Adiabatic law for self-focusing of optical beams

Gadi Fibich

Department of Mathematics, University of California, Los Angeles, Los Angeles, California 90095-1555

Received May 23, 1996

An adiabatic approach is used to derive a new law for self-focusing in the nonlinear Schrödinger equation that is valid from the early stages of self-focusing until the blowup point. The adiabatic law leads to an analytical formula for the location of the blowup point and can be used to estimate the effects of various small perturbations on self-focusing. The results of the analysis are confirmed by numerical simulations. © 1996 Optical Society of America

The study of blowup of solutions of the nonlinear Schrödinger equation (NLS)

$$i\psi_z + \Delta_\perp \psi + |\psi|^2\psi = 0, \qquad \psi(0, r) = \psi_0(r) \tag{1}$$

has been ongoing for more than 30 years, since Kelley used Eq. (1) to predict the possibility of catastrophic self-focusing of optical beams.[1] Here $\psi(z, r)$ is the electric field envelope of a laser beam propagating in a medium with Kerr nonlinearity, z is the distance in the direction of the propagation, $r = \sqrt{x^2 + y^2}$ is the radial coordinate, and $\Delta_\perp = \partial^2/\partial r^2 + (1/r)(\partial/\partial r)$ is the Laplacian in the transverse two-dimensional plane. The initial approach to self-focusing analysis was to assume that the solution maintains a Gaussian profile. This approach was successful in predicting the critical power for self-focusing (but only up to a constant), finding the critical dimension for blowup, etc.[2] However, in critical transverse dimension $D = 2$ the Gaussian approximation fails to capture the delicate balance between the focusing nonlinearity and radial dispersion, which increase in magnitude while almost completely canceling each other. Indeed, resolving the local structure of ψ near the blowup point Z_c had long defied research efforts until Fraiman and, independently, Landman *et al.* and LeMesurier *et al.* showed that as the beam approaches Z_c it follows the log–log law[3]

$$L(z) \sim \left[\frac{2\pi(Z_c - z)}{\ln\ln 1/(Z_c - z)}\right]^{1/2}, \tag{2}$$

where L is the beam width and is also inversely proportional to the amplitude $|\psi|$. Although NLS singularity was resolved mathematically, it turned out that the log–log law is still not valid even after the solution has been focused by a factor of 10^{10}. However, the validity of the NLS as a model for beam propagation breaks down much earlier when the field intensity reaches the material breakdown threshold. Even at subthreshold intensities, some terms that have been neglected during the derivation of the NLS from Maxwell's equations (non-paraxial terms,[4,5] time dispersion,[6,7] etc.) may become important. These terms may be small in magnitude yet have a large effect on self-focusing and even lead to its arrest. Therefore there is still a need for a description of NLS self-focusing that is valid in the domain of physical interest and that can be extended to the analysis of small perturbations. In this Letter a new adiabatic law is derived that satisfied both requirements.

Previous studies[3,8] showed that, as the beam propagates forward, it splits into an inner part ψ_s that self-focuses toward the center axis and an outer part $\psi_{\rm nf}$ that diffracts and diverges. Until the beam gets close to the blowup point, self-focusing is a nonadiabatic process in which ψ_s transfers most of its excess power above critical to $\psi_{\rm nf}$ while focusing and approaching the quasi-self-similar form

$$\psi_s(r, z) \sim \frac{V(\xi, \zeta)}{L(z)} \exp\left(i\zeta + i\frac{L_z}{L}\frac{r^2}{4}\right), \tag{3}$$

where $L(z)$, the function to be determined, is used to rescale ψ_s and the independent variables:

$$\xi = \frac{r}{L}, \qquad \frac{{\rm d}\zeta}{{\rm d}z} = \frac{1}{L^2}.$$

From the corresponding equation for V it follows that $V \sim R + O(\beta)$, where $R(\xi)$ is the positive solution of

$$\Delta_\perp R - R + R^3 = 0, \qquad R'(0) = 0, \qquad R(\infty) = 0 \tag{4}$$

and β is the adiabaticity parameter:

$$\beta = -L^3 L_{zz}. \tag{5}$$

During self-focusing $\beta \searrow 0$. Near the blowup point the rate of self-focusing accelerates and the following conditions hold[8]: (i) $0 < \beta \ll 1$. (ii) β is proportional to the excess power of ψ_s above critical:

$$\beta \sim \frac{N_s - N_c}{M}, \qquad N_s = \int |\psi_s|^2 r{\rm d}r, \tag{6}$$

where $N_c = \int R^2 r{\rm d}r \cong 1.86$ is the critical power for self-focusing and $M = (1/4)\int R^2 r^3 {\rm d}r \cong 0.55$. (iii) The Hamiltonian of ψ_s is given by

$$H_s \sim M\left(L_z^2 - \frac{\beta}{L^2}\right),$$

$$H_s = \int |\nabla\psi_s|^2 r{\rm d}r - {}^1\!/\!_2 \int |\psi_s|^4 r{\rm d}r, \qquad \nabla = \partial/\partial r. \tag{7}$$

