Sequential Auctions and Externalities

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Abstract

In many settings agents participate in multiple different auctions that are not necessarily implemented simultaneously. Future opportunities affect strategic considerations of the players in each auction, introducing externalities. Motivated by this consideration, we study a setting of a market of buyers and sellers, where each seller holds one item, bidders have combinatorial valuations and sellers hold item auctions sequentially.

Our results are qualitatively different from those of simultaneous auctions, proving that simultaneity is a crucial aspect of previous work. We prove that if sellers hold sequential first price auctions then for unit-demand bidders (matching market) every subgame perfect equilibrium achieves at least half of the optimal social welfare, while for submodular bidders or when second price auctions are used, the social welfare can be arbitrarily worse than the optimal. We also show that a first price sequential auction for buying or selling a base of a matroid is always efficient, and implements the VCG outcome.

An important tool in our analysis is studying first and second price auctions with externalities (bidders have valuations for each possible winner outcome), which can be of independent interest. We show that a Pure Nash Equilibrium always exists in a first price auction with externalities.

1 Introduction

The first and second price auctions for a single item, and their corresponding strategically equivalent ascending versions are some of the most commonly used auctions for selling items. Their popularity has been mainly due to the fact that they combine simplicity with efficiency: the auctions have simple rules and in many settings lead to efficient allocation.

The simplicity and efficiency tradeoff is more difficult when auctioning multiple items, especially so when items to be auctioned are possibly owned by different sellers. The most well-known auction is the truthful VCG auction, which is efficient, but is not simple: it requires coordination among sellers, requires the sellers to agree on how to divide the revenue, and in many situations requires solving computationally hard problems. In light of these issues, it is important to design simple auctions with good performance, it is also important to understand properties of simple auction designs used in practice.

Several recent papers have studied properties of simple item-bidding auctions, such as using simultaneous second price auctions for each item [7, 4], or simultaneous first price auction [14, 16]. Each of these papers study the case when all the items are auctioned simultaneously, a property essential for all of their results. The simplest, most natural, and most common way to auction items by different owners is to run individual single item auctions (e.g., sell each item separately on eBay). No common auction environment is running simultaneous auctions (first price or second price) for large sets of items. To evaluate to what extent is the simultaneity important for the good properties of the above simple auctions [7, 4, 14], it is important to understand the sequential versions of item bidding auctions.

There is a large body of work on online auctions (see [27] for a survey), where players have to make strategic decisions without having any information about the future. In many auctions participants have information about future events, and engage in strategic thinking about the upcoming auctions. Here we take the opposite view from online auctions, and study the full information version of this game, when the players have full information about all upcoming auctions.

Driven by this motivation we study sequential simple auctions from an efficiency perspective. Sequential auctions are very different from their simultaneous counterparts. For example, it may not be dominated to bid above the actual value of an item, as the outcome of this auction can have large effect for the player

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in future auctions beyond the value of this item. We focus on two different economic settings. In the first set of results we study the case of a market of buyers and sellers, where each seller holds one item and each bidder has a combinatorial valuation on the items. In the second setting we study the case of procuring a base of a matroid from a set of bidders, each controlling an element of the ground set.

For item auctions, we show that subgame perfect equilibrium always exists, and study the quality of the resulting outcome. While the equilibrium is not unique in most cases, in some important classes of games, the quality of any equilibrium is close to the quality of the optimal solution. We show that in the widely studied case of matching markets [29, 8, 20], i.e. when bidders are unit-demand, the social welfare achieved by a subgame perfect equilibrium of a sequential first price auction is at least half of the optimal. Thus in a unit-demand setting sequential implementation causes at most a factor of 2 loss in social welfare. On the other hand, we also show that the welfare loss due to sequential implementation is unbounded when bidders have arbitrary submodular valuations, or when the second price auction is used, hence in these cases the simultaneity of the auctions is essential for achieving the positive results [7, 4, 14].

For the setting of auctioning a base of a matroid to bidders that control a unique element of the ground set, we show that a natural sequential first price auction has unique outcome, that achieves the same allocation and price outcome as VCG.

An important building block in all of our results is a single item first or second price auction in a setting with externalities, i.e., when players have different valuations for different winner outcomes. Many economic settings might give rise to such externalities among the bidders of an auction:

- We are motivated by externalities that arise in sequential auctions. In this setting, bidders might know of future auctions, and realize that their surplus in the future auction depends on who wins the current auction. Such consideration introduces externalities, as players can have a different expected future utility according to the winner of the current auction.

  To illustrate how externalities arise in sequential auction, consider the following example of a sequential auction with two items and two players and with valuations: $v_1(1) = v_1(2) = 5, v_1(1,2) = 10$ for player 1, and $v_2(1) = v_2(2) = v_2(1,2) = 4$ for player 2. Player 1 has higher value for both item. However, if she wins item 1 then she has to pay 4 for the second item, while by allowing player 2 to win the first item, results in a price of 0 for the second item. We can summarize this by saying that his value for winning the first item is 6 (value 5 for the first item itself, and an additional expected value of 1 for subsequently winning the second item at a price 4), while her value for player 2 winning the first item is 5 (for subsequently winning the second item for free).

- Bidders might want to signal information through their bids so as to threat or inform other bidders and hence affect future options. This is the cause of the inefficiency in our example of sequential second price auction. Such phenomena have been observed at Federal Communication Commission (FCC) auctions where players were using the lower digits of their bids to send signals discouraging other bidders to bid on a particular license (see chapter 1 of Milgrom [25]).

- If bidders are competitors or collaborators in a market then it makes a difference whether your friend or your enemy wins. One very vivid portrayal of such an externality is an auction for nuclear weapons [19].

The properties of such an auction are of independent interest outside of the sequential auction scope.

1.1 Our Results

Existence of equilibrium. In section 2 we show that the first price single item auction always has a pure Nash equilibrium that survives iterated elimination of dominated strategies, even in the presence of arbitrary externalities, strengthening the result in [17, 10].

In section 3 we use such external auction to show that sequential first price auctions have pure subgame perfect Nash equilibria for any combinatorial valuations. This is in contrast to simultaneous first price auctions that may not have pure equilibria even for very simple valuations.

Quality of outcomes in sequential first price item auctions. Next we study the quality of outcome of sequential first price auctions in various settings. Our main result is that with unit demand bidders the price of anarchy is bounded by 2. In contrast, when valuations are submodular, the price of anarchy can be arbitrarily high. This differentiates sequential auctions from simultaneous auctions, where pure Nash equilibria are socially optimal [14].

These results extend to sequential auctions, where multiple items are sold at each stage using independent first price auctions. Further, the efficiency guarantee

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degrades smoothly as we move away from unit demand. Moreover, the results also carry over to mixed strategies with a factor loss of 2. Unfortunately, the existence of pure equilibria is only guaranteed when auctioning one item at-a-time.

Sequential second price auctions. Our positive results depend crucially on using the first price auction format. In the appendix, we show that sequential second price auctions can lead to unbounded inefficiency, even for additive valuations, while for additive valuations sequential first price, or simultaneous first or second price auctions all lead to efficient outcomes.

Sequential Auctions for selling basis of a matroid. In section 4 we consider matroid auctions, where bidders each control a unique element of the ground set, and we show that a natural sequential first price auction achieves the same allocation and price outcome as VCG. Specifically, motivated by the greedy spanning tree algorithm, we propose the following sequential auction: At each iteration pick a cut of the matroid that doesn't intersect previous winners and run a first price auction among the bidders in the cut. This auction is a more distributed alternative to the ascending auction proposed by Bikhchandani et al [5]. For the interesting case of a procurement auction for buying a spanning tree of a network from local constructors, our mechanism takes the form of a geographically local and simple auction.

We also study the case where bidders control several elements of the ground set but have a unit-demand valuation. This problem is a common generalization of the matroid problem, and the unit demand auction problem considered in the previous section. We show that the bound of 2 for the price of anarchy of any subgame perfect equilibrium extends also to this generalization.

1.2 Related Work

Externalities. The fact that one player might influence the other without direct competition has been long studied in the economics literature. The textbook model is due to Meade [24] in 1952, and the concept has been studied in various contexts. To name a few recent ones: [19] study it in the context of weapon sales, [12, 13] in the context of AdAuctions, and [21] in the context of combinatorial auctions. Our externalities model is due to Jehiel and Moldovanu [17]. They show that a pure Nash equilibrium exists in a full information game of first price auction with externalities, but use dominated strategies in their proof. Funk [10] shows the existence of an equilibrium after one round of elimination of dominated strategies, but argues that this refinement alone is not enough for ruling out unnatural equilibria - and gives a very compelling example of this fact. Iterative elimination of dominated strategies would eliminate the unnatural equilibria in his example, but instead of analyzing it, Funk analyzes a different concept: locally undominated strategies, which he defines in the paper. We show the existence of an equilibrium surviving any iterated elimination of dominated strategies - our proof is based on a natural ascending price auction argument, that provides more intuition on the structure of the game. Our first price auction equilibrium is also an equilibrium in a second price auction. Previous work studying second price auctions include Jehiel and Moldovanu [18], who study a simple case of second price auctions with two buyers and externalities between the buyers, derive equilibrium bidding strategies, and point out the various effects caused by positive and negative externalities, while in [15] the same authors study a simple case of second price auction with two types of buyers.

Sequential Auctions. A lot of the work in the economic literature studies the behavior of prices in sequential auctions of identical items where bidders participate in all auctions. Weber [30] and Milgrom and Weber [26] analyze first and second price sequential auctions with unit-demand bidders in the Bayesian model of incomplete information and show that in the unique symmetric Bayesian equilibrium the prices have an upward drift. Their prediction was later refuted by empirical evidence (see e.g. [1]) that show a declining price phenomenon. Several attempts to describe this "declining price anomaly" have since appeared such as McAfee and Vincent [23] that attribute it to risk averse bidders. Although we study full information games with pure strategy outcomes, we still observe declining price phenomena in our sequential auction models without relying to risk-aversion. Boutilier el al [6] studies first price auction in a setting with uncertainty, and gives a dynamic programming algorithm for finding optimal auction strategies assuming the distribution of other bids is stationary in each stage, and shows experimentally that good quality solutions do emerge when all players use this algorithm repeatedly.

