On a Ramsey-type problem of Erdős and Pach

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Abstract

In this paper we show that there exists a constant \( C > 0 \) such that for any graph \( G \) on \( Ck \ln k \) vertices either \( G \) or its complement \( \overline{G} \) has an induced subgraph on \( k \) vertices with minimum degree at least \( \frac{1}{2}(k-1) \). This affirmatively answers a question of Erdős and Pach from 1983.

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1 Introduction

Recall that the (diagonal, two-colour) Ramsey number is defined to be the smallest integer \( R(k) \) for which any graph on \( R(k) \) vertices is guaranteed to contain a homogeneous set of order \( k \) — that is, a set of \( k \) vertices corresponding to either a complete or independent subgraph. The search for better bounds on \( R(k) \), particularly asymptotic bounds as \( k \to \infty \), is a challenging topic that has long played a central role in combinatorial mathematics (see [4, 7]).

We are interested in a degree-based generalisation of \( R(k) \) where, rather than seeking a clique or coclique of order \( k \), we seek instead an induced subgraph of order (at least) \( k \) with high minimum degree (clique-like graphs) or low maximum degree (coclique-like graphs). Erdős and Pach [1] introduced this class of problems in 1983 and called them quasi-Ramsey problems. By gradually relaxing the degree requirement, a spectrum of Ramsey-type problems arise, and Erdős and Pach showed that this spectrum exhibits a sharp change in behaviour at a certain point. Naturally, this point corresponds to a degree requirement of half the order of the subgraph sought. Three of the authors recently revisited this topic together with Pach [5], and refined our understanding of the threshold for mainly what is referred to in [5] as the variable quasi-Ramsey numbers (corresponding to the parenthetical ‘at least’ above). In the present paper we focus on the harder version of this problem, the

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determination of what is called the fixed quasi-Ramsey numbers (where ‘exactly’ is implicit instead of ‘at least’ above).

Using a result on graph discrepancy, Erdős and Pach [1] proved that there is a constant \( C > 0 \) such that for any graph \( G \) on at least \( Ck^2 \) vertices either \( G \) or its complement \( \overline{G} \) has an induced subgraph on \( k \) vertices with minimum degree at least \( \frac{1}{2}(k - 1) \). With an unusual random graph construction, they also showed that the previous statement does not hold with \( C'k\ln k/\ln \ln k \) in place of \( Ck^2 \) for some constant \( C' > 0 \). They asked if it holds instead with \( Ck\ln k \). (This was motivated perhaps by the fact that this bound holds for the corresponding variable quasi-Ramsey numbers.) Our main contribution here is to confirm this, by showing the following.

**Theorem 1.** There exists a constant \( C > 0 \) such that for any graph \( G \) on \( Ck\ln k \) vertices, either \( G \) or its complement \( \overline{G} \) has an induced subgraph on \( k \) vertices with minimum degree at least \( \frac{1}{2}(k - 1) \).

Although it is short, our proof of Theorem 1 has a number of different ingredients, including the use of graph discrepancy in Section 2, an application of the celebrated ‘six standard deviations’ result of Spencer [8] in Section 3 and a greedy algorithm in Section 4 that was inspired by similar procedures for max-cut and min-bisection. It is interesting to remark that the two discrepancy results we use are of a different nature; the one in Section 2 is an anti-concentration result while the result of Spencer is a concentration result.

### 2 An auxiliary result via graph discrepancy

Our first step in proving Theorem 1 will be to apply the following result. This is a bound on a variable quasi-Ramsey number which is similar to Theorem 3(a) in [5]. The idea of the proof of this auxiliary result is inspired by the sketch argument for Theorem 2 in [1], in spite of the error contained in that sketch (cf. [5]).

