepsilon U)$.

Now we give an example of differential inclusion which satisfies the conditions of theorem 4.1 and the right-hand side is neither Lipschitz nor strengthened one-sided Lipschitz ([20, 21]).

Example 5.4

$$\begin{aligned} \dot{x}(t) &= -x^{1/3} + y^{2/3},\ x(0) = 0 \\ \dot{y}(t) &= -y^{1/3} + x^{2/3},\ y(0) = 0. \end{aligned}$$

We will show that the right hand side is OSL. Notice that

$$\begin{aligned} &\sigma((x,y)-(u,v),F(x,y)) - \sigma((x,y)-(u,v),F(u,v)) \\ &= -(x^{1/3}-u^{1/3})(x-u) - (y^{1/3}-v^{1/3})(y-v) \\ &+(x-u)(y^{2/3}-v^{2/3}) + (y-v)(x^{2/3}-u^{2/3}) \end{aligned}$$

Furthermore

$$\begin{aligned} &(x-u)(y^{2/3}-v^{2/3}) \le \frac{1}{2}[(x-u)^2 + (y^{2/3}-v^{2/3})^2]. \text{ Besides} \\ &(y^{2/3}-v^{2/3})^2 = (y^{1/3}-v^{1/3})^2(y^{1/3}+v^{1/3})^2 \\ &\le \frac{4}{3}(y^{1/3}-v^{1/3})^2(y^{2/3}+y^{1/3}v^{1/3}+v^{2/3}) = \frac{4}{3}(y-v)(y^{1/3}-v^{1/3}). \end{aligned}$$

Hence

$$\begin{aligned} &\sigma((x,y)-(u,v),F(x,y)) - \sigma((x,y)-(u,v),F(u,v)) \\ &\le \frac{1}{2}[(x-u)^2 + (y-v)^2] - \frac{1}{3}[(x-u)(x^{1/3}-u^{1/3}) + (y-v)(y^{1/3}-v^{1/3})]. \end{aligned}$$

Therefore F is OSL with a constant $\frac{1}{2}$. Obviously F is Hölder continuous of degree $\frac{1}{3}$. It is not hard to check that the right-hand side is not strengthened OSL ([20, 21]). The numerical experiments showed, however, a linear rate of convergence of the Euler method.

An important open question is whether the order $\frac{1}{2}$ of the convergence is sharp for OSL differential inclusions. Our numerical experiments have shown global covergence of first order even for OSL equations (inclusions), that are not SOSL.

What is the true order of the Euler scheme for OSL inclusions? Is it $\mathcal{O}(h^{\frac{1}{2}})$ as Theorem 4.1 states, or $\mathcal{O}(h)$?

To find an example having the order $\mathcal{O}(h^{\frac{1}{2}})$ or to prove a better convergence theorem should be the final step of this research.

Remark 5.1 In the literature the following concept of Pseudo-Lipschitz multifunctions with nonnecessarilly bounded values is considered:

Let Y be a normed space, $F : X \to 2^Y$ be a multifunction, $x_o \in X$ be given and $y_o \in F(x_o)$. Then F is said to be Pseudo-Lipschitz with respect to the set $U \times V \subset X \times Y$ iff *for every* $x_1, x_2 \in U$ *and every* $y_1 \in F(x_1) \bigcap V$ *there exists* $y_2 \in F(x_2)$ *with*

$$|y_1 - y_2| \leq L|x_1 - x_2| \tag{11}$$

One can easily extend this definition to Pseudo-OSL multifunctions replacing (11) by $\langle x_1 - x_2, y_1 - y_2 \rangle \leq L|x_1 - x_2|^2$. Additional efforts are necessary to prove similar approximation results and especially to ensure solutions' extendability.

Remark 5.2 It is well-known that the subgradient of a convex function is a monotone mapping. The question of characterizing of the set of scalar functions having one-sided Lipschitz subgradients (or generalized gradients) is another intersting question.

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Tzanko Donchev
Department of Mathematics
University of Mining and Geology
1100 Sofia, Bulgaria

Elza Farkhi
School of Mathematical Sciences
Sackler Faculty of Exact Sciences
Tel Aviv University
69978 Tel Aviv, Israel