The multi-unit demands case has been studied under the complete information model as well. Several papers (e.g. [11, 28]) study the case of two bidders. In the case of two bidders they show that there is a unique subgame perfect equilibrium that survives the iterated elimination of weakly dominated strategies, which is not the case for more than two bidders. Bae et al. [3, 2] study the case of sequential second price auctions of identical items to two bidders with concave valuations on homogeneous items. They show that the unique outcome that survives the iterated elimination of weakly dominated strategies is inefficient, but achieves a social

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welfare at least $1-e^{-1}$ of the optimal. Here we consider more than two bidders, which makes our analysis more challenging, as the uniqueness argument of the Bae et al. [3, 2] papers depends heavily on having only two players: when there are only two players, the externalities that naturally arise due to the sequential nature of the auction can be modeled by standard auction with no externalities using modified valuations.

Item Auctions. Recent work from the Algorithmic Game Theory community tries to propose the study of outcomes of simple mechanisms for multi-item auctions. Christodoulou, Kovacs and Schapira [7] and Bhawalkar and Roughgarden [4] study the case of running simultaneous second price item auctions for combinatorial auction settings. Christodoulou et al. [7] prove that for bidders with submodular valuations and incomplete information the Bayes-Nash Price of Anarchy is 2. Bhawalkar and Roughgarden [4] study the more general case of bidders with subadditive valuations and show that under complete information the Price of Anarchy of any Pure Nash Equilibrium is 2 and under incomplete information the Price of Anarchy of any Bayes-Nash Equilibrium is at most logarithimic in the number of items. Hassidim et al. [14] and Immorlica et al [16] study the case of simultaneous first price auctions and show that the set of pure Nash equilibria of the game correspond to exactly the Walrasian equilibria. Hassidim et al. [14] also show that mixed Nash equilibria have a price of anarchy of 2 for submodular bidders and logarithmic, in the number of items, for subadditive valuations.

Unit Demand Bidders. Auctions with unit demand bidders correspond to the classical matching problem in optimization. They have been studied extensively also in the context of auctions, starting with the classical papers of Shapley and Shubik [29] and Demange, Gale, and Sotomayor [8]. The most natural application of unit demand bidders is the case of a buyer-seller network market. A different interesting application where sequential auction is also natural, is in the case of scheduling jobs with deadlines. Suppose we have a set of jobs with different start and end times (that are commonly known) and each has a private valuation for getting the job done, not known to the auctioneer. Running an auction for each time slot sequentially is natural since, for example, it doesn't require for a job to participate in an auction before its start time.

Matroid Auctions. The most recent and related work on Matroid Auctions is that of Bikhchandani et al [5] who propose a centralized ascending auction for selling bases of a Matroid that results in the VCG outcome. In their model each bidder has a valuation for several elements of the matroid and the auctioneer is interested in selling a base. Kranton and Minehart [20] studied the case of a buyer-seller bipartite network market where each buyer has a private valuation and unit-demand. They also propose an ascending auction that simulates the VCG outcome. Their setting can be viewed as a matroid auction, where the matroid is the set of matchable bidders in the bipartite network. Under this perspective their ascending auction is a special case of that of Bikhchandani et al. [5]. We study a sequential version of this matroid basis auction game, but consider only the case when bidders are interested in a specific element of the matroid, and show that the sequential auction also implements the VCG outcome.

2 Auctions with Externalities

In this section we consider a *single-item auction with externalities*, and analyze a simple first price auction for this case. Variations on the same concept of externalities can be found, in Jehiel and Moldovanu [17], Funk [10] and in Bae et al [2] - the last one also motivated by sequential auctions, but considered auctions with two players only. Here we show that pure Nash equilibrium exists using strategies that survives the iterated elimination of dominated strategies. The single-item auction with externalities will be used as the main building block in the study of sequential auctions.

DEFINITION 2.1. *A* ***first-price single-item auction with externalities*** *with n players is a game such that the type of player i is a vector $[v_i^1, v_i^2, \ldots, v_i^n]$ and the strategy set is $[0, \infty)$. Given bids $b = (b_1, \ldots, b_n)$ the first price auction allocates the item to the highest bidder, breaking ties using some arbitrary fixed rule, and makes the bidder pay his bid value. If player i is allocated, then experienced utilities are $u_i = v_i^i - b_i$ and $u_j = v_j^i$ for all $j \neq i$.*

For technical reasons, we allow a player do bid x and $x+$ for each real number $x \geq 0$. The bid $b_i = x+$ means a bid that is infinitesimally larger than x. This is essential for the existence of equilibrium in first-price auctions. Alternatively, one could consider limits of ϵ-Nash equilibria (see Hassidim et al [14] for a discussion).

Next, we present a natural constructive proof of the existence of equilibrium that survives iterated elimination of weakly-dominated strategies. See the formal definition concept of iterated elimination of weakly-dominated strategies we use in Appendix A. Our proof is based on ascending auctions.

THEOREM 2.1. *Each instance of the first-price single-item auction with externalities has a pure Nash equilibrium that survives iterated elimination of weakly-dominated strategies.*

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Proof. For simplicity we assume that all v_i^j are multiples of ϵ. Further, we may assume without loss of generality that $v \geq 0$ and $\min_j v_i^j = 0$. We say that an item is toxic, if $v_i^i < v_i^j$ for all $i \neq j$. If the item is toxic, then $b_i = 0$ for all players and player 1 getting the item is an equilibrium. If not, assume $v_1^1 \geq v_1^2$.

Let $\langle i, j, p \rangle$ denote the state of the game where player i wins for price p and j is the price setter, i.e., $b_i = p+$, $b_j = p$, $b_k = 0$ for $k \neq i, j$. The idea of the proof is to define a sequence of states which have the following invariant property: $p \leq \gamma_i$, $p \leq \gamma_j$ and $v_i^i - p \geq v_i^j$, where $\gamma_i = \max_j v_i^i - v_i^j$. We will define the sequence, such that states don't appear twice on the sequence (so it can't cycle) and when the sequence stops, we will have reached an equilibrium.

Start from the state $\langle 1, 2, 0 \rangle$, which clearly satisfies the conditions. Now, if we are in state $\langle i, j, p \rangle$, if there is no k such that $v_k^k - p > v_k^i$ then we are at an equilibrium satisfying all conditions. If there is such k, move to the state $\langle k, i, p \rangle$ if this state hasn't appeared before in the sequence, and otherwise, move to $\langle k, i, p + \epsilon \rangle$. We need to check that the new states satisfy the invariant conditions: first $v_k^k - (p+\epsilon) \geq v_k^i$. Now, we need to check the two first conditions: $p + \epsilon \leq v_k^k - v_k^i \leq \gamma_k$. Now, the fact that i is not overbidding is trivial for $\langle k, i, p \rangle$, since i wasn't overbidding in $\langle i, j, p \rangle$. If, $\langle k, i, p \rangle$ already appeared in this sequence, it means that i took the item from player j, so: $v_i^i - p > v_i^j$ so: $p < v_i^i - v_i^j \leq \gamma_i$ so $p + \epsilon \leq \gamma_i$.

Now, notice this sequence can't cycle, and prices are bounded by valuations, so it must stop somewhere and there we have an equilibrium. To show the existence of an equilibrium surviving iterative elmination of weakly dominated strategies, we need a more careful argument: we refer to appendix A for a proof. ∎

It is not hard to see that any equilibrium of the *first-price* auction with externalities is also an equilibrium in the *second-price* version. The subclass of second-price equilibria that are equivalent to a first-price equilibria (producing same price and same allocation), are the second-price equilibria that are **envy-free**, i.e., no player would rather be the winner by the price the winner is actually paying. So, an alternative way of looking at our results for first price is to see them as outcomes of second-price when we believe that envy-free equilibria are selected at each stage.

3 Sequential Item Auctions

Assume there are n players and m items and each player has a monotone (free-disposal) combinatorial valuation $v_i : 2^{[m]} \to \mathbb{R}_+$. We will consider sequential auctions. First assume that at each time step only a single item is being auctioned off: item t is auctioned in step t. We define the sequential first (second) price auction for this case as follows: in time step $t = 1 \ldots m$ we ask for bids $b_i(t)$ from each agent for the item being considered in this step, and run a first (second) price auction to sell this item. Generally, we assume that after each round, the bids of each agent become common knowledge, or at least the winner and the winning price become public knowledge. The agents can naturally choose their bid in time t as a function of the past history. We will also consider the natural extension of these games, when each round can have multiple items on sale, bidders submit bids for each item on sale, and we run a separate first (second) price auction for each item.

This setting is captured by **extensive form games** (see Appendix B for a formal definition and [9] for a more comprehensive treatment). The strategy of each player is an adaptive bidding policy: the policy specifies what a player bids when the t^{th} item (or items) is auctioned, depending on the bids and outcomes of the previous $t-1$ items. More formally a strategy for player i is a bidding function $\beta_i(\cdot)$ that associates a bid $\beta_i(\{b_i^\tau\}_{i,\tau<t}) \in \mathbb{R}_+$ with each sequence of previous bidding profiles $\{b_i^\tau\}_{i,\tau<t}$.

Utilities are calculated in the natural way: utility for the set of items won, minus the sum of the payments from each round. In each round the player with largest bid wins the item and pays the first (second) price. We are interested in the **subgame perfect equilibria** (SPE) of this game: which means that the profile of bidding policies is a Nash equilibrium of the original game and if we arbitrarily fix what happens in the first t rounds, the policy profile also remains a Nash equilibrium of this induced game.

Our goal is to measure the **Price of Anarchy**, which is the worse possible ratio between the optimal welfare achievable (by allocating the items optimally) and the welfare in a subgame perfect equilibrium. Again, we invite the reader to Appendix B for formal definitions.

3.1 First Price Auctions: existence of pure equilibria

First we show that sequential first price single item auctions have pure equilibria for all valuations.

THEOREM 3.1. *Sequential first price auction when each round a single item is auctioned has a* SPE *that doesn't use dominated strategies, and in which bids in each node of the game tree depend only on who got the item in the previous rounds.*

We use backwards induction, and apply our result on the existence of Pure Nash Equilibria in first price auctions with externalities to show the theorem. Given

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outcomes of the game starting from stage $k+1$ define a game with externalities for stage k, and by Theorem 2.1 this game has a pure Nash equilibrium. It interesting to notice that we have existence of a pure equilibrium for arbitrary combinatorial valuation. In contrast, the simultaneous item bidding auctions, don't always possess a pure equilibrium even for subadditive bidders ([14]).