**Theorem 2.** For any constant \( \nu \geq 0 \), there exists a constant \( C = C(\nu) > 1 \) such that for any graph \( G \) on \( Ck\ln k \) vertices, either \( G \) or its complement \( \overline{G} \) has an induced subgraph on \( \ell \geq k \) vertices with minimum degree at least \( \frac{1}{2}(\ell - 1) + \nu\sqrt{\ell - 1} \).

Note that the \( O(k\ln k) \) quantity is tight up to an \( O(\ln \ln k) \) factor by the unusual construction in [1] (cf. also Theorem 4 in [5]). The astute reader may later notice that the second-order term \( \nu\sqrt{\ell - 1} \) in the minimum degree guarantee of Theorem 2 can be straightforwardly improved to an \( \Omega(\sqrt{(\ell - 1)\ln \ln \ell}) \) term. Since this does not seem to help in our results, we have omitted this improvement to minimise technicalities. On the other hand, a standard random graph construction yields the following, which certifies that the second-order term cannot be improved to a \( \omega(\sqrt{(\ell - 1)\ln \ln \ell}) \) term.

**Proposition 3.** For any \( c > 0 \), for large enough \( k \) there is a graph \( G \) with at least \( k\ln^c k \) vertices such that the following holds. If \( H \) is any induced subgraph of \( G \) or \( \overline{G} \) on \( \ell \geq k \) vertices, then \( H \) has minimum degree less than \( \frac{1}{2}(\ell - 1) + \sqrt{3c(\ell - 1)\ln \ln \ell} \).

**Proof.** Substitute \( \nu(\ell) = \sqrt{2c\ln \ln \ell}/\ln \ell \) into the proof of Theorem 3(b) in [5]. (We may not use Theorem 3(b) in [5] directly as stated as it needs \( \nu(\ell) \) to be non-decreasing in \( \ell \).)

We use a result on graph discrepancy to prove Theorem 2. Given a graph \( G = (V, E) \), the discrepancy of a set \( X \subseteq V \) is defined as

\[
D(X) := e(X) - \frac{1}{2}\binom{|X|}{2},
\]
where \( e(X) \) denotes the number of edges in the subgraph \( G[X] \) induced by \( X \). We use the following result of Erdős and Spencer \cite{erdos1960discrepancy}.

**Lemma 4** (Theorem 7.1 of \cite{erdos1960discrepancy}). Provided \( n \) is large enough and \( t \in \mathbb{N} \) satisfies \( \frac{1}{2} \log_2 n < t \leq n \), then any graph \( G = (V, E) \) of order \( n \) satisfies

\[
\max_{S \subseteq V, |S| \leq t} |D(S)| \geq \frac{t^{3/2}}{10^3} \sqrt{\ln(5n/t)}.
\]

**Proof of Theorem 2.** Let \( G = (V, E) \) be any graph on at least \( N = \lceil Ck \ln k \rceil \) vertices for a sufficiently large choice of \( C \). We may assume that \( k > \frac{1}{4} \log_2 N \) because otherwise \( G \) or \( \overline{G} \) contains a clique of order \( k \) by the Erdős-Szekeres bound \cite{erdos1935geometrical} on ordinary Ramsey numbers.

For any \( X \subseteq V \) and \( v > 0 \), we define the following skew form of discrepancy:

\[
D_v(X) := |D(X)| - v|X|^{3/2}.
\]

We now construct a sequence \( (H_0, H_1, \ldots, H_t) \) of graphs as follows. Let \( H_0 = G \) or \( \overline{G} \). At step \( i + 1 \), we form \( H_{i+1} \) from \( H_i = (V_i, E_i) \) by letting \( X_i \subseteq V_i \) attain the maximum skew discrepancy \( D_v \) and setting \( V_{i+1} := V_i \setminus X_i \) and \( H_{i+1} := H|_{V_{i+1}} \). We stop after step \( t + 1 \) if \( |V_{t+1}| < \frac{1}{2}N \). Let \( I^+ \subseteq \{1, \ldots, t\} \) be the set of indices \( i \) for which \( D(X_i) > 0 \). By symmetry, we may assume

\[
\sum_{i \in I^+} |X_i| \geq \frac{1}{4}N. \tag{1}
\]

**Claim 1.** For any \( i \in I^+ \) and \( x \in X_i \), \( \deg_{H_i}(x) \geq \frac{1}{2}(|X_i| - 1) + v(|X_i| - 1)^{1/2} \).

