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LARGE ASYMMETRIC FIRST-PRICE AUCTIONS— A BOUNDARY-LAYER APPROACH*

GADI FIBICH† AND NIR GAVISH‡

Abstract. The inverse equilibrium bidding strategies $\{v_i(b)\}_{i=1}^n$ in a first-price auction with n asymmetric bidders, where v_i is the value of bidder i and b is the bid, are solutions of a system of n first-order ordinary differential equations, with $2n$ boundary conditions and a free boundary on the right. In this study we show that when the number of bidders is large ($n \gg 1$), this problem has a boundary-layer structure with several nonstandard features: (1) The small parameter does not multiply the highest-order derivative. (2) The number of equations goes to infinity as the small parameter goes to zero. (3) The boundary-layer structure is for the derivatives $\{v_i'(b)\}_{i=1}^n$ but not for $\{v_i(b)\}_{i=1}^n$. (4) In the boundary-layer region, the solution is the sum of an outer solution in the original variable and an inner solution in the rescaled boundary-layer variable. Using boundary-layer theory, we compute an $O(1/n^3)$ uniform approximation for $\{v_i(b)\}_{i=1}^n$. The accuracy of the boundary-layer approximation is confirmed numerically, for both moderate and large values of n.

Key words. asymmetric auctions, first-price auctions, boundary value problems, boundary-layer theory, singular perturbations, simulations, backward shooting

AMS subject classifications. 91B26, 34B15, 34E15

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1. Introduction.

Auction is an important economic mechanism, which is central to the modern economy. For example, in 2013 the US treasury auctioned securities in a total sum of 7.9 trillion dollars. Google makes most of its profits by selling sponsored links via online auctions. The first systematic analysis of auctions was done in 1961 by Vickrey [15]. Since then, auctions have been the subject of an intense study. For an introduction to auction theory, see, e.g., [8, 9].

In this study we analyze a boundary value problem that arises in the study of asymmetric first-price private-value auctions, in which n risk-neutral bidders compete for a single object. Each of the bidders submits his bid in a closed envelope; the highest bidder wins the object and pays his bid, while all other bidders pay nothing. In this case, the inverse equilibrium bidding strategies $\{v_i(b)\}_{i=1}^n$ are the solutions of [10, 12]

$$(1.1a)\qquad v_i'(b) = \frac{F_i(v_i(b))}{f_i(v_i(b))}\left[\left(\frac{1}{n-1}\sum_{j=1}^{n}\frac{1}{v_j(b)-b}\right) - \frac{1}{v_i(b)-b}\right], \qquad i=1,\ldots,n,$$

for $0 < b < \bar{b}$, subject to the n left boundary conditions

$$(1.1b)\qquad v_i(0) = 0, \qquad i = 1,\ldots,n,$$

and the n right boundary conditions

$$(1.1c)\qquad v_i(\bar{b}) = 1, \qquad i = 1,\ldots,n.$$

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†Department of Applied Mathematics, Tel Aviv University, Ramat Aviv, Tel Aviv, 69978 Israel (fibich@tau.ac.il).
‡Department of Mathematics, Technion, 923 Amado Bldg., Haifa 32000, Israel (ngavish@tx.technion.ac.il).

Here $F_i(v_i)$, the cumulative distribution function (CDF) of the private value of bidder i, is a twice differentiable function that satisfies

$$F_i(0) = 0, \qquad F_i(1) = 1, \qquad f_i(v_i) := F_i'(v_i) \geq 0. \tag{1.2}$$

Equation (1.1) is a free boundary value problem, since the location $\bar{b}$ of the right boundary is unknown. This problem is nonstandard, because

1. the n first-order ODEs are subject to $2n$ boundary conditions;
2. by (1.1b) and (1.2), the right-hand side of (1.1a) at $b = 0$ is of the form "0/0." Hence, one cannot apply the standard local existence and uniqueness theorem to the initial value problem (1.1a)–(1.1b).