In the remainder of this section we consider three classes of valuations: additive, unit-demand and submodular. For additive valuations, the sequential first-price auction always produces the optimal outcome. This is in contrast to second price auctions, as we show in Appendix D.

In the next two subsections we consider unit-demand bidders, and prove a bound of 2 for the Price of Anarchy, and then show that for submodular valuations the price of anarchy is unbounded (while in the simultaneous case, the price of anarchy is bounded by 2 [14]).

3.2 First Price Auction for Unit-Demand Bidders

We assume that there is free disposal, and hence say that a player i is unit-demand if, for a bundle $S \subseteq [m]$, $v_i(S) = \max_{j \in S} v_{ij}$, where v_{ij} is the valuation of player i for item j.

To see that inefficient allocations are possible, consider the example given in Figure 1. There is a sequential first price auction of three items among four players. Player b prefers to loose the first item, anticipating that he might get a similar item for a cheaper price later. This gives an example where the Price of Anarchy is 3/2. Notice that this is the only equilibrium using non-dominated strategies.

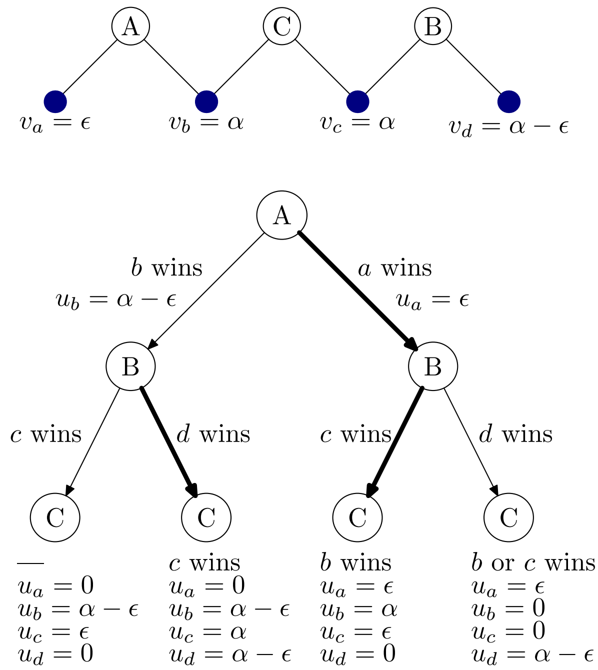

Figure 1: Sequential Multi-unit Auction generating PoA 3/2: there are 4 players $\{a,b,c,d\}$ and three items that are auctioned first A, then B and then C. The optimal allocation is $b \to A$, $c \to C$, $d \to B$ with value $3\alpha - \epsilon$. There is a Spe that has value $2\alpha + \epsilon$. In the limit when ϵ goes to 0 we get PoA $= 3/2$.

Theorem 3.2. *For unit-demand bidders, the* PoA *of pure subgame perfect equilibria of Sequential First Price Auctions of individual items is bounded by* 2*, while for mixed equilibria it is at most* 4*.*

Proof. Consider the optimal allocation and a subgame perfect equilibrium, and let Opt denote the social value of the optimum, and Spe the social value of the subgame perfect equilibrium. Let N be the set of players allocated in the optimum. For each $i \in N$, let $j^*(i)$ be the element it was allocated to in the optimal, and let $j(i)$ be the element he was allocated in the subgame perfect equilibrium and let $v_{i,j(i)}$ be player i's value for this element (if player i got more than one element, let $j(i)$ be his most valuable element). If player i wasn't allocated at all, let $v_{i,j(i)}$ be zero. Let $p(j(i))$ be the price for which item $j(i)$ was sold in equilibrium. Consider three possibilities:

1. i gets $j^*(i)$, then clearly $v_{i,j(i)} \geq v_{i,j^*(i)}$
2. i gets $j(i)$ after $j^*(i)$ or doesn't get allocated at all, then $v_{i,j(i)} \geq v_{i,j^*(i)} - p(j^*(i))$, otherwise he could have improved his utility by winning $j^*(i)$
3. i gets $j(i)$ before $j^*(i)$, then either $v_{i,j(i)} \geq v_{i,j^*(i)}$ or he can't improve his utility by getting $j^*(i)$, so it must be the case that his marginal gain from $j^*(i)$ was smaller than the maximum bid in $j^*(i)$, i.e. $p(j^*(i)) \geq v_{i,j^*(i)} - v_{i,j(i)}$

Therefore, in all the cases, we got $p(j^*(i)) \geq v_{i,j^*(i)} - v_{i,j(i)}$. Summing for all players $i \in N$, we get:

$$\text{Opt} = \sum_i v_{i,j^*(i)} \leq \sum_i v_{i,j(i)} + p(j^*(i)) \leq 2\text{Spe}$$

where in the last inequality is due to individual rationality of the players.

Next we prove the bound of 4 for the mixed case. We focus of a player i and let $j = j^*(i)$ denote item assigned to i in the optimal matching. In the case of mixed Nash equilibria, the price $p(j)$ is a random variable, as well as A_i the set of items player i wins in the auction. Consider a node n of the extensive form game, where j is up for auction, i.e., a possible

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history of play up to j being auctioned. Let P_i^{n-} be the expected value of the total price i paid till this point in the game, and let $P^n(j) = \mathbf{E}[p(j)|n]$ be the expected price for item j at this node n, and note that $P(j) = \mathbf{E}[p(j)] = \mathbf{E}[P^n(j)]$, where the right expectation is over the induced distribution on nodes n where j is being auctioned.

Player i deviating by offering price $2P^n(j)$ at every node n that j comes up for auction, and then dropping out of the auction, gets him utility at least $1/2(v_i(j) - 2P^n(j)) - P_i^{n-}$, as he wins item j with probability at least $1/2$ and paid P_i^{n-} to this point. Using the Nash inequality we get

$$\mathbf{E}[v_i(A_i)] - P_i \geq \mathbf{E}\left[1/2(v_i(j) - 2P^n(j)) - P_i^{n-}\right],$$

where P_i is the expected payment of player i, and the expectation on the right hand side is over the induced distribution on the the nodes of the game tree where j is being auctioned. Note that the proposed deviation does not effect the play before item j is being auctioned, so the expected value of $\mathbf{E}\left[P_i^{n-}\right]$ over the nodes n is at least the expected payment P_i of player i, and that the expected value of P_j^n over the nodes u is the expected price $P(j)$ of item j. Using these we get

$$\mathbf{E}[v_i(A_i)] \geq \frac{1}{2}v_i(j) - P(j).$$

Now summing over all players, and using that $\sum_j P(j) \leq \sum_i \mathbf{E}[v_i(A_i)]$ due to individual rationality, we get the claimed bound of 4.

■

The proof naturally extends to sequential auctions when in each round multiple items are being auctioned. We can also generalize the above positive result to any class of valuation functions that satisfy the property that the optimal matching allocation is close to the optimal allocation.

Theorem 3.3. *Let* Opt_M *be the optimal matching allocation and* Opt *the optimal allocation of a Sequential First Price Auction. If* $\text{Opt} \leq \gamma\text{Opt}_M$ *then the* PoA *is at most* 2γ *for pure equilibria and at most* 4γ *for mixed Nash, even if each round multiple items are auctioned in parallel (using separate first price auctions).*

Proof. Let $j^*(i)$ be the item of bidder i in the optimal matching allocation and A_i his allocated set of items in the Spe. Let A_i^- be the items that bidder i wins prior or concurrent to the auction of $j^*(i)$ and A_i^+ the ones that he wins after. Consider a bidder i that has not won his item in the optimal matching allocation. Bidder i could have won this item when it appeared by bidding above its current price $p_{j^*(i)}$ and then abandon all subsequent auctions. Hence:

$$v_i(A_i^- \cup \{j^*(i)\}) - p_{j^*(i)} - \sum_{j\in A_i^-} p_j \leq v_i(A) - \sum_{j\in A} p_j$$

$$v_i(A_i^- \cup \{j^*(i)\}) - p_{j^*(i)} \leq v_i(A) - \sum_{j\in A_i^+} p_j \leq v_i(A)$$

$$v_i(j^*(i)) - p_{j^*(i)} \leq v_i(A)$$

If a player did acquire his item in the optimal matching allocation then the above inequality certainly holds. Hence, summing up over all players we get:

$$\begin{aligned}\text{Opt}_M = \sum_i v_i(j^*(i)) &\leq \sum_i v_i(A_i) + \sum_i p_{j^*(i)} \\ &\leq \text{Spe} + \sum_j p_j = \text{Spe} + \sum_i \sum_{j\in A_i} p_j \\ &\leq \text{Spe} + \sum_i v_i(A_i) = 2\text{Spe}\end{aligned}$$

which in turn implies:

$$\text{Opt} \leq \gamma\text{Opt}_M \leq 2\gamma\text{Spe}$$

The bound of 4γ for the mixed case is proved along the lines as the mixed proof of Theorem 3.2. ■

The above general result can be applied to several natural classes of bidder valuations. For example, we can derive the following corollary for multi-unit auctions with submodular bidders: a bidder is said to be uniformly submodular if his valuation is a submodular function on the number of items he has acquired and not on the exact set of items. Thus a submodular valuation is defined by a set of decreasing marginals $v_i^1, \ldots, v_i^m$.

Corollary 3.1. *If bidders have uniformly submodular valuations and* $\forall i, j : |v_i^1 - v_j^1| \leq \delta \max(v_i^1, v_j^1)$ $(\delta < 1)$ *and there are more bidders than items then the* PoA *of a Sequential First Price Auction is at most* $2/(1-\delta)$.

3.3 First Price Auctions, Submodular Bidders

In sharp contrast to the simultaneous item bidding auction, where both first and second price have good price of anarchy whenever Pure Nash equilibrium exist [7, 4, 14], we show that for certain submodular valuations, no welfare guarantee is possible in the sequential case. While there are multiple equilibria in such auctions, in our example the natural equilibrium is arbitrarily worse then the optimal allocation.