(iv) Power losses of ψ_s (to $\psi_{\rm nf}$) become exponentially small compared with its focusing rate $[\partial N_s/\partial\zeta \sim$

0146-9592/96/211735-03$10.00/0

$-\exp(-\pi/\sqrt{\beta})]$, indicating that near the blowup point self-focusing is essentially an adiabatic process.

The new approach reported here is to use the dual interpretations for β [Eqs. (5) and (6)] and the multiple-scales method. If we ignore the slow-scale power loss $[\beta_z \sim -\exp(-\pi/\sqrt{\beta})/L^2]$, adiabatic self-focusing follows the fast-scale equation

$$-L^3 L_{zz} = \beta, \qquad \beta \equiv \beta_0 := \beta(0). \tag{8}$$

If we multiply Eq. (8) by $2L_z L^{-3}$, integrate, and use relation (7), we observe that in addition to N_s, H_s is also constant over the fast scale:

$$L_z^2 = \frac{\beta}{L^2} + \frac{H_s}{M}, \qquad H_s \equiv H_s(0).$$

Multiplying by L^2 and integrating one more time lead to the new adiabatic law

$$L = \left[L_0^2 \pm 2\left(\beta + \frac{H_s L_0^2}{M} z\right)_z^{1/2} + \frac{H_s}{M} z^2\right]^{1/2},$$
$$L_0 = L(0). \tag{9}$$

By setting $L = 0$ in Eq. (9) we can get an equation for the blowup point Z_c:

$$Z_c = \begin{cases} \dfrac{L_0^2}{\sqrt{\beta} + (\beta + H_s L_0^2/M)^{1/2}} & L_z(0) \le 0 \\ \dfrac{L_0^2}{\sqrt{\beta} - (\beta + H_s L_0^2/M)^{1/2}} & L_z(0) > 0, \quad H_s < 0 \\ \text{no blowup} & L_z(0) > 0, \quad H_s > 0 \end{cases} \cdot \tag{10}$$

In the case of a collimated beam (ψ_0 real) $L_z(0) = 0$, $H_s(0) \sim -M\beta_0/L_0^2$ [relation (7)], and the pure adiabatic law is

$$L \sim L_0\left(1 - \frac{z^2}{Z_c^2}\right)^{1/2}, \qquad Z_c = \frac{L_0^2}{\sqrt{\beta_0}} \cdot \tag{11}$$

If we add a lens with focal length F at $z = 0$, the initial condition becomes $\tilde{\psi}_0 = \psi_0 \times \exp(-ir^2/4F)$. Therefore $\tilde{L}_0 = L_0$, $\tilde{\beta}_0 \sim \beta_0$, and $\tilde{H}_s(0) \sim H_s(0) + ML_0^2/F^2$ [relations (6) and (7)], where the tildes denote the corresponding parameters for $\tilde{\psi}_0$. Thus the new blowup point is [Eq. (10)]

$$\tilde{Z}_c = \frac{L_0^2}{\sqrt{\beta_0} + L_0^2/F} \cdot$$

Note that $1/\tilde{Z}_c = 1/Z_c + 1/F$, in agreement with Talanov's lens transformation property for the NLS.[9]

The adiabatic law [Eq. (9)] can be rewritten in the form

$$L \sim \left[2\sqrt{\beta}(Z_c - z) + \frac{H_s}{M}(Z_c - z)^2\right]^{1/2}. \tag{12}$$

As z approaches the singularity point the quadratic term becomes negligible, and Eq. (12) reduces to Malkin's law[8]:

$$L \sim \left[2\sqrt{\beta}(Z_c - z)\right]^{1/2}. \tag{13}$$

Thus laws (12) and (13) agree asymptotically but Eq. (12) becomes valid earlier, because in addition to beam power it also incorporates the initial focusing angle. Likewise, the log–log law can be derived as the asymptotic limit of Eq. (13).[8] Therefore the three laws are not in disagreement; only their domains of validity differ.

Note that all adiabatic relations [(9)–(13)] are only $O(\beta)$ accurate owing to the approximations [(6) and (7)] used in their derivation. To maintain this accuracy in Eq. (12) or (13) one must include the slow-scale (nonadiabatic) changes in β and H_s.