Consequently, there is little understanding of (1.1). This is in contrast with the symmetric case of identical bidders ($F_i = F$ for $i = 1, \ldots, n$), where the equation for $v_i = v_{\mathrm{sym}}(b)$ can be solved explicitly (see section 2.1) and so "everything" is known. Indeed, most of auction theory concerns the symmetric case not because bidders are believed to be symmetric, but simply because the analysis of asymmetric auctions is much harder.

Existence and uniqueness of solutions of (1.1) were proved by Maskin and Riley [12] and by Lebrun [10]. In [1], Fibich and Gavious analyzed (1.1) in the case of a weak asymmetry ($F_i = F + \varepsilon H_i$, where $\varepsilon \ll 1$), by using perturbation analysis to expand $v_i(b)$ in a power series in ε about the known symmetric solution $v_{\mathrm{sym}}(b)$. In [5], Fibich and Gavious considered the special case where $F_i = v^{\alpha_i}$ for $i = 1, \ldots, n$, since in that case (1.1) can be transformed into an autonomous dynamical system (section 7). This transformation revealed that the nonuniqueness at $b = 0$ is "equivalent" to a saddle point with an $(n-1)$-dimensional unstable manifold, and the original solution corresponds to a trajectory that exits from the saddle point. Over the years, asymmetric first-price auctions were also studied numerically (section 8). Nevertheless, despite all the research effort, the present understanding of the solutions of (1.1) is limited, and so the effect of bidders' asymmetry remains unclear.

In [16], Wilson showed that $\lim_{n\to\infty} v_i(b) = b$ under quite general conditions. In this study we show that when the number of asymmetric bidders is large ($n \gg 1$), we can exploit the presence of the small parameter $1/(n-1)$ in (1.1a) to go beyond this limiting result. Specifically, we obtain $O(1/n^3)$ explicit approximations of the solutions of (1.1). Since our approximations are already valid when n is moderately large, and not only in the limit as $n \to \infty$, they are relevant to real-life first-price auctions.

1.1. Nonstandard boundary-layer problem. The paper is organized as follows. In section 2 we present the mathematical model for asymmetric first-price auctions. In section 2.1 we review the symmetric case, where explicit solutions are available. In section 3 we derive some technical results that will be used later on.

In section 4 we use regular perturbations to compute a solution of (1.1) of the form

$$v_i(b) = v_i^{\mathrm{outer}}(b) + O\left(\frac{1}{n^3}\right), \qquad v_i^{\mathrm{outer}}(b) = b + \frac{1}{n-1} u_i(b) + \frac{1}{(n-1)^2} w_i(b).$$

This computation yields $u_i = F_G/f_G$ and $w_i = (F_G/f_G)^2 \frac{f_i}{F_i} - F_G/f_G$, where F_G is the geometric average of $\{F_i\}_{i=1}^n$ and $f_G = F_G'$. Therefore, we conclude that (1) the asymmetry among $\{v_i\}_{i=1}^n$ is only $O(n^{-2})$, (2) the correct averaging of asymmetric bidders is the geometric one, and (3) under this averaging, one can "replace" asymmetric bidders with symmetric ones with $O(n^{-2})$ accuracy.

Since the approximate solution v_i^{outer} satisfies the ODEs (1.1a), the boundary conditions (1.1b), and the boundary conditions (1.1c), all to $O(n^{-2})$, it seems that the problem is "solved." Surprisingly, however, if one first substitutes the boundary conditions (1.1c) into the ODEs (1.1a) and then substitutes $v = v_i^{\text{outer}}$, then v_i^{outer} does not satisfy these equations at $\bar{b}$, even to leading order! In section 5.1 we resolve this "anomaly" by showing that there is a narrow boundary layer near the right boundary $\bar{b}$, in which v_i "bifurcates" from v_i^{outer}. A priori, this suggests that, as is the case in boundary-layer theory, the solution in the boundary-layer is of the form $v_i \sim v_i^{\text{bl}}(\xi)$, where $\xi = (\bar{b} - b)/\delta$ is the rescaled boundary-layer variable and δ is the width of the boundary layer. It turns out, however, that this is not the case. Rather, the solution in the boundary layer has the nonstandard form of being the *sum* of an outer solution in the original variable and a small inner solution in the rescaled variable; i.e.,

$$v_i^{\text{bl}} = v_i^{\text{outer}}(b) + \delta \cdot v_i^{\text{inner}}(\xi), \qquad \xi = \frac{\bar{b} - b}{\delta}.$$