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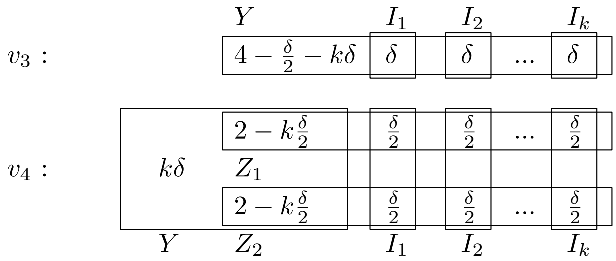

Figure 2: Valuations v_3 and v_4.

THEOREM 3.4. *For submodular players, the Price of Anarchy of the sequential first-price auction is unbounded.*

The intuition is that there is a misalignment between social welfare and player's utility. A player might not want an item for which he has high value but has to pay a high price. In the sequential setting, a bidder may prefer to let a smaller value player win because of the benefits she can derive from his decreased value on future items, allowing her to buy future items at a smaller price, or diverting a competitor, and hence decreasing the price.

Proof. Consider four players and $k+3$ items where 2 of the payers have additive valuations and 2 of them has a coverage function as a valuation. Call the items $\{I_1, \dots, I_k, Y, Z_1, Z_2\}$ and let players 1, 2 have additive valuations. Their valuations are represented by the following table:

	I_1	$\dots$	I_k	Y	Z_1	Z_2
1	$1+\epsilon$	$\dots$	$1+\epsilon$	0	$2-k\delta/2$	0
2	1	$\dots$	1	0	0	$2-k\delta/2$

The valuations of players 3 and 4 are given by the coverage functions defined in Figure 2: each item corresponds to a set in the picture. If the player gets a set of items, his valuation for those items (sets) is the sum of the values of the elements covered by the sets corresponding to the items.

In the optimal allocation, player 1 gets all the items $I_1, \dots, I_k$, player 3 gets Y and player 4 gets Z_1, Z_2. The resulting social welfare is $k + 8 + k\epsilon - \delta/2$. We will show that there is a subgame perfect equilibrium such that player 3 wins all the items $I_1, \dots, I_k$, even though it has little value for them, resulting in a social welfare of approximately 8 only. The intuition is the following: in the end of the auction, player 4 has to decide if he goes for item Y or goes for items Y_1, Y_2. If he goes for item Y, he competes with player 3 and afterwards lets players 1 and 2 win items Z_1, Z_2 for free. This decision of player 4 depends on the outcomes of the first k auctions. In particular, we show that if all items $I_1, \dots, I_k$ go to either 3 or 4, then player 4 will go for item Y, otherwise, he will go for items Z_1, Z_2. If either players 1 or 2 acquire any of the items $I_1, \dots, I_k$, they will be guaranteed to lose item Z_1, Z_2, and therefore both will start bidding truthfully on all subsequent I_i auctions, deriving very little utility. In equilibrium agent 3 gets all items $I_1, \dots, I_k$, resulting in a social welfare of approximately 8 only.

In the remainder of this section, we provide a more formal analysis: We begin by examining what happens in the last three auctions of Y, Z_1 and Z_2 according to what happened in the first k auctions. Let $k_{1,2}, k_3, k_4$ be the number of items won by the corresponding players in the first k auctions.

- Case 1: $k_{1,2} = 0$. Thus $k_3 = k - k_4$. Player 3 has a value of $4 - \frac{\delta}{2} - k\delta + (k - k_3)\delta = 4 - \frac{\delta}{2} - (k - k_4)\delta$ for item Y. Player 4 has a value of 4 for Y and a value of $2 - k\frac{\delta}{2} + (k - k_4)\frac{\delta}{2}$ for each of Z_1 and Z_2. Thus if player 4 loses auction Y he will get a utility of $(k - k_4)\delta$ from the auctions of Z_1 and Z_2 since players 1 and 2 will bid $2 - k\delta/2$. Thus at auction Y player 4 is willing to win for a price of at most $4 - (k - k_4)\delta$ and player 3 will bid $4 - \frac{\delta}{2} - (k - k_4)\delta$. Thus, player 4 will win Y and will only bid $(k - k_4)\frac{\delta}{2}$ in each of Z_1, Z_2. Therefore, in this case we get that the utilities of all the players from the last three auctions are: $u_1 = u_2 = 2 - (2k - k_4)\frac{\delta}{2}, u_3 = 0, u_4 = \frac{\delta}{2} + (k - k_4)\delta$

- Case 2: $k_{1,2} > 0$. Player 3 has a value of $4 - \frac{\delta}{2} - k\delta + (k - k_3)\delta$ for Y. Since $k_{12} \geq 1$, we have $k - k_3 = k_4 + k_{1,2} \geq k_4 + 1$, hence the value of 3 for Y is at least $4 + \frac{\delta}{2} - (k - k_4)\delta$. Player 4 has a value of 4 for Y and a value of $2 - k\frac{\delta}{2} + (k - k_4)\frac{\delta}{2}$ for each of Z_1 and Z_2. Hence, again player 4 wants to win at auction Y for at most $4 - \frac{\delta}{2} - (k - k_4)\delta$, hence he will lose to 3 and will go on to win both Z_1 and Z_2. Thus the utilities of all the players from the last three auctions are: $u_1 = u_2 = 0, u_3 = \delta, u_4 = (k - k_4)\delta$

We show by induction on i that as long as players 1 and 2 haven't won any of the $k-i$ items auctioned so far then they will bid 0 in the remaining i items and one of players 3 or 4 will win marginally with zero profit. For $i = 1$ since both 1 and 2 haven't won any previous item, by losing the k'th item we know by the above analysis that they both get utility of ≈ 2, while if any of them wins then they get utility of 0. The external auction that is played at the k'th item is represented by the

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following $[v^i_j]$ matrix:

$$\begin{bmatrix} 1+\epsilon & 0 & \approx 2 & \approx 2 \\ 0 & 1 & \approx 2 & \approx 2 \\ \delta & \delta & \delta & 0 \\ (k-k_4)\delta & (k-k_4)\delta & \frac{\delta}{2}+(k-k_4)\delta & \frac{3\delta}{2}+(k-k_4)\delta \end{bmatrix}$$

It is easy to observe that the following bidding profile is an equilibrium of the above game that doesn't involve any weakly dominated strategies: $b_1 = b_2 = 0, b_3 = \delta, b_4 = \delta+$. Thus, player 4 will marginally win with no profit from the current auction (alternatively we could have player 3 win with no profit).

Now we prove the induction step. Assume that it is true for the $i-1$. We know that if either player 1 or 2 wins the $k-i$ item then whatever they do in subsequent auctions, from the case 2 of the analysis, player 4 will go for Z_1 and Z_2 and they will get 0 utility in the last 3 auctions. Hence, in the $i-1$ subsequent auctions they will bid truthfully, making player 1 win marginally at zero profit every auction. On the other hand if they lose, by the induction hypothesis they will lose all subsequent auctions leading them to utility of ≈ 2. Moreover, players 3 and 4 have the same exactly utilities as in the base case, since they never acquire any utility from the first k auctions. Thus the external auction played at the $k-i$ item is exactly the same as the auction of the base case and hence has the same bidding equilibrium.

Thus in the above Spe players 1 and 2 let some of the players 3 and 4 win all the first k items. This leads to an unbounded PoA. ■

4 Matroid Auctions

In this section we first consider a matroid auction where each matroid element is associated with a separate bidder, then in Section 4.1 we consider a problem that generalizes matroid auctions and item auctions with unit demand bidders.

4.1 Sequential Matroid Auctions Suppose that a telecommunications company wants to build a spanning tree over a set of nodes. At each of the possible links of the network there is a distinct set of local constructor's that can build the link. Each constructor has a private cost for building the link. So the company has to hold a procurement auction to get contract for building edges of a spanning tree with minimum cost. In this section, we show that by running a sequential first price auction, we get the outcome equivalent to the Vcg auction in a distributed and asynchronous fashion.

A version of the well-known greedy algorithm for this optimization problem is to consider cuts of this graph sequentially, and for each cut we consider, include the minimum cost edge of the cut. Our sequential auction is motivated by this greedy algorithm: we run a sequence of first price auctions among the edges in a cut. More formally, at each stage of the auction, we consider a cut where no edge was included so far, and hold a first price sealed bid auction, whose winner is contracted. More generally, we can run the same auction on any matroid, not just the graphical matroid considered above. The goal of the procurement auction is to select a minimum cost matroid basis, and at each stage we run a sealed bid first price auction for selecting an element in a co-circuit.

Alternately, we can also consider the analogous auction for selling some service to a basis of a matroid. As before, the bidders correspond to elements of a matroid. Their private value v_i is their value for the service. Due to some conflicts, not all subsets of the bidders can be selected. We assume that feasible subsets form a matroid, and hence the efficient selection chooses the basis of maximum value. As before, it may be simpler to implement smaller regional auctions. Our method sequentially runs first price auctions for adding a bidder from a co-circuit. For the special case of the dual of graphical matroid, this problem corresponds to the following. Suppose that a telecommunications company due to some mergers ended up with a network that has cycles. Thus the company decides to sell off its redundant edges so that it ends up owning just a spanning tree. The sequential auction we propose runs a sequence of first price auctions, each time selecting an edge of a cycle in the network for sale. If more than one bidder is interested in an edge we can simply think of it as replacing that edge with a path of edges, each controlled by a single individual.

The main result of this section is that the above sequential auction implements the Vcg outcome both for procurement and direct auctions. To unify the presentation with the other sections, we will focus here on direct auction. In the final subsection, we will consider a common generalization of the unit-demand auction and this matroid auction. In the procurement version, we make the small technical assumption that every cut of the matroid contains at least two elements, otherwise the Vcg price of a player could be infinity. Such assumption was also made in previous work on matroid auctions [5].

Theorem 4.1. *In a sequential first price auction among players in the co-circuit of a matroid (as described above), subgame perfect equilibria in undominated strategies emulate the* Vcg *outcome (same allocations and prices).*

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For completeness, we summarize some some definitions regarding matroids and review notation.