**Proof.** Write \(|X_i| = n_i\). We are trivially done if \( n_i = 1 \), so assume \( n_i \geq 2 \). Suppose \( x \in X_i \) has strictly smaller degree than claimed and set \( X'_i := X_i \setminus \{x\} \). Then, since \( i \in I^+ \),

\[
D_v(X'_i) \geq e(X'_i) - \frac{1}{2} \binom{n_i - 1}{2} - v(n_i - 1)^{3/2} > e(X_i) - \frac{1}{2} \binom{n_i}{2} - v\sqrt{n_i - 1} - v(n_i - 1)^{3/2}.
\]

Note that \( n_i^{3/2} > n_i^{1/2} + (n_i - 1)^{3/2} \), which by the above implies \( D_v(X'_i) > D_v(X_i) \), contradicting the maximality of \( D_v(X_i) \).

\( \hfill \Box \)

Claim 1 implies that we may assume for each \( i \in I^+ \) that \(|X_i| \leq k - 1 \), or else we are done. This gives for any \( i_1, \ldots, i_4 \in I^+ \) that

\[
\left( \sum_{i=1}^{4} |X_{i_s}| \right)^{3/2} \leq 8(k - 1)^{3/2}. \tag{2}
\]

Writing \( I^+ = \{i_1, \ldots, i_m\} \), we next show the following.

**Claim 2.** For any \( \ell \in \{1, \ldots, m - 3\} \), \( D(X_{i_{\ell+3}}) \leq \frac{5}{6}D(X_{i_{\ell}}) \).
Proof. For $X \subseteq V$, let us write $v(X) := v|X|^{3/2}$ so that $D_v(X) = |D(X)| - v(X)$. For $i_1, \ldots, i_r \in I^+$, we may write $X_{i_1, \ldots, i_r} := \bigcup_{s=1}^r X_{i_s}$. For disjoint $X, Y \subseteq V$, we define the relative discrepancy between $X$ and $Y$ to be

$$D(X, Y) := e(X, Y) - \frac{1}{2}|X||Y|,$$

where $e(X, Y)$ denotes the number of edges between $X$ and $Y$.

Now let $i, j \in I^+$ with $i < j$. Then, by the maximality of $D_v(X_i)$, we have $D_v(X_i \cup X_j) \leq D_v(X_i)$, i.e.

$$|D(X_i) + D(X_i, X_j) + D(X_j)| - v(X_{ij}) \leq |D(X_i)| - v(X_i) = D(X_i) - v(X_i),$$

and hence

$$D(X_i) \leq -D(X_i, X_j) + v(X_{ij}).$$

Applying (3) (and the fact that $v(X_{i_1, \ldots, i_{s+2}}) \leq v(\bigcup_{s=0}^{3} X_{i_{s+3}})$ for any $r, s \in \{0, 1, 2, 3\}$), we find that

$$D(X_{i_{s+3}}) + 2D(X_{i_{s+2}}) + 3D(X_{i_{s+3}}) \leq - \sum_{0 \leq r < s \leq 3} D(X_{i_r, i_{s+3}}) + 6v(\bigcup_{s=0}^{3} X_{i_{s+3}}).$$