Therefore, $v_i^{\text{bl}} - v_i^{\text{outer}} = O(\delta)$ but $(v_i^{\text{bl}})' - (v_i^{\text{outer}})' = O(1)$, thus resolving the "anomaly" observed earlier. In other words, the boundary layer is for v_i' and not for v_i. Another difference from standard boundary-layer problems is that the small parameter $1/(n-1)$ does not multiply the highest-order derivative. Therefore, it is not a priori clear why there should be a boundary layer.

Further analysis shows that the boundary layer width is $\delta = 1/(n-1)^2$. The inner solution satisfies a constant-coefficient eigenvalue problem (section 5.2), and its coefficients are determined from the boundary conditions at the right boundary and from matching of the boundary-layer solution v_i^{bl} with the solution v_i^{outer} outside the boundary layer (section 5.3). Summarizing the results, we obtain explicit $O(1/n^3)$ approximations for $\{v_i(b)\}_{i=1}^n$ and for $\bar{b}$ (section 5.4).

The $O(1/n^3)$ accuracy of these approximations is confirmed numerically in section 6, even for moderate levels of n. In section 7 we view the boundary-layer analysis from a dynamical systems perspective. In particular, we show that the outer solution is equivalent to the saddle-point trajectory, and the boundary-layer solution corresponds to the trajectory that leaves the saddle point. In section 8 we consider numerical methods for solving (1.1). In particular, we utilize the boundary-layer analysis to gain further insight on the instability of the popular backward shooting method. Section 9 concludes with some final remarks.

1.2. Level of rigor. For lack of a better terminology, the asymptotic analysis results in this paper are referred to as "lemmas" and "propositions," and their derivations are referred to as "proofs." We stress, however, that these results are not rigorous, since we did not prove that the order of magnitude of the error terms is the one which we obtained using formal asymptotic analysis. Nevertheless, the agreement between the asymptotic results and the numerical simulations provides strong support for the correctness of these results.

2. Model formulation. Consider n risk-neutral bidders (players) that compete in a first-price auction, in which the highest bidder wins the object and pays his bid while all other bidders pay nothing. Let us denote by v_i the valuation of the ith bidder for the object (roughly speaking, this is the maximal price that a rational bidder is willing to pay for the object). We assume that v_i is private information to bidder i, and that v_i is drawn independently according to a twice continuously differentiable

CDF $F_i(v_i)$, whose support $[0,1]$ is common to all bidders; see (1.2). We also denote by $f_i = F_i'$ the corresponding density function.

Let $b_i = b_i(v_i)$ be the bid function of bidder i in equilibrium. Since the equilibrium bids are strictly monotonic [12], we can define the inverse equilibrium bid functions $v_i = v_i(b_i)$. The first-order condition for equilibrium yields (1.1a).[1] Since a bidder with zero valuation will not submit a positive bid, we have (1.1b). In addition, in equilibrium all bidders with the highest valuation (i.e., $v_i = 1$) place the same (unknown) maximal bid, denoted by $\overline{b}$; see [10, 12].[2] Therefore, we have (1.1c).

2.1. Symmetric bidders. When bidders are symmetric (i.e., $F_i = F$ for $i = 1, \dots, n$), the inverse bid functions are identical; i.e., $v_i(b) = v_{\mathrm{sym}}(b)$ for all i. Since the right boundary condition (1.1c) is automatically satisfied, the boundary value problem (1.1) reduces to the initial value problem

$$v_{\mathrm{sym}}'(b) = \frac{[F(v_{\mathrm{sym}}(b))]}{(n-1)f(v_{\mathrm{sym}}(b))} \frac{1}{v_{\mathrm{sym}}(b) - b}, \qquad v_{\mathrm{sym}}(0) = 0.$$