A Matroid $\mathcal{M}$ is a pair $(\mathcal{E}_M, \mathcal{I}_M)$, where $\mathcal{E}_M$ is a ground set of elements and $\mathcal{I}_M$ is a set of subsets of $\mathcal{E}_M$ with the following properties: (1) $\emptyset \in \mathcal{I}_M$, (2) If $A \in \mathcal{I}_M$ and $B \subset A$ then $B \in \mathcal{I}_M$, (3) If $A, B \in \mathcal{I}_M$ and $|A| > |B|$ then $\exists e \in A - B$ such that $B + e \in \mathcal{I}_M$. The subsets of $\mathcal{I}_M$ are the independent subsets of the matroid and the rest are called dependent.

The rank of a set $S \subset \mathcal{E}_M$, denoted as $r_{\mathcal{M}}(S)$, is the cardinality of the maximum independent subset of S. A base of a matroid is a maximum cardinality independent set and the set of bases of a matroid $\mathcal{M}$ is denoted with $\mathcal{B}_{\mathcal{M}}$. the An important example of a matroid is the graphical matroid on the edges of a graph, where a set S of edges is independent if it doesn't contain a cycle, and bases of this matroid are the spanning trees.

A circuit of a matroid $\mathcal{M}$ is a minimal dependent set. and we denote with $\mathcal{C}(\mathcal{M})$ the set of circuits of $\mathcal{M}$. Circuits in graphical matroids are exactly the cycles of the graph. A cocircuit is a minimal set that intersects every base of $\mathcal{M}$. Cocircuits in graphical matroids corresond to cuts of the graph.

Definition 4.1. (Contraction) *Given a matroid $\mathcal{M} = (\mathcal{E}_M, \mathcal{I}_M)$ and a set $X \subset \mathcal{E}_M$ the contraction of $\mathcal{M}$ by X, denoted $\mathcal{M}/X$, is the matroid defined on ground set $\mathcal{E}_M - X$ with $\mathcal{I}_{M/X} = \{S \subseteq \mathcal{E}_M - X : S \cup X \in \mathcal{I}_M\}$.*

If we are given weights for each element of the ground set of a matroid $\mathcal{M}$ then it is natural to define the following optimization problem: Find the base $\text{Opt}(\mathcal{M}) \in \mathcal{B}_{\mathcal{M}}$ that has minimum/maximum total weight (we might sometimes abuse notation and denote with Opt both the set and its total weight). A well known algorithm for solving the above optimization problem is the following (see [22]): At each iteration consider a cocircuit that doesn't intersect the elements already picked in previous iterations, then add its minimum/maximum element to the current solution.

The most well known mechanism for auctioning items to a set of bidders is the Vickrey-Clarke-Groves Mechanism (Vcg). The Vcg mechanism selects the optimal basis $\text{Opt}(\mathcal{M})$. It is easy to see that the Vcg price of a player $i \in \text{Opt}(\mathcal{M})$, denoted as $\text{Vcg}_i(\mathcal{M})$, is the valuation of the highest bidder $j(i)$ that can be exchanged with i in $\text{Opt}(\mathcal{M})$, i.e. $\text{Vcg}_i(\mathcal{M}) = \max\{v_j : \text{Opt}(\mathcal{M}) - i + j \in \mathcal{I}_M\}$, or alternately the above price is the maximum over all cycles of the matroid that contain i of the minimum value bidder in each cycle: $\text{Vcg}_i(\mathcal{M}) = \max_{C \in \mathcal{C}(\mathcal{M}): i \in C} \min_{i \neq j \in C} v_j$. To unify notation we say that $\text{Vcg}_i(\mathcal{M}) = \infty$ for a bidder $i \notin \text{Opt}(\mathcal{M})$, although the actual price assigned by the Vcg mechanism is 0.

The proof of Theorem 4.1 is based on an induction on matroids of lower rank. After a few stages of the sequential game, we have selected a subset of elements X. Notice that the resulting subgame is exactly a matroid basis auction game in the contracted matroid $\mathcal{M}/X$. To understand such subgames, we first prove a lemma that relates the Vcg prices of a player in a sequence of contracted matroids.

Lemma 4.1. *Let $\mathcal{M}$ be a matroid, and consider a player $i^* \in \text{Opt}(\mathcal{M})$. Consider a co-circuit D, and assume our auction selects an element $k \neq i^*$, and let $\mathcal{M}'$ be the matroid that results from contracting k. Then $\text{Vcg}_{i^*}(\mathcal{M}') \geq \text{Vcg}_{i^*}(\mathcal{M})$ and the two are equal if $k \in \text{Opt}(\mathcal{M})$.*

Proof. First we show that the VCG prices do not change when contracting an element from the optimum. From matroid properties it holds that for any set X: $\text{Opt}(\mathcal{M}/X) \subset \text{Opt}(\mathcal{M})$. In this case $\text{Opt}(\mathcal{M}') = \text{Opt}(\mathcal{M}/\{k\}) \subset \text{Opt}(\mathcal{M})$, which directly implies that $\text{Opt}(\mathcal{M}') = \text{Opt}(\mathcal{M}) - \{k\}$. Hence, $\{j : \text{Opt}(\mathcal{M}) - i^* + j \in \mathcal{I}_M\} = \{j : \text{Opt}(\mathcal{M}') - i^* + j \in \mathcal{I}'_M\}$, and thus, $\text{Vcg}_{i^*}(\mathcal{M}) = \text{Vcg}_{i^*}(\mathcal{M}')$.

As mentioned in the previous section, the Vcg price can also be defined as the maximum over all cycles of the matroid that contain i^* of the minimum value bidder in each cycle: $\text{Vcg}_{i^*}(\mathcal{M}) = \max_{C \in \mathcal{C}(\mathcal{M}): i^* \in C} \min_{i^* \neq j \in C} v_j$. Let C be the cycle that attains this maximum in $\mathcal{M}$. The element i^* is dependent on the set $C \setminus \{i^*\}$ in $\mathcal{M}$, and as a result i^* is dependent of the set $C \setminus \{i^*, k\}$ in $\mathcal{M}'$, hence there is a cycle $C' \subset C$ in $\mathcal{M}'$ with $i^* \in C'$. This proves that the Vcg price can only increase due to contracting an element. ■

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1 : For clarity, we assume the values of the players for being allocated to be all different. We will prove the theorem by induction on the rank of the matroid. Let $\mathcal{M}$ be our initial matroid prior to some auction. Notice that for any outcome of the current auction the corresponding subgame is exactly a sequential matroid auction on a contracted matroid $\mathcal{M}'$. The proposed auction for rank 1 matroids is exactly a standard (no external payoffs) first price auction.

Let D be the co-circuit auctioned. Using the induction hypothesis, we can write the induced game on this node of the game tree exactly. For $i \in D - \text{Opt}(\mathcal{M})$, if he doesn't win the current auction then by the induction hypothesis, he is not going to win in any of the subsequent auctions, and hence $v_i^i = v_i$ and $v_i^j = 0$ for $j \neq i$. For player a player $i \in \text{Opt}(\mathcal{M}) \cap D$. Again

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$v_i^i = v_i$ and for $j \neq i$, we have that $v_i^j = v_i - \text{VCG}_i(\mathcal{M}/j)$ if $i \in \text{OPT}(\mathcal{M}/j)$ and $v_i^j = 0$ otherwise.

We claim that in all equilibria of this game, where no player bids $b_i > \gamma_i := \max_j v_i^i - v_i^j$ (notice that bidding above $\max_j v_i^i - v_i^j$ is dominated strategy), a player $i \in \text{OPT}(\mathcal{M})$ wins by his VCG price. Suppose some player $k \notin \text{OPT}(\mathcal{M})$ wins the auction. Then, there is some player $j \in \text{OPT}(\mathcal{M}) \setminus \text{OPT}(\mathcal{M}/k)$. This is not an equilibrium, as player has $v_j > v_k$, and hence j could overbid k and get the item, since the price is $p \leq v_k < v_j$.

Next we claim that the winner $i \in \text{OPT}(\mathcal{M})$ gets the item by his VCG price, and the winner is $i \in D$ with highest VCG price. Suppose he gets the item by some value strictly smaller than his VCG price. If we can show that there is some player t such that $v_t^t - \text{VCG}_i(\mathcal{M}) \geq v_t^i$. Let C and j be the cycle and element j that define the VCG price of i in:

$$\text{VCG}_i(\mathcal{M}) = \max_{C \in \mathcal{C}(\mathcal{M}): i \in C} \min_{i \neq j \in C} v_j$$

Now, since $|C \cap D| \geq 2$, there is some $t \neq i$, $t \in C \cap D$. Notice that $v_t \geq \text{VCG}_i(\mathcal{M})$, so if $t \notin \text{OPT}(\mathcal{M})$, then he would overbid i and get the item. If $t \in \text{OPT}(\mathcal{M})$, then notice $\text{VCG}_t(\mathcal{M}) \geq \text{VCG}_i(\mathcal{M})$, so again he would prefer to overbid i and get the item. This also shows that some player i whose VCG price is not maximum winning by at most his VCG price is not possible in equilibrium.

At last, suppose the winner gets the item for some price p above his VCG price. Then $b_i = p+$ and there is some player $j \in D$ such that $b_j = p$. It can't be that $j \notin \text{OPT}(\mathcal{M})$, then his value v_j can't be higher than then maximum VCG price. So, it must be that $j \in \text{OPT}(\mathcal{M})$, then player i can improve his utility by decreasing his bid, letting j win and win for $\text{VCG}_i(\mathcal{M}/j) = \text{VCG}_i(\mathcal{M}) < p$ (by Lemma 4.1). ∎

The above optimality result tells us that VCG can be implemented in a distributed and asynchronous way. Although the auctions happen locally, the final price of each auction (the VCG price) is a global property. It should be noted, nevertheless, that this is a common feature in network games in general. The previous theorem concerns with the state of the game after equilibrium is reached. If one considers a certain (local) dynamics and believes it will eventually settle in an equilibrium, the VCG outcome is the only possible such stable state.

4.2 Unit-demand matroid auction In this section we sketch a common generalization of the auction for unit demand bidders from section 3.2 and the matroid auction of section 4.1. Suppose that the items considered form the ground set of some matroid $\mathcal{M} = ([m], \mathcal{I}_{\mathcal{M}})$ and the auctioneer wants to sell an independent set of this matroid, while buyers remain unit-demand and are only interested in buying a single item.