Although Eq. (11) was derived under the assumptions that ψ_s has approached its asymptotic form [Eq. (3)] and $\beta \ll 1$, we can try to extrapolate it to predict Z_c for general initial conditions. We determine the value of β from relation (6) with $N_s \sim \int |\psi_0|^2$ and that of L_0 by matching $\psi_0(r) \sim R_{L_0} := L_0^{-1} R(r/L_0)$. For example, if we impose $\int |\nabla\psi_0|^2 = \int |\nabla R_{L_0}|^2$, then $L_0 = (N_c/\int |\nabla\psi_0|^2)^{1/2}$ and

$$Z_c \sim \left(\frac{MN_c}{p-1}\right)^{1/2} \Big/ \int |\nabla\psi_0|^2 r\mathrm{d}r, \qquad p = \frac{\int |\psi_0|^2 r\mathrm{d}r}{N_c} \cdot \tag{14}$$

In the simulations Eq. (1) was solved by the method of dynamic rescaling[10] and β was evaluated by relation (6). In Fig. 1 we plot the relative error in the prediction for L based on the new adiabatic law [Eq. (12)], Malkin's adiabatic law [Eq. (13)], and the log–log law [Eq. (2)]. The initial condition is $\psi_0 = 1.02R(r)$, whose power is 4% above critical and whose profile is close to the asymptotic one [Eq. (3)]. The two adiabatic laws become $O(\beta)$ accurate and agree asymptotically, with Eq. (12) accurate from the beginning and Eq. (13) after focusing by a factor of 10. After focusing by 100,000 β has decreased only by 30%, and the log–log law is still not valid. Note that, if we add a focusing lens term $[\psi_0 = 1.02 \exp(-ir^2/4F)R(r)]$, only the new adiabatic law will maintain the same accuracy. In Fig. 2 we compare pure adiabatic self-focusing [Eq. (11)] and almost adiabatic self-focusing [Eq. (12) with the slowly varying $\beta(z)$ and $H_s(z)$ and $Z_c - z$ from the numerics] for the same initial condition. Whereas both Eqs. (11) and (12) are in reasonable agreement with the numerical

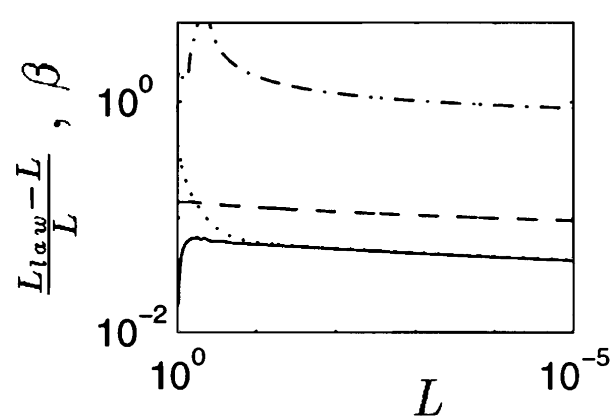

Fig. 1. Relative error in L based on the new adiabatic law [Eq. (12); solid curve], Malkin's law [Eq. (13); dotted curve], the log–log law [relation (2); dashed–dotted curve], and β [relation (6); dashed curve]. The initial condition is $\psi_0 = 1.02R(r)$.

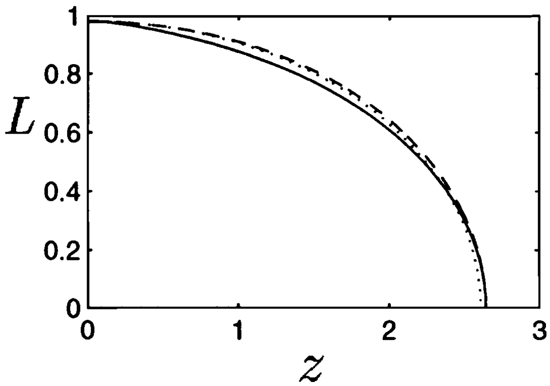

Fig. 2. Comparison of the pure adiabatic law [Eq. (11); dotted curve] and the adiabatic law [Eq. (12); dashed curve] with the numerical solution of the NLS (solid curve). The initial condition is $\psi_0 = 1.02R(r)$.