This equation can be easily solved [15], yielding

$$b_{\mathrm{sym}}(v) = v - \frac{1}{F^{n-1}(v)} \int_0^v F^{n-1}(s)\, ds. \tag{2.1}$$

3. Notation and preliminary technical lemmas. Two matrices play an important role in the analysis of (1.1). The first is

$$\mathbf{B}_0 := \frac{1}{n-1} \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & -1 & \ddots & \vdots \\ -1 & \cdots & 1 & n-1 \end{bmatrix}. \tag{3.1}$$

Lemma 3.1. *The matrix $\mathbf{B}_0$ has the positive eigenvalue $\lambda_i = n/(n-1)$ with multiplicity $n-1$, whose corresponding eigenvectors $\mathbf{u}_i$ are spanned by the $(n-1)$-dimensional space of vectors $\mathbf{u} = (u_1, u_2, \dots, u_n)$ for which $\sum u_i = 0$ and by the eigenvalue $\lambda_n = 0$ with the corresponding eigenvector $\mathbf{u}_n = \mathbf{1}/\|\mathbf{1}\|$, where $\mathbf{1} := (1, \dots, 1)^T$.*

Lemma 3.2. *Consider the equation*

$$\mathbf{B}_0 \mathbf{w} = \mathbf{d}, \tag{3.2}$$

where $\mathbf{w} := (w_1, \dots, w_n)^T$ and $\mathbf{d} := (d_1, \dots, d_n)^T$. If $\sum_{i=1}^n d_i = 0$, the general solution of (3.2) is

$$\mathbf{w} = \frac{n-1}{n}\mathbf{d} + c\mathbf{1}, \tag{3.3}$$

where c is an arbitrary constant.

Proof. By Lemma 3.1, $\mathbf{d}$ is an eigenvector of $\mathbf{B}_0$ with eigenvalue $\lambda = n/(n-1)$, and $\mathbf{1}$ spans the null space of $\mathbf{B}_0$. ◻

The second matrix which is central to the analysis is

$$\mathbf{B} := \mathbf{B}_0 - \frac{1}{n-1}\mathbf{I} = \frac{1}{n-1} \begin{bmatrix} n-2 & -1 & \cdots & -1 \\ -1 & n-2 & \cdots & -1 \\ \vdots & -1 & \ddots & \vdots \\ -1 & \cdots & -1 & n-2 \end{bmatrix}. \tag{3.4}$$

[1] See, e.g., [1] for a derivation.

[2] An elementary proof of this condition is given in [6, 13].

LEMMA 3.3. *The matrix* $\mathbf{B}$ *has the positive eigenvalue* $\lambda = 1$ *with multiplicity* $n-1$*, whose corresponding eigenvectors* $\mathbf{u}_i$ *are spanned by the* $(n-1)$*-dimensional space of vectors* $\mathbf{u} = (u_1, u_2, \dots, u_n)$ *for which* $\sum u_i = 0$ *and the negative eigenvalue* $\lambda_n = -\frac{1}{n-1}$ *with the corresponding eigenvector* $\mathbf{u}_n = \mathbf{1}/\|\mathbf{1}\|$*.*

Proof. This follows directly from Lemma 3.1. ∎

An important role will be played by the geometric mean of the CDFs, as follows.

LEMMA 3.4. *Let*

$$F_G(v) := \left(\prod_{i=1}^{n} F_i(v)\right)^{1/n}, \qquad f_G(v) := F_G'(v). \tag{3.5a}$$

Then

(3.5b)

$$f_G(v) = \frac{1}{n} F_G \sum_{i=1}^{n} \frac{f_i(v)}{F_i(v)}, \qquad f_G'(v) = \frac{f_G^2(v)}{F_G(v)} + \frac{1}{n} F_G(v) \sum_{i=1}^{n} \left(\frac{f_i'(v)}{F_i(v)} - \frac{f_i^2(v)}{F_i^2(v)} \right).$$

In particular,

$$f_G(1) = \frac{1}{n} \sum_{i=1}^{n} f_i(1), \qquad f_G'(1) = f_G^2(1) + \frac{1}{n} \sum_{i=1}^{n} \left(f_i'(1) - f_i^2(1) \right). \tag{3.5c}$$