We define the Sequential Matroid Auction with Unit-Demand Bidders to be the game induced if in the above setting we run the Sequential First Price Auction on co-circuits of the matroid as defined in previous section.

THEOREM 4.2. *The price of anarchy of a subgame perfect equilibrium of any Sequential Matroid Auction with Unit-Demand Bidders is* 2.

To adopt our proof from the auction with unit-demand bidder to the more general Theorem 4.2 we define the notion of the **participation graph** $\mathcal{P}(B)$ of a base B to be a bipartite graph between the nodes in the base and the auctions that took place. An edge exists between an element of the base and an auction if that element participated in the auction. Now the proof is a combination of the proof of Theorem 3.2 and of the following lemma.

LEMMA 4.2. *For any base B of the matroid $\mathcal{M}$, $\mathcal{P}(B)$ contains a perfect matching.*

Proof. We will prove that given any k-element independent set, there were k auctions that had at least one of those elements participating. Then by applying Hall's theorem we get the lemma.

Let $I_k = \{x_1, \dots, x_k\}$ be such an independent set of the matroid. Let $\mathcal{A}_{-k} = \{A_1, \dots, A_t\}$ be the set of auctions (co-circuits) that contain no element of I_k ordered in the way they took place in the game and $\mathcal{A}_k$ its complement. Let $a_1, \dots, a_t$ be the winners of the auctions in A_{-k}. Let $r(\mathcal{M})$ be the rank of the matroid.

Since I_k is an independent set, it is a subset of some basis and by the properties of co-circuits: for any $x_i \in I_k$ there exists a co-circuit X_i that contains x_i and no other x_j. The sequence of elements $(x_1, \dots, x_k, a_1, \dots, a_t)$ and co-circuits $(X_1, \dots, X_k, A_1, \dots, A_t)$ have the property that each element belongs to its corresponding co-circuit and no co-circuit contains any previous element. Hence, the set $\{x_1, \dots, x_k, a_1, \dots, a_t\}$ is an independent set and therefore $t + k \leq r(\mathcal{M})$. Since the total number of auctions is $r(\mathcal{M})$, $|\mathcal{A}_k| \geq k$. ∎

Now the proof of Theorem 4.2.

Proof of Theorem 4.2 (sketch) : Using the last lemma, there is a bijection between the elements allocated in the efficient outcome and the co-circuits auctioned. For a player i that is assigned an item $j^*(i)$ in the efficient outcome, let $A(i)$ be the auction (co-circuit) matched with $j^*(i)$ in the above bijection. Now

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if in the proof of Theorem 3.2 we replace any reasoning about the auction of item $j^*(i)$ with the auction $A(i)$, we can extend the arguments and prove that $p(A(i)) \geq v_{i,j^*(i)} - v_{i,j(i)}$, where $p(A(i))$ is the value of the bid that won auction $A(i)$. Summing these inequalities over the auctions completes the proof. ■

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A Iterated elimination of dominated strategies

First we define precisely the concept of a strategy profile that survives iterated elimination of weakly dominated

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strategies. Then we characterize such profiles for the first-price auction with externalities using a graph-theoretical argument.

DEFINITION A.1. *Given an n-player game define by strategy sets $S_1, \dots, S_n$ and utilities $u_i : S_1 \times \dots \times S_n \to \mathbb{R}$ we define a* valid procedure for eliminating weakly-dominated strategies *as a sequence $\{S_i^t\}$ such that for each t there is i such that $S_j^t = S_j^{t-1}$ for $j \neq i$, $S_i^t \subseteq S_i^{t-1}$ and for all $s_i \in S_i^{t-1} \setminus S_i^t$ there is $s_i' \in S_i^t$ such that $u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_{-i} \in \prod_{j \neq i} S_j^t$ and the inequality is strict for at least one s_{-i}. We say that an strategy profile s survives iterated elimination of weakly-dominated strategies if for any valid procedure $\{S_i^t\}$, $s_i \in \cap_t S_i^t$.*

The concept above is very strong as different elimination procedures can lead to elimination of different strategies. This can possibly lead to no strategy (at least no Nash equilibrium) surviving iterated elimination of weakly-dominated strategies. We show that the first price auction game has equilibria that satisfies this strong definition, which makes the equilibria a very robust prediction.

As a warm up, consider the first price auction without externalities, i.e., $v_i^i = v_i$ and $v_i^j = 0$ for $j \neq i$ with $v_1 \geq v_2 \geq \dots$. It is easy to see that the set of strategies surviving any iterated elimination procedure is $[0, v_i)$ for player $i > 1$ and $[0, v_2]$ for player 1. Bidding $b_i > v_i$ is clearly dominated by bidding v_i. By the definition, bidding v_i is dominated by bidding any value smaller then v_i, since by bidding v_i, the player can never get positive utility. After we eliminate $b_i \geq v_i$ for all the players, it is easy to see that $b_1 = v_2$ dominates any bid $b_1 > v_2$, since player 1 wins anyway (since all the other players have eliminated their strategies $b_i \geq v_i$). The natural equilibrium to expect in this case is player 1 getting the item for price v_2, which is a result of $b_1 = v_2+$ and $b_2 = v_2$. However, $b_2 = v_2$ is eliminated for player 2, but any strategy arbitrarly close to $b_2 = v_2$ is not.

This motivates us to pass to the topological closure when discussing iterative elimination of weakly dominated strategies for first price auctions:

DEFINITION A.2. *In a first-price auction with externalities, a bid b_i for player i is* compatible with iterated elimination of weakly dominated strategies, *if b_i is in the topological closure of the set of bids that survive any procedure of elimination. In other words, for each $\delta > 0$ there is a bid b_i' that survives any procedure of elimination such that $|b_i - b_i'| < \delta$.*

Now, we are ready to characterize the set of Nash equilibria that are compatible with iterated elimination. In order to do that, we define an *overbidding-graph* in the following way: for each price p, consider a directed graph G_p on n nodes such that there is an edge from i to j if $v_j^j - p > v_j^i$, i.e., if player i were getting the item at price p, player j would rather overbid him and take the item. Now, notice that the graph $G_{p+\epsilon}$ is a subgraph of G_p.

Let's assume that all nodes have positive in-degree and out-degree in G_0. If there are nodes with zero in-degree, simply remove the players that have in-degree zero in G_0 (which mean that they can't possibly want the item, i.e. they bidding zero is a dominant strategy). If there are players with zero out-degree, then the problem is trivial, since there are nodes for who we can give the item and get an equilibrium with zero price.

THEOREM A.1. *The strategies for player i that survive iterated elimination of weakly dominated strategies are $S_i = [0, \tau_i)$ where τ_i can be computed by the following algorithm: begin with $p = 0$ and $V = [n]$. In each step, if there is a node $i \in V$ of in-degree zero in $G_p[V]$ (i.e., G_p defined on the nodes V), then set $\tau_i = p$ and remove i from V and recurse. If there is no such node, increase the value of p until some node's in-degree becomes zero.*

Proof. Consider that the players are numbered such that $\tau_1 \leq \tau_2 \leq \dots \leq \tau_n$. Now, we will prove by induction that no element of $[0, \tau_i)$ can be eliminated from the strategy set of player $j \geq i$ by recursive elimination of weakly dominated strategies. And that there is one procedure that eliminates all bids $b \geq \tau_i$ for player i strategy set.

For the base case, suppose there is some process of iterated elimination that removes some strategy $b \in [0, \tau_1)$ for player i, imagine the first time it happens in this process and say that the strategy that eliminates it is some b'. If $b' < b$, consider the profile for the other players where everyone plays some value between b' and b, and given that player i has positive in-degree in G_b, suppose that the highest bid is submitted by a player j such that (j, i) is an edge of G_b. Then clearly b generates strictly higher utility then b'. Now, suppose $b' > b$, then b performs strictly better then b' in the profile where all the other players bid zero. Now, notice that all the bids $b_1 > \tau_1$ for player 1 are dominated and bidding $b_1 = \tau_1$ is dominated by playing any smaller bid.

Now the induction step is along the same lines: We know that no elimination procedure can eliminate bids in $[0, \tau_k)$ for player k, $k < i$. Now, suppose there is some procedure in which we are able to eliminate some bid $b \in [0, \tau_i)$ for some player $j \geq i$. Then again, consider the first time it happens and let b' be the bid that dominates b. We analyze again two cases. If $b' < b$. consider a profile where the other players $j' \geq i, j' \neq j$

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bid between b' and b where the highest bidder is a player k such that the edge (k, j') is in G_b. It is easy to see that b outperforms b' for this profile. If $b' > b$, we can use the same argument as in the base case. Also, given that the strategies $b_j \geq \tau_j$ were already eliminated for players $j < i$, clearly $b_i > \tau_i$ is dominated by τ_i. ■

COROLLARY A.1. *The bids $b_i \in [0, \tau_i]$ are exactly the bids that are compatible with iterative elimination of weakly dominated strategies for the first price auction with externalities.*

Now, given the result above, it is simple to prove that there are Nash equilibria that are compatible with iterated elimination. Consider the algorithm used to calculate τ_i. Consider that at point p, V, the active edges are the edges in $G_p[V]$. Now, in the execution of the algorithm, we can keep track of the in-degree and out-degree of each node with respect to active egdes. Those naturally decrease with the execution of the algorithm. Since in each step some edges become inactive, there is at least one node such that its out-degree becomes zero before or at the same time that his in-degree becomes zero. So, for the corresponding player i, there is one price p such that $\tau_i \geq p$, there is an edge (j, i) in $G_{p'}[V']$, where p', V' is the state of the algorithm just before the out-degree of i became zero. So, clearly $\tau_j \geq p$. Now, it is easy to see that the strategy profile $b_i = p+$, $b_j = p$ and $b_k = 0$ for all $k \neq i, j$ is a Nash equilibrium and it is compatible with iterative elimination.