Using $-D(\bigcup_{s=0}^{3} X_{i_{s+3}}) - v(\bigcup_{s=0}^{3} X_{i_{s+3}}) \leq D_v(\bigcup_{s=0}^{3} X_{i_{s+3}}) \leq D_v(X_{i_s})$, we obtain

$$- \sum_{s=0}^{3} D(X_{i_{s+3}}) - \sum_{0 \leq r < s \leq 3} D(X_{i_r, i_{s+3}}) \leq D(X_{i_s}) + v(\bigcup_{s=0}^{3} X_{i_{s+3}}),$$

which combined with (4) implies that $D(X_{i_{s+3}}) + 2D(X_{i_{s+3}}) \leq 2D(X_{i_s}) + 7v(\bigcup_{s=0}^{3} X_{i_{s+3}})$. From this, we obtain that

$$3D(X_{i_{s+3}}) \leq 2D(X_{i_s}) + 8v(\bigcup_{s=0}^{3} X_{i_{s+3}}),$$

where we have used the fact that $D(X_{i_{s+3}}) \leq D(X_{i_{s+2}}) + v(\bigcup_{s=0}^{3} X_{i_{s+3}})$, which follows since $D_v(X_{i_{s+2}}) \leq D_v(X_{i_{s+2}})$. Using the fact that the graph $H_i$ for any $s \in \{1, \ldots, m\}$ has at least $\frac{1}{2}N \geq \frac{k}{2} \ln k$ vertices, it follows by Lemma 4 (using our assumption on $k$) that there exists a subset $Y_s \subseteq V_i$ of size at most $k$ which satisfies

$$|D(Y_s)| \geq k^{3/2} \frac{\sqrt{\ln(C \ln k)}}{10^3}.$$

However, by our choice of $X_{i_s}$, we have

$$D(X_{i_s}) \geq D_v(X_{i_s}) \geq D_v(Y_s) \geq |D(Y_s)| - v k^{3/2} \geq k^{3/2} \left( \frac{\sqrt{\ln(C \ln k)}}{10^3} - v \right) \geq 2 \left( 8v \left( \bigcup_{s=0}^{3} X_{i_{s+3}} \right) \right),$$

by (2), provided $C$ is sufficiently large. Therefore, from (5) we find that $3D(X_{i_{s+3}}) \leq 2D(X_{i_s}) + \frac{1}{2} D(X_{i_s})$, proving the claim. \hfill \diamond
Claim 2 now implies that \((5/6)^{(m-1)/3}D(X_i) \geq D(X_{i_0}) \geq 1\) (assuming for simplicity \(m \equiv 1 \pmod{3}\)), which then implies

\[
m - 1 \leq \frac{3\ln(D(X_i))}{\ln(6/5)} \leq \frac{6}{\ln(6/5)} \ln(k - 1).
\]

By (I), we deduce that at least one of the \(m\) sets \(X_i\) with \(i \in I\) satisfies

\[
|X_i| \geq \frac{N\ln(6/5)}{25\ln k}.
\]

This last quantity is at least \(k\) by a choice of \(C\) sufficiently large, contradicting our assumption that \(|X_i| \leq k - 1\) for each \(i \in I^+\). This completes the proof.

3 Subgraphs of high minimum degree via set-system discrepancy

In this section we prove, based on a well known discrepancy result of Spencer [8], that from a graph on \(\ell = Ck\) vertices with minimum degree at least \(\ell/2 + C'\sqrt{\ell}\) (with \(C'\) depending on \(C\)) we can select a subgraph on \(k\) vertices that has minimum degree at least \(k/2\).

We start by defining the various standard notions of discrepancy that we need. Suppose \(\mathcal{H} = \{A_1, \ldots, A_n\}\) where \(A_i \subseteq V = [n]\). Let \(\chi : V \to \{-1, 1\}\) be a colouring of \(V\) with the colours \(-1\) and 1. For any \(S \subseteq V\), we write \(\chi(S) := \sum_{i \in S} \chi(i)\) and we define the discrepancy of \(\mathcal{H}\) to be

\[
\text{disc}(\mathcal{H}) := \min_{\chi \in \{-1, 1\}^V} \max_{S \in \mathcal{H}} \chi(S).
\]

The result of Spencer [8] states that for any such \(\mathcal{H}\) we have \(\text{disc}(\mathcal{H}) \leq 6\sqrt{n}\).

