LARGE ASYMMETRIC FIRST-PRICE AUCTIONS — A BOUNDARY-LAYER APPROACH SUPPLEMENTARY MATERIAL

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High order analysis: $\bar{w}(b) \equiv 0$. Let us consider an expansion of the form

$$v_i(b) = b + \frac{u(b)}{n-1} + \frac{w_i(b)}{(n-1)^2} + \frac{y_i(b)}{(n-1)^3} + O\left(\frac{1}{n^4}\right), \qquad i = 1, \dots, n,$$

where $u = \frac{F_G}{f_G}$ and $w_i = u^2 \frac{f_i}{F_i} - u + \bar{w}(b)$. Then,

$$\frac{1}{v_i(b) - b} = \frac{1}{\frac{u(b)}{n-1} + \frac{w_i(b)}{(n-1)^2} + \frac{y_i(b)}{(n-1)^3} + O\left(\frac{1}{n^4}\right)} = \frac{n-1}{u} - \frac{w_i}{u^2} + \frac{1}{n-1} \frac{w_i^2 - u y_i}{u^3} + O\left(\frac{1}{n^2}\right).$$

Thus,

$$\begin{aligned} &\frac{1}{n-1} \sum \frac{1}{v_j(b) - b} - \frac{1}{v_i(b) - b} = \\ &\frac{1}{u} - \frac{1}{u^2} \left(\frac{1}{n-1} \sum w_j - w_i \right) \\ &+ \frac{1}{n-1} \left[\frac{1}{u^3} \left(\frac{1}{n-1} \sum w_j^2 - w_i^2 \right) - \frac{1}{u^2} \left(\frac{1}{n-1} \sum y_j - y_i \right) \right] = \\ &\frac{f_i}{F_i} + \frac{1}{n-1} \frac{\bar{w}(b)}{u^2} + \frac{1}{n-1} \left[u \left\langle \frac{f_i^2}{F_i^2} \right\rangle_i - 2 \left(1 - \frac{\bar{w}}{u}\right) \left\langle \frac{f_i}{F_i} \right\rangle_i - \frac{1}{u^2} \langle y_i \rangle_i \right], \end{aligned} \tag{A}$$

where we denote $\langle u_i \rangle_i := \frac{1}{n-1} \sum_{j=1}^n u_j - u_i$, and where in the last equality used

$$\frac{1}{n-1} \sum w_j - w_i = -u^2 \frac{f_i}{F_i} + u - \frac{\bar{w}(b)}{n-1}.$$

In addition,

$$w_i^2 = u^4 \frac{f_i^2}{F_i^2} - 2u^2 (u - \bar{w}) \frac{f_i}{F_i} + (u - \bar{w})^2.$$

hence,

$$\frac{1}{n-1} \sum w_j^2 - w_i^2 = u^4 \left\langle \frac{f_i^2}{F_i^2} \right\rangle_i - 2u^2 (u - \bar{w}) \left\langle \frac{f_i}{F_i} \right\rangle_i + O\left(\frac{1}{n}\right).$$

Substituting (A) into (1.1a) yields

$$\begin{aligned} &1 + \frac{u'}{n-1} = \\ &\left(\frac{F_i}{f_i} + \frac{u}{n-1} \left(\frac{F_i}{f_i} \right)' \right) \left(\frac{f_i}{F_i} + \frac{1}{n-1} \left[u \left\langle \frac{f_i^2}{F_i^2} \right\rangle_i - 2 \left(1 - \frac{\bar{w}}{u}\right) \left\langle \frac{f_i}{F_i} \right\rangle_i - \frac{1}{u^2} \langle y_i \rangle_i + \frac{\bar{w}}{u^2} \right] \right). \end{aligned}$$

The $O\left(\frac{1}{n}\right)$ terms are

$$u'(b) = \left(\frac{F_i}{f_i} \right)' \frac{f_i}{F_i} u(b) + \frac{F_i}{f_i}(b) \left[u \left\langle \frac{f_i^2}{F_i^2} \right\rangle_i - 2 \left(1 - \frac{\bar{w}}{u}\right) \left\langle \frac{f_i}{F_i} \right\rangle_i - \frac{1}{u^2} \langle y_i \rangle_i + \frac{\bar{w}(b)}{u^2} \right].$$

Summing over i yields

$$\bar{w}(b) = \frac{u^2}{n} \sum_{i=1}^{n} \left[\frac{f_i}{F_i} u'(b) - \left(\frac{F_i}{f_i} \right)' \frac{f_i^2}{F_i^2} u(b) \right].$$

Since $f_G = \frac{1}{n} F_G \sum_{i=1}^{n} \frac{f_i}{F_i}$ and $f_G' = \frac{f_G^2}{F_G} + \frac{1}{n} F_G \sum_{i=1}^{n} \left(\frac{f_i'}{F_i} - \frac{f_i^2}{F_i^2} \right)$, then

$$\bar{w}(b) = u^2 u' \frac{1}{n} \sum_{i=1}^{n} \frac{f_i}{F_i} - u^3 \frac{1}{n} \sum_{i=1}^{n} \left(\frac{f_i^2}{F_i^2} - \frac{f_i'}{F_i} \right) = u^2 u' \frac{f_G}{F_G} - u^3 \left(\frac{f_G^2}{F_G^2} - \frac{f_G'}{F_G} \right).$$

Therefore, since $u := \frac{F_G}{f_G}$,

$$\bar{w}(b) = \frac{F_G^2}{f_G^2} \left(1 - \frac{F_G f_G'}{f_G^2} \right) \frac{f_G}{F_G} - \frac{F_G^3}{f_G^3} \left(\frac{f_G^2}{F_G^2} - \frac{f_G'}{F_G} \right) = 0.$$