In fact, the reasoning above allows us to fully characterize and enumerate all outcomes that are a Nash equilibrium compatible with iterated elimination:

THEOREM A.2. *The outcome of player i winning the item for price p can be expressed as a Nash equilibrium that is compatible with iterated elimination iff $p \leq \tau_i$, player i has out-degree zero in G_p and there is some player j with $\tau_j \geq p$ such that the edge (j, i) is in $G_{p'}$ for all $p' < p$.*

B Formal definition of extensive form games

We provide in this session a formal mathematical description of the concepts described in session 3: We can represent an **extensive-form game** via a game-tree, where nodes of the tree correspond to different histories of play. At each stage of the game, players make simultaneous moves, that can depend on the history of play so far. So a player's strategy in an extensive form game is a strategy for each possible history, i.e., each node of the tree. More formally,

- Let N denote the set of players, and let $n = |N|$
- A k-stage game is represented by a directed game tree $\mathcal{T} = (V, E)$ of $k + 1$ levels. Let V^t be the nodes in level t, where V^t denotes possible partial histories at the start of stage t. So V^1 contains only the root and V^{k+1} contains all the leaves, i.e., the outcomes of the game. Note that the tree can be infinite, if for example some player has an infinite strategy set.
- for each $v \in V \setminus V^{k+1}, i \in N$, a strategy set $S_i(v)$ is the set of possible strategies of player i
- for each $v \in V$, the out-going edges of v correspond to strategy profiles $s(v) \in \times_i S_i(v)$, the outcome of this stage when players play strategies $s(v) = (s_1(v), \ldots, s_n(v))$.
- for each $i \in N$, we have the utility function $u_i : V^{k+1} \to \mathbb{R}$, that denotes the utility of the outcome corresponding to node $v \in V^{k+1}$ for player i.

The **pure strategy** of a player consists of choosing $s_i(v) \in S_i(v)$ for each node $v \in V$, i.e. a function $s_i : V \to \cup_v S_i(v)$ such that $s_i(v) \in S_i(v)$. In other words, it is a strategy choice for each round, given the history of play so far, which is encoded by node v. A strategy profile is a n-tuple $s = (s_1, \ldots, s_n)$. It defines the **actual history of play** $h = (h_1, h_2, \ldots, h_k)$, where $h_1 = s(r)$ is the strategy profile played at the root, and h_i is the strategy profile played at the node that corresponds to history $h_1, \ldots, h_{i-1}$. Notice that h corresponds to a leaf of the tree, which allows to define the utility of i for a strategy profile:

$$u_i(s) = u_i(h(s))$$

We use **subgame perfect equilibrium** (SPE) as our main solution concept. A subgame of sequential game is the game resulting after fixing some initial history of play, i.e., starting the game from a node v of the game tree. Let $u_i^v(s)$ denote the utility that i gets from playing s starting from node v in the tree. We say that a profile s is a SPE if it is a Nash equilibrium for each subgame of the game, that is, for all nodes v we have:

$$\forall s_i' : u_i^v(s_i, s_{-i}) \geq u_i^v(s_i', s_{-i}).$$

Given a node v in the game tree and fixing $s_i(v')$ for all v' below v, we can define an induced normal-form game in node v by s as the game with strategy space $\times S_i(v)$ such that the utility for player i by playing $\tilde{s}(v), \tilde{s}_i(v) \in S_i(v)$ is $u_i^v(s_i, s_{-i})$ where player i plays $\tilde{s}_i(v)$ in node v and according to $s_i(v')$ in all nodes v' below v. Kuhn's Theorem states that s is a subgame perfect equilibrium iff $s(v)$ is a Nash equilibrium on the induced normal-form game in node v for all v.

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The main tool we will use to analyse those games is the **price of anarchy**. Consider a welfare function defined on the leaves of the tree, i.e. $W : V^{k+1} \to \mathbb{R}$. Given a certain strategy profile s and its induced history $h(s)$, the social welfare of this game play is given by $W(v) = \sum_{i\in N} u_i(h(s))$. We define the optimal welfare as $W^* = \max_{v\in V^{k+1}} W(v)$, and the pure Price of Anarchy (PoA) as:

$$\text{PoA} = \max_{s\in E} \frac{W^*}{W(s)}$$

where E is the set of all subgame perfect equilibria.

There sequential auctions we study are m-stage games and strategy space on each node v for player i is a bid $b_i(v) \in [0,\infty)$. In other words, the strategy of each player in this game is a function that maps the bid profiles in the first $k-1$ items to his bid in the k-th item. Their utility is the total value they get for the bundle they acquired minus the price paid. The welfare is the sum of the values of all players.

C Non-Existense of SPE in Multi-Item Auctions

We give an example of a multi-item sequential auction with no SPE in pure strategies. The example has 4 players and 5 items. The first two items X_1, X_2 are auctioned simultaneously first and the remaining items are auctioned sequentialy afterwards in the order W, Y, Z. Players 1 and 4 are single minded. Player 1 has value v only for item Z and player 4 has value $\frac{2\delta}{3}+\epsilon$ only for item W. Players 2 and 3 have coverage submodular valuations that are depicted in Figure 3. One can check that the following allocation and prices constitutes a walrasian equilibrium of the above instance: $A_1 = \emptyset, A_2 = \{X_1, Z\}, A_3 = \{X_2, Y\}, A_4 = \{W\}), p_{X_1} = p_{X_2} = \delta/3, p_Y = v + \delta/6, p_W = 2\delta/3$. However, we will show that there is no subgame perfect equilibrium in pure strategies.

We will show that the subgame perfect equilibrium in the last three auctions is always unique given the outcome in the first two item auction and is such that player 1 has a huge value for winning both X_1, X_2 and almost 0 otherwise and player 3 has huge value for winning any of X_1 or X_2 and almost 0 otherwise. Thus ignoring players 2 and 4 since they have negligent value for X_1, X_2 we observe that the first two-item auction is an example of an AND and an OR bidder that is well known to not have walrasian equilibria and hence pure nash equilibria in the first price item auction.

So we examine what happens after any outcome of the first two-item auction:

- Case 1: Player 1 won both X_1, X_2.

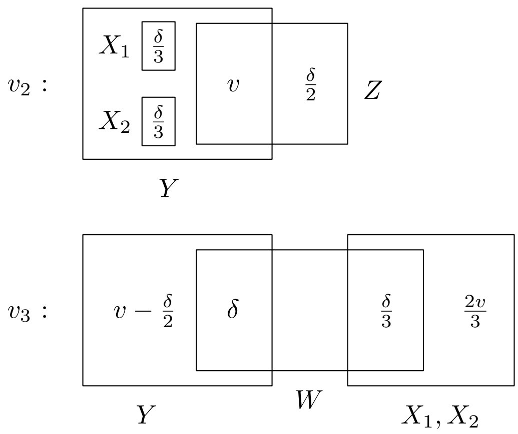

Figure 3: Valuations v_2 and v_3.

In this case player 2 has a value of $v+\frac{2\delta}{3}$ for Y and a value of $v+\delta/2$ for Z given that he loses Y. In the Z auction player 2 will bid v. Hence, player 2 will gain a profit of $\delta/2$ from the Z auction if he loses Y. Moreover, the value of player 3 for W is $\delta + \delta/3$.

  - Case 1a: Player 3 won W. In this case the value of 3 for Y is $v - \delta/2$. Hence, the game played at the Y auction is the following (we ignore player 4):

$$[v_i^j] = \begin{bmatrix} 0 & v-\frac{\delta}{2} & 0 \\ \delta/2 & v+\frac{2\delta}{3} & \delta/2 \\ 0 & 0 & v-\delta/2 \end{bmatrix}$$

    Thus player 2 wants to win for a price of at most $v+\frac{2\delta}{3}-\frac{\delta}{2}$. Player 3 will bid $v-\delta/2$ and player 2 will win. In the last auction player 2 will just bid $\delta/2$. Hence, player 1 will get utility $v-\delta/2$, player 2 utility $\frac{\delta}{2}+\frac{2\delta}{3}$ and player 3 utility 0.

  - Case 1b: Player 3 lost W. In this case the value of 3 for Y is $v+\delta/2$ and the game played is:

$$[v_i^j] = \begin{bmatrix} 0 & v-\frac{\delta}{2} & 0 \\ \delta/2 & v+\frac{2\delta}{3} & \delta/2 \\ 0 & 0 & v+\delta/2 \end{bmatrix}$$

    Thus player 3 now wants to win for a value at most $v+\delta/2$ and player 2 for a value at most $v+\frac{2\delta}{3}-\frac{\delta}{2}$. Hence, in the unique no-overbidding equilibrium player 3 will win. Therefore, player 1 will get utility 0,player 2 utility $\delta/2$ and player 3 utility $\frac{\delta}{3}$.

Thus we see that at auction W the following game

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is played:

$$[v_i^j] = \begin{bmatrix} 0 & 0 & v - \frac{\delta}{2} & 0 \\ \frac{\delta}{2} & \frac{\delta}{2} & \frac{\delta}{2} + \frac{2\delta}{3} & \frac{\delta}{2} \\ \frac{\delta}{3} & \frac{\delta}{3} & \delta + \frac{\delta}{3} & \frac{\delta}{3} \\ 0 & 0 & 0 & \frac{2\delta}{3} + \epsilon \end{bmatrix}$$

Players 1 and 2 will bid 0 and player 4 wants to win for at most $\frac{2\delta}{3} + \epsilon$. Player 3 wants to win for at most δ. Hence, in the unique equilibrium player 3 will win W. Consequently, player 2 will win Y and player 1 will win Z. Thus, player 1 will get utility $v - \frac{\delta}{2}$, player 2 utility $\frac{2\delta}{3}$, player 3 utility $\frac{2\delta}{3} - \epsilon$ and player 4 utility 0.

- Case 2: Player 3 won at least one of X_1 or X_2.

  In this case player 2 has a value of at least $v + \frac{\delta}{3}$ and at most $v + \frac{2\delta}{3}$ for Y and a value of $v + \delta/2$ for Z given that he loses Y. In the Z auction player 2 will bid v. Hence, player 2 will gain a profit of $\delta/2$ from the Z auction if he loses Y. Moreover, the value of player 3 for W is δ.

  - Case 2a: Player 3 won W. In this case the value of 3 for Y is $v - \delta/2$. Hence, the game played at the Y auction is the following (we ignore player 4):

    $$[v_i^j] = \begin{bmatrix} 0 & v - \frac{\delta}{2} & 0 \\ \delta/2 & v + \frac{\delta}{3} \text{ or } v + \frac{2\delta}{3} & \delta/2 \\ 0 & 0 & v - \delta/2 \end{bmatrix}$$

    Thus player 2 wants to win for a price of at most $v + \frac{\delta}{3} - \frac{\delta}{2}$. Player 3 will bid $v - \delta/2$ and player 2 will win. In the last auction player 2 will just bid $\delta/2$. Hence, player 1 will get utility $v - \delta/2$, player 2 utility $\geq \frac{\delta}{2} + \frac{\delta}{3}$ and player 3 utility 0.