For \(X \subseteq V\), we define \(\mathcal{H}|_X := \{A_1 \cap X, \ldots, A_n \cap X\}\). Then the hereditary discrepancy of \(\mathcal{H}\) is defined by

\[
\text{herdisc}(\mathcal{H}) := \max_{X \subseteq V} \text{disc}(\mathcal{H}|_X).
\]

The result of Spencer also immediately implies that \(\text{herdisc}(\mathcal{H}) \leq 6\sqrt{n}\) for any \(\mathcal{H}\).

Let \(A\) be the incidence matrix of \(\mathcal{H}\), i.e. \(A\) is the \(n \times n\) matrix given by

\[
A_{ij} = \begin{cases} 1 & \text{if } j \in A_i \\ 0 & \text{otherwise.} \end{cases}
\]

Then we clearly have

\[
\text{disc}(\mathcal{H}) = \min_{x \in \{-1, 1\}^V} \|Ax\|_\infty = 2 \min_{x \in \{0, 1\}^V} \left\| A \left( x - \frac{1}{2} \mathbb{1} \right) \right\|_\infty,
\]

where \(\mathbb{1}\) is the all 1 vector.

Now we define the linear discrepancy by

\[
\text{lindisc}(\mathcal{H}) := \max_{c \in \{0, 1\}^V} \min_{x \in \{0, 1\}^V} \|A(x - c)\|_\infty. \quad (6)
\]

Note that here we are using \(\{0, 1\}\)-colourings again. Similarly, we define the hereditary linear discrepancy of \(\mathcal{H}\) by

\[
\text{herlindisc}(\mathcal{H}) := \max_{X \subseteq V} \text{lindisc}(\mathcal{H}|_X).
\]
A result of Lovász, Spencer, and Vestergombi [6] states that herlindisc(ℋ) ≤ herdisc(ℋ). (Note that the factor of 2 from [6] is missing to adjust for the slightly different definition we are using.) Combining with Spencer’s result, we have

\[ \text{lindisc}(ℋ) \leq \text{herlindisc}(ℋ) \leq \text{herdisc}(ℋ) \leq 6\sqrt{n}. \]

If we set \( c \) to be the all \( p \) vector (for some \( p \in [0, 1] \)) in (6), we obtain the following result.

**Lemma 5.** Let \( A_1, \ldots, A_n \subseteq V = [n] \) and \( p \in [0, 1] \). Then there exists \( Y \subseteq V \) such that, for all \( i \in [n] \),

\[ ||A_i \cap Y| - p|A_i|| \leq 6\sqrt{n}. \]

We use the previous lemma to prove the following result.

**Lemma 6.** Suppose \( G = (V, E) \) is a graph with \( \ell = Pk \) vertices for some \( P > 1 \) and \( k \) a positive integer, and suppose

\[ \delta(G) \geq \frac{1}{2}k + \eta \sqrt{\ell} \]

for some \( \eta > 0 \). Then \( G \) has an induced subgraph \( H \) on \( k \) vertices with minimum degree

\[ \delta(H) \geq \frac{1}{2}k + \left( \frac{\eta}{\sqrt{P}} - 19\sqrt{P} \right) \sqrt{k}. \]

