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e-companion

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Electronic Companion—"Aggregate Diffusion Dynamics in Agent-Based Models with a Spatial Structure" by Gadi Fibich and Ro'i Gibori, *Operations Research*, doi 10.1287/opre.1100.0818.

Appendix

A. Proof of Lemma 10

Since

$$f_{1D} = 1 - e^{\left(- (p+q)t + q\frac{1-e^{-pt}}{p} \right)},$$

the adoption rate in the 1D model is given by

$$\dot{f}_{1D}(t) = (1 - f_{1D})[p + q(1 - e^{-pt})]. \tag{49}$$

Therefore, by equation (27), for any $q > 0$ and $t > 0$,

$$\dot{f}_{1D}(t) < (1 - f_{1D})(p + q f_{1D}). \tag{50}$$

For comparison, the adoption rate in the Bass model is given by

$$\dot{f}_{\text{Bass}}(t) = (1 - f_{\text{Bass}})(p + q f_{\text{Bass}}). \tag{51}$$

Equation (28) follows from explicit integration of equation (50). Indeed, from equation (50) we have that

$$\frac{\dot{f}_{1D}}{(1-f_{1D})(p+qf_{1D})} = \frac{\dot{f}_{1D}}{p+q}\left(\frac{1}{1-f_{1D}} + \frac{q}{p+qf_{1D}}\right) < 1.$$

Taking the integral between 0 and t gives

$$\frac{1}{p+q}\Big(- \ln(1-f_{1D}) + \ln(p+qf_{1D})\Big)\Big|_0^t < t.$$

Since $f_{1D}(0) = 0$, we have that

$$\ln(p + qf_{1D}(t)) - \ln(1 - f_{1D}(t)) - \ln p < t(p+q).$$

Therefore, $p + qf_{1D}(t) < p(1 - f_{1D})e^{t(p+q)}$. Hence,

$$f_{1D}(t) < \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}} = f_{\text{Bass}}(t).$$

B. Proof of equation (35)

Let $f(t)$ be the solution of

$$f(t) = p\int_0^t (1-f(\tau))(1+q(t-\tau))\,d\tau. \tag{52}$$

and let $F(s) = \mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-st}\,dt$ be the Laplace transform of $f(t)$. Equation (52) can be rewritten as

$$f = p(1-f)\star(1+qt),$$

where $\star$ is the Laplace transform convolution. Therefore, if we take the Laplace transform of both sides and use the relation

$$\mathcal{L}(t^{k-1}) = \frac{\Gamma(k)}{s^k}, \qquad k>0,$$

we get that

$$F = p\left(\frac{1}{s} - F\right)\left(\frac{1}{s} + \frac{q}{s^2}\right).$$

Therefore,

$$F = \frac{p}{s}\frac{s+q}{s^2+ps+pq}.$$

In order to transform back, we first rewrite F as

$$F = \frac{p}{s}\frac{s+q}{(s-s_1)(s-s_2)}, \qquad s_{1,2} = \frac{-p\pm\sqrt{p^2-4pq}}{2}. \tag{53}$$

Recall that

$$\mathcal{L}^{-1}\left(\frac{1}{(s-s_1)(s-s_2)}\right) = \frac{1}{s_1-s_2}\left(e^{s_1t}-e^{s_2t}\right), \quad \mathcal{L}^{-1}\left(\frac{s}{(s-s_1)(s-s_2)}\right) = \frac{1}{s_1-s_2}\left(s_1e^{s_1t}-s_2e^{s_2t}\right),$$

and

$$\mathcal{L}^{-1}\left(\frac{1}{s}G\right) = \int_0^t g(\tau)\,d\tau, \qquad G = \mathcal{L}(g).$$

Therefore, transforming equation (53) back gives

$$f = p\int_0^t \frac{1}{s_1-s_2}\Big((s_1+q)e^{s_1\tau} - (s_2+q)e^{s_2\tau}\Big)\,d\tau.$$

Integrating the right-hand-side gives, after some technical calculations, the result.