  - Case 2b: Player 3 lost W. In this case the value of 3 for Y is $v + \delta/2$ and the game played is:

    $$[v_i^j] = \begin{bmatrix} 0 & v - \frac{\delta}{2} & 0 \\ \delta/2 & v + \frac{\delta}{3} \text{ or } v + \frac{2\delta}{3} & \delta/2 \\ 0 & 0 & v + \delta/2 \end{bmatrix}$$

    Thus player 3 now wants to win for a value at most $v + \delta/2$ and player 2 for a value at most $v + \frac{2\delta}{3} - \frac{\delta}{2}$. Hence, in the unique no-overbidding equilibrium player 3 will win. Therefore, player 1 will get utility 0, player 2 utility $\delta/2$ and player 3 utility at least $\frac{\delta}{3}$.

    Thus we see that at auction W the following game is played:

    $$\begin{bmatrix} 0 & 0 & v - \frac{\delta}{2} & 0 \\ \frac{\delta}{2} & \frac{\delta}{2} & \frac{\delta}{2} + \frac{\delta}{3} \text{ or } \frac{\delta}{2} + \frac{2\delta}{3} & \frac{\delta}{2} \\ \frac{\delta}{3} \text{ or } \frac{2\delta}{3} & \frac{\delta}{3} \text{ or } \frac{2\delta}{3} & \delta & \frac{\delta}{3} \text{ or } \frac{2\delta}{3} \\ 0 & 0 & 0 & \frac{2\delta}{3} + \epsilon \end{bmatrix}$$

    Players 1 and 2 will bid 0 and player 4 wants to win for at most $\frac{2\delta}{3} + \epsilon$. Player 3 wants to win for at most $\delta - \frac{\delta}{3}$. Hence, in the unique equilibrium player 4 will win W. Consequently, player 3 will win Y and player 2 will win Z. Thus, player 1 will get utility 0, player 2 utility $\delta/2$, player 3 utility at least $\frac{\delta}{3}$ and at most $\frac{2\delta}{3}$ and player 4 utility at least ϵ and at most $\frac{\delta}{3} + \epsilon$ (according to whether player 2 won one of X_1, X_2 or not).

- Case 3: Player 3 didn't win any of X_1, X_2 and player 2 won some of X_1, X_2.

  In this case we just need to observe that 2 expects a profit of at most $\delta/2$ from Z hence he will set a price of at least $v - \delta/2$ at the Y auction. Thus player 3 expects to get utility at most 2δ from the Y and W auctions.

Now we examine the existence of equilibrium in the two-item auction. Both players 2 and 4 get utilities at most 2δ from the Y, W and Z auctions and have at most δ value for X_1 and X_2. Thus they will bid at most 3δ. On the other hand player 1 has a utility of $v - \delta/2$ from subsequent auctions if he wins both items and utility 0 if player 3 wins some of them. Moreover, player 3 has a utility at most 2δ from subsequent auctions in any outcome, but has a value of $2v/3 + \frac{\delta}{3}$ for winning some of X_1 or X_2. Hence, player 3 is willing to win some of X_1 or X_2 at a price of $2v/3 - 2\delta$. Since we assume that $\delta \to 0$ we can ignore players 2 and 4 in the first auction.

If player 1 wins both items and both at a price smaller than $2v/3 - 2\delta$ then player 3 has a profitable deviation to bid higher than that at one auction and outbid 1. Thus if player 1 wins both items he must be paying at least $4v/3 - 4\delta$ which is much more than the utility he receives. Hence, this cannot happen.

Thus player 3 must be winning some auction. If that is true then player 2 receives 0 utility in any possible outcome and since he has no direct value for X_1 or X_2 he doesn't won to win any of the auctions. Moreover, if player 3 bids more than 2δ in both auctions and wins both auctions then he has a profitable deviation to bid 0 in one of them since given that he wins one item his marginal valuation for the second is 0. Thus in equilibrium player 3 will bid less than 2δ in some of the two auctions. Moreover, he is bidding at most $2v/3 + 2\delta$

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in the auction he is winning. However, in that case player 1 has a profitable deviation of marginally outbidding player 3 in both auctions. Hence, player 3 winning some auction cannot happen either at equilibrium and therefore no pure nash equilibrium can exist in the first round.

D Second vs First price in Sequential Auctions

In order to stress how essential is the design decision of adopting first price instead of second price in the sequential auctions[1], we present two examples that show how sequential second price auction fail to provide any welfare guarantee even for elementary valuations. It is important to notice that this happens even though we restrict ourselves to equilibria where no player overbids in any game induced in a node of the game-tree. The second example is even stronger: even if we restrict out attention to equilibria that remain after iterative elimination of weakly dominated strategies of all induced games, still no welfare guarantee is possible.

D.1 Additive Valuations Consider a sequential auction of m items among n players using a sequential second price auction, where each player has additive valuation $v_i : 2^{[m]} \to \mathbb{R}_+$, i.e., $v_i(S) = \sum_{j \in S} v_j(\{j\})$. It is tempting to believe that this is equivalent to m independent Vickrey auctions. Using SPE as a solution concept, however, allows the possibility of signaling.

Consider the following example with 3 players, where the Price of Anarchy is infinite, which happens due to a miscoordination of the players. Consider $t+2$ items $\{A_1, \ldots, A_t, B, C\}$ and valuations given by the following table:

	A_1	A_2	$\ldots$	A_t	B	C
1	1	1	$\ldots$	1	0	1
2	$1-\epsilon$	$1-\epsilon$	$\ldots$	$1-\epsilon$	1	$1-\epsilon$
3	δ	δ	$\ldots$	δ	$1-\epsilon$	0

Now, notice that in each subtree, it is an equilibrium if everyone plays truthfully in the entire subtree and notice that under this players get only very small utility. Now, consider the outcome where player 3 gets items $A_1 \ldots A_t$, player 2 gets item B and player 1 gets item C. This outcome has social welfare $SW = 2 + t\epsilon$ while $Opt = t + 2$. Now we argue that there is an SPE that produces this outcome, showing therefoe that the Price of Anarchy is unbounded.

In the game tree, in the path corresponding to the equilibrium described above, consider the winner bidding truthfully and all other bidding zero. Now, in all other decision nodes of the tree outside that path, let everyone bid truthfully. It is easy to check that this is a SPE according to the definition above.

Notice that this is a feature of second price. For example, in the last auction, player 3 couldn't have gotten this item for free in the first price version, since player 2 would have been overbidded him and got it instead. Second price auctions have the bug that a player can win an item for some price p, but some other player to take the item, he may need to pay $p' > p$ and it may make (as in the example) the equilibrium be non-envy free.

D.2 Unit-demand players In this section we present a unit-demand sequential second price instance that exhibits arbitrarily high PoA. The instance we present involves signaling behaviour from the players. Moreover, the second price nature of the auction enables players to signal for a zero price and as much as they want, a combination that has devastating effects on the efficiency.

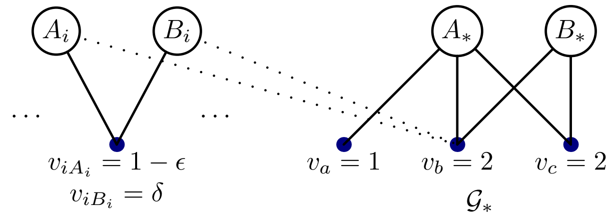

Figure 4: Sequential Second Price instance with high PoA. Auctions happen from left to right. If a bidder is not connected to an item that implies 0 value. If a bidder has a single value then that is his value for any of the items he is connected to. Dashed lines mean 0 value but imply that the bidder might bid at that auction despite the 0 value.

The instance is depicted in Fig. 4 is an auction with $2k+2$ items auctioned in the order $A_1, B_1, A_2, B_2, \ldots, A_k, B_k, A_*, B_*$ and $n+3$ player called $1, 2, \ldots, k, a, b, c$. The main component of the instance is gadget $\mathcal{G}_*$, which comprises of the last two auctions of the game. As a subgame $\mathcal{G}_*$ has two possible subgame perfect equilibria: In the first equilibrium, which we denote SPE$_1$, b wins A_* at price 1 and c wins B_* at price 0. In the second SPE c wins A_* and b wins B_*. Hence, player b's utility is 1 unit higher in SPE$_1$.

In what follows we construct a SPE of the whole instance that survives iterated elimination of weakly dominated strategies and exhibits unbounded price of anarchy. We describe what happens in the last 2 external auctions A_k, B_k. If player b or c win at auction A_k and at 0 price then in last two auctions SPE$_1$ is

[1] or altenatively, how crucial the envy-free assumption in second-price auctions is

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implemented. If player k wins auction A_k then $\textsc{Spe}_2$ is implemented. If player k loses and sets a positive price then if either b or c win at auction B_k then $\textsc{Spe}_1$ is implemented otherwise $\textsc{Spe}_2$. Now using backwards induction we see that if player c has no incentive to bid at any of A_k, B_k. Moreover, if player b wins A_k at any price then at B_k he has a utility of 2 for winning and 1 for losing. Thus at B_k he bids 1 and player k bids δ. Thus, at A_k player b has a utility of $1-\epsilon$ for winning at any price. Hence, he will bid $1-\delta > 1-\epsilon$. Now, player k knows that he is going to lose at A_k, and if he sets a positive price he is going to also lose at B_k. On the other hand if he sets a price of 0 at A_k then none of b, c have any incentive to outbid him on B_k which will give him a utility of δ. Thus, player k will bid 0 on A_k. We can copy this behaviour by adding several auctions A_i, B_i happening before A_*, B_*. At each of these auctions player b is going to be winning auction A_i at a price of 0 and the corresponding player i will be winning auction B_i. This leads to a $\textsc{PoA} = \frac{k(1-\epsilon)+4}{k\delta+4} = O(\frac{1-\epsilon}{\delta})$ which can be arbitrarily high.

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