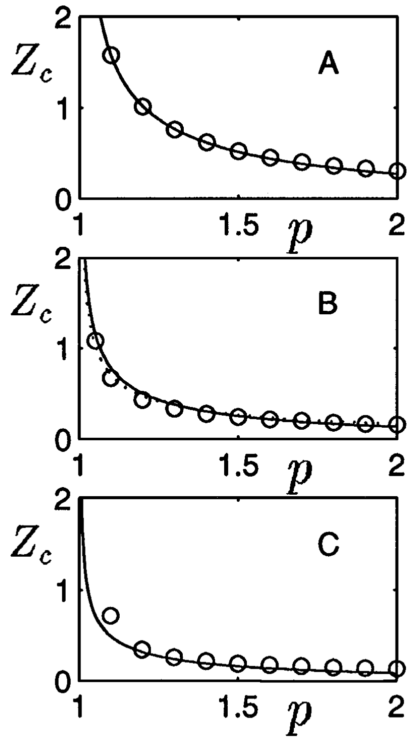

Fig. 3. Location of blowup point Z_c as a function of beam power p according to adiabatic theory [relation (14); solid curve] and numerical simulations (circles) for A, $cR(r)$; B, $c\ \exp(-r^2)$ [the dotted curve is Eq. (15)]; and C, $c\ \exp(-r^4)$.

solution in the prefocal region, only Eq. (12) maintains $O(\beta)$ accuracy near the focal point (Fig. 1). In Fig. 3 the adiabatic predictions for Z_c [Eq. (14)] are compared with simulations for the following initial conditions: Fig. 3A, $\psi_0 = cR(r)$; Fig. 3B, $\psi_0 = c\ \exp(-r^2)$; and Fig. 3C, $\psi_0 = c\ \exp(-r^4)$, where c is varied so that $1 < p \leq 2$. Naturally, the best agreement is in Fig. 3A, where ψ_0 is closest to the asymptotic profile. However, even in Figs. 3B and 3C the agreement is quite good, considering that we have neglected nonadiabatic changes, that ψ_0 is not close to Eq. (3), and that the excess power above critical is not small. As Fig. 3B indicates, the accuracy of the formula of Dawes and Marburger,[11]

$$Z_c = 0.367[(p^{1/2} - 0.852)^2 - 0.0219]^{-1/2}, \qquad (15)$$

and of relation (14) is of comparable magnitude. However, Eq. (15) is valid only for the special case of Gaussian initial conditions and was derived by curve fitting values of Z_c obtained from simulations.

The adiabatic approach can be extended to analyze the effects of small perturbations on self-focusing.[9] For example, it was recently shown[5] that nonparaxial effects become important and lead to the arrest of self-focusing when $a/\lambda = O(\sqrt{N_c/M\beta_0}/4\pi)$, where a and λ are the pulse radius and wavelength, respectively. Because simulations of Eq. (1) suggest that typically in the adiabatic regime $\beta = O(0.1)$, self-focusing arrest that is due to nonparaxiality will occur when $a \sim \lambda/2$. One can use a similar approach to analyze at which point small normal time dispersion will affect self-focusing by combining the results of this Letter and Ref. 6. Therefore for given initial conditions it is possible to determine which of these two mechanisms will be the first to affect self-focusing.

The author thanks G. Papanicolaou and W. Morokoff for useful discussions. This study has been supported in part by National Science Foundation grant DMS-9623087.

References

1. P. Kelley, Phys. Rev. Lett. **15,** 1005 (1965).
2. S. Akhmanov, A. Sukhorukov, and R. Khokhlov, JETP **23,** 1025 (1966); F. Cooper, C. Lucheroni, and H. Shepard, Phys. Lett. A **70,** 184 (1992).
3. G. Fraiman, Sov. Phys. JETP **61,** 228 (1985); M. Landman, G. Papanicolaou, C. Sulem, and P. Sulem, Phys. Rev. A **38,** 3837 (1988); B. LeMesurier, G. Papanicolaou, C. Sulem, and P. Sulem, Physica D **32,** 210 (1988).
4. M. Feit and J. Fleck, J. Opt. Soc. Am. B **5,** 633 (1988).
5. G. Fibich, Phys. Rev. Lett. **76,** 4356 (1996).
6. G. Fibich, V. Malkin, and G. Papanicolaou, Phys. Rev. A **52,** 4218 (1995).
7. J. E. Rothenberg, Opt. Lett. **17,** 1340 (1992); G. Luther, A. Newell, and J. Moloney, Physica D **74,** 59 (1994).
8. V. Malkin, Physica D **64,** 251 (1993).
9. V. Talanov, JETP Lett. **11,** 199 (1970).
10. D. McLaughlin, G. Papanicolaou, C. Sulem, and P. Sulem, Phys. Rev. A **34,** 1200 (1986).
11. E. Dawes and J. Marburger, Phys. Rev. **179,** 862 (1969); J. Marburger, Prog. Quantum Electron. **4,** 35 (1975).