**Proof.** Write \( V = \{v_1, \ldots, v_\ell\} \), let \( A_0 = V \) and for each \( i \in [\ell] \) let \( A_i \subseteq V \) be the neighbourhood of \( v_i \) in \( G \). We apply Lemma 3 to the sets \( A_0, \ldots, A_{\ell-1} \) with \( p = (k + 6\sqrt{\ell})/\ell \). (Note that if \( p > 1 \) then with a simple calculation it is easy to see we can obtain the desired graph \( H \) simply by deleting any \( \ell - k \) vertices from \( G \).) Thus there exists \( Y \subseteq V \) satisfying

\[ ||A_i \cap Y| - p|A_i|| \leq 6\sqrt{\ell} \]

for all \( i \in \{0, \ldots, \ell - 1\} \). Applying this for \( i = 0 \) and noting \( A_0 \cap Y = Y \) gives

\[ k + 1 = p|A_0| - 6\sqrt{\ell} \leq |Y| \leq p|A_0| + 6\sqrt{\ell} = k + 1 + 12\sqrt{Pk} \]

and applying it for \( i \in [\ell - 1] \) gives

\[ |A_i \cap Y| \geq p|A_i| - 6\sqrt{\ell} \geq \frac{k}{\ell} \left( \frac{1}{2}k + \eta \sqrt{\ell} \right) - 6\sqrt{\ell} = \frac{1}{2}k + \eta \frac{k}{\sqrt{\ell}} - 6\sqrt{\ell} \]

\[ = \frac{1}{2}k + \left( \frac{\eta}{\sqrt{P}} - 19\sqrt{P} \right) \sqrt{k}. \]

Thus \( Y \) has between \( k + 1 \) and \( k + 1 + 12\sqrt{Pk} \sqrt{k} \) vertices. Let \( Z \) be an arbitrary subset of \( Y \setminus \{v_\ell\} \) of size \( k \) and let \( H = G[Z] \). Then since we have removed at most \( 12\sqrt{Pk} + 1 \leq 13\sqrt{Pk} \) vertices from \( Y \) to obtain \( Z \), we have for each \( i \in [\ell - 1] \) that

\[ |A_i \cap Z| \geq \frac{1}{2}k + \left( \frac{\eta}{\sqrt{P}} - 19\sqrt{P} \right) \sqrt{k}. \]

In particular this means

\[ \delta(H) \geq \frac{1}{2}k + \left( \frac{\eta}{\sqrt{P}} - 19\sqrt{P} \right) \sqrt{k}, \]

as desired. \( \square \)
4 Proof of Theorem 1

To prove the theorem, we use as a subroutine the following algorithm, which is inspired by the greedy algorithm for max-cut or min-bisection.

**Lemma 7.** Let G = (V, E) be a graph of order n with δ(G) ≥ ½(n−1)+t for some number t. Let x ∈ [0,1] and let a, b ∈ N such that a + b = n. Then either there exists A ⊆ V of size a such that δ(G[A]) ≥ ½a − 1 + at, or there exists B ⊆ V of size b such that δ(G[B]) ≥ ½b − 1 + (1−x)t.

**Proof.** Take any A ⊆ V of size a and let B := V \ A. If there exists x ∈ A with degₐ(x) < ½a − 1 + at and y ∈ B with degₐ(y) < ½b − 1 + (1−x)t, then move x to B and y to A, i.e., swap x and y. Note that when there is no such pair of vertices x, y we are done. We just need to prove that, if we keep iterating, then this procedure must stop at some point.

Consider the number of edges in G[A] before and after we swap x and y. The number of edges in G[A] increases by at least

\[ \text{deg}_A(y) - \text{deg}_A(x) - 1 \geq \delta(G) - \text{deg}_B(y) - \text{deg}_A(x) - 1 \geq 1/2, \]

(where we subtracted 1 in case x and y are adjacent). This shows that we cannot continue to swap pairs indefinitely. \(\square\)

At last we are ready to prove the main result. In fact, we prove something stronger.

**Theorem 8.** There exist constants D, D′ > 0 such that for k ≥ 2 and any graph G on Dk ln k vertices, G or its complement \(\overline{G}\) has an induced subgraph on k vertices with minimum degree at least \(\frac{1}{2}(k−1) + D′\sqrt{(k−1)/\ln k}\).

**Proof.** Set \(v = 160, C = C(v)\) as defined according to Theorem \(2\) and \(D := 4C\). Also set \(D′ := 1/\sqrt{D}\).

By Theorem \(2\) since \(C \cdot 2k \ln(2k) \leq 4Ck \ln k = Dk \ln k \leq |V(G)|\), we find G or \(\overline{G}\) has an induced subgraph H on \(\ell \geq 2k\) vertices with \(\delta(H) \geq \frac{1}{2}(\ell−1) + \sqrt{\ell−1}\).

Let \(x = \ell \mod k\) (so \(x ∈ \{0, \ldots, k−1\}\)). We can now apply Lemma \(7\) to H with \(a = k + x, b = \ell − k − x, t = \sqrt{\ell−1}\) and \(a = 1/2\). Suppose this gives us a subset \(A \subseteq V(H)\) of size a such that

\[ \delta(H[A]) \geq \frac{1}{2}a − 1 + \frac{1}{2}\sqrt{\ell−1} \geq \frac{1}{2}a + \frac{1}{4}\sqrt{\ell} \geq \frac{1}{2}a + \frac{1}{4}\sqrt{a}. \]

Then \(k \leq a < 2k\) and, so applying Lemma \(6\) (with \(P = a/k \in [1,2]\) and \(\eta = v/4 = 40\)) yields a subset \(A' \subseteq A\) of size k such that

\[ \delta(H[A']) \geq \frac{1}{2}k + \left(\frac{40}{\sqrt{P}} - 19\sqrt{P}\right)\sqrt{k} \geq \frac{1}{2}k + \left(\frac{40}{\sqrt{2}} - 19\sqrt{2}\right)\sqrt{k} \geq \frac{1}{2}k + \sqrt{2k}, \]

which is more than required. In case Lemma \(7\) does not produce such a subset A, it gives instead a subset B of size \(b = \ell − k − x \equiv 0 \pmod{k}\) such that \(\delta(H[B]) \geq \frac{1}{2}(b−1) + \frac{1}{2}\sqrt{\ell−1} + \frac{1}{4}\sqrt{a}\). We iteratively apply Lemma \(7\) to \(H[B]\) in a binary search to find a desired induced subgraph as follows.

Set \(G_0 = H[B]\). Let \(\ell_0 := |V(G_0)| = b\) (so that \(k \leq \ell_0 \leq Dk \ln 2k\) and \(\ell_0 \equiv 0 \pmod{k}\)) and set \(t_0 := \frac{1}{2}\sqrt{\ell−1} - \frac{1}{4}\sqrt{\ell_0−1} - \frac{1}{2}\) (so that \(\delta(G_0) \geq \frac{1}{2}(\ell_0−1) + t_0\). Suppose that \(G_t\) is given, where \(G_{t+1}\) has \(\ell_t\) vertices with \(\ell_t \equiv 0 \pmod{k}\) and \(\delta(G_{t+1}) \geq \frac{1}{2}(\ell_{t+1}−1) + t_{t+1}\) for some number \(t_{t+1}\). Set \(a_t = \lfloor \ell_t/2k \rfloor k\) and \(b_t = \lfloor \ell_t/2k \rfloor k\) so that \(a_t + b_t = \ell_t\) and \(a_t \equiv b_t \equiv 0 \pmod{k}\).
(mod $k$). Apply Lemma 7 with $G = G_{i\nu}$, $a = a_{i\nu}$, $b = b_{i\nu}$, $t = t_{i\nu}$, and $\alpha = \frac{1}{2}$. Then we either obtain a set of vertices $A_i$ of size $a_i$ such that $\delta(G_i[A_i]) \geq \frac{1}{2}a_i - 1 + \frac{1}{2}t_{i\nu}$, in which case we set $G_{i+1} := G_i[A_i] = H[A_i]$, or we obtain a set of vertices $B_i$ of size $b_i$ such that $\delta(G_i[B_i]) \geq \frac{1}{2}b_i - 1 + \frac{1}{2}t_{i\nu}$, in which case we set $G_{i+1} := G_i[B_i] = H[B_i]$. Now set $\ell_i+1 = |V(G_{i+1})|$ and note that $\ell_{i+1} \equiv 0 \pmod{k}$ and $\delta(G_{i+1}) \geq \frac{1}{2}(\ell_{i+1} - 1) + t_{i+1}$, where $t_{i+1} = \frac{1}{2}(t_i - 1)$. Note also that $\ell_i/k \leq |\ell_i/2k|$. In this way we obtain subgraphs $G_0, G_1, \ldots$ of $G_0 = H[B]$ and we see from the recursion for $\ell_i$ above that if $\ell_i > k$ then $\ell_{i+1} < \ell_i$. Thus there exists some $j$ such that $\ell_j = k$ (since $\ell_i \equiv 0 \pmod{k}$ for all $i$) and an easy computation shows we can assume that $j \leq \log_2(\ell_0/k) + 1$. The recursion for $t_i$ implies that $t_i \geq t_02^{-i} - 1$ so that

$$t_j \geq \frac{t_0k}{2t_0} - 1 \geq \frac{v(\sqrt{\ell_0 - 1} - 1)k}{4t_0} \geq \frac{k}{\sqrt{D\ln k}} = D'\sqrt{\frac{k}{\ln k}}$$

(where we used that $t_0 \geq \frac{1}{2}v\sqrt{\ell_0 - 1} - \frac{1}{2}$, that $\ell_0 \geq k \geq 2$ with $v = 160$, and that $\ell_0 \leq Dk \ln k$). Thus $G_j$ has $k$ vertices and minimum degree at least $\frac{1}{2}(k-1) + D'\sqrt{(k-1)/\ln k}$ and is an induced subgraph of $H[B]$ and hence of $G$ or $\overline{G}$.

$\square$

5 Concluding remarks

It is tempting to try using the greedy subroutine (Lemma 7) in a binary search on the output of Theorem 3(a) of [5], but since we cannot control the order of this output graph, the search might require $O(\log k)$ steps, which would destroy the minimum degree bounds.

Determination of the second-order term in the minimum degree threshold for polynomial to super-polynomial growth of the fixed quasi-Ramsey numbers is an open problem. (The corresponding term for the variable quasi-Ramsey numbers was determined in [5].) To pose the problem concretely, we recall notation of Erdős and Pach. For $c \in [0,1]$ and $k \in \mathbb{N}$, let $R^*_c(k)$ be the least number $n$ such that for any graph $G = (V,E)$ on at least $n$ vertices, there exists $S \subseteq V$ with $|S| = k$ such that either $\delta(G[S]) \geq c(k-1)$ or $\delta(\overline{G}[S]) \geq c(k-1)$. Now consider $c = \frac{1}{2} + \epsilon$ where $\epsilon = \epsilon(k)$ is a function of the size $k$ of the subset sought by Theorem 8. If $\epsilon(k) = O(\sqrt{\ln(k-1)\ln k})$ then $R^*_c(k)$ is polynomial in $k$, and by Proposition 3 if $\epsilon(k) = \omega(\sqrt{\ln\ln k(k-1)})$ then $R^*_c(k)$ is superpolynomial in $k$. Hence the choice of $\epsilon$ for which we find a transition between polynomial and super-polynomial growth in $k$ of $R^*_c(k)$ is essentially determined to within a $\sqrt{\ln k \ln k}$ factor of $\sqrt{T(k-1)}$. What is it precisely?

Last, we remark that, in the above notation, our main result is that $R^*_{1/2}(k) \leq Ck \ln k$ for some $C > 0$, while Erdős and Pach showed that $R^*_{3/2}(k) \geq C'k \ln k / \ln \ln k$ for some $C' > 0$. They also asked if $R^*_{3/2}(k) \geq C'k \ln k$ for some $C' > 0$. This question remains open.

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References