4. Analysis of large auctions—Naïve (outer) solution. As the number of bidders increases, the competition between bidders becomes more intense, and therefore they have to increase their bids in order to maintain their chances of winning the object. Consequently $\lim_{n\to\infty} v_i(b) = b$; see [16]. Motivated by the asymptotic expansion in the symmetric case (see [2]), we look for a solution of (1.1a) in a power series in $\frac{1}{n-1}$, i.e.,

$$v_i = b + \frac{1}{n-1} u_i(b) + \frac{1}{(n-1)^2} w_i(b) + O\left(\frac{1}{n^3}\right), \qquad i = 1, \dots, n. \tag{4.1}$$

PROPOSITION 4.1. *Let* $\{v_i\}_{i=1}^n$ *be a solution of* (1.1a) *of the form* (4.1)*. Then* $v_i = v_i^{\text{outer}} + O\left(\frac{1}{n^3}\right)$*, where*

$$v_i^{\text{outer}}(b) = b + \frac{u(b)}{n-1} + \frac{w_i(b)}{(n-1)^2}, \qquad u = \frac{F_G}{f_G}, \quad w_i = u^2 \frac{f_i}{F_i} - u. \tag{4.2}$$

Proof. Substituting (4.1) into the large brackets expression in (1.1a) gives

$$\begin{aligned}
&\left[\left(\frac{1}{n-1} \sum_{j=1}^{n} \frac{1}{v_j(b) - b} \right) - \frac{1}{v_i(b) - b} \right] \\
&= \frac{1}{n-1} \sum_{j=1}^{n} \left[\frac{n-1}{u_j} - \frac{w_j}{u_j^2} + O\left(\frac{1}{n}\right) \right] - \left[\frac{n-1}{u_i} - \frac{w_i}{u_i^2} + O\left(\frac{1}{n}\right) \right] \\
&= (n-1) \left[\frac{1}{n-1} \sum_{j=1}^{n} \frac{1}{u_j} - \frac{1}{u_i} \right] - \left[\frac{1}{n-1} \sum_{j=1}^{n} \frac{w_j}{u_j^2} - \frac{w_i}{u_i^2} \right] + O\left(\frac{1}{n}\right).
\end{aligned}$$

Therefore, substitution of (4.1) into (1.1a) gives

$$1+O\left(\frac{1}{n}\right)= \tag{4.3}$$

$$\left[\frac{F_i(b)}{f_i(b)}+O\left(\frac{1}{n}\right)\right]\left[(n-1)\left[\frac{1}{n-1}\sum_{j=1}^{n}\frac{1}{u_j}-\frac{1}{u_i}\right]-\left[\frac{1}{n-1}\sum_{j=1}^{n}\frac{w_j}{u_j^2}-\frac{w_i}{u_i^2}\right]+O\left(\frac{1}{n}\right)\right].$$

A priori, to leading order the left-hand side is $O(1)$, whereas the right-hand side is $O(n)$. Therefore, we first impose the condition that the right-hand side also be $O(1)$. This implies that

$$\frac{1}{n-1}\sum_{j=1}^{n}\frac{1}{u_j}-\frac{1}{u_i}=O\left(\frac{1}{n}\right), \qquad i=1,\dots,n, \tag{4.4}$$

or

$$\frac{1}{u_i(b)}=\frac{1}{n-1}\sum_{j=1}^{n}\frac{1}{u_j(b)}+O\left(\frac{1}{n}\right).$$

The $O\left(\frac{1}{n}\right)$ difference among $\{u_i\}_{i=1}^n$ can be absorbed in $\{w_i\}_{i=1}^n$. Therefore,

$$u_i(b)=u(b), \qquad i=1,\dots,n.$$

Substituting this into (4.3) gives

$$1=\frac{F_i(b)}{f_i(b)}\left(\frac{1}{u(b)}-\frac{1}{u^2(b)}\left[\frac{1}{n-1}\sum_{j=1}^{n}w_j-w_i\right]\right)+O\left(\frac{1}{n}\right), \qquad i=1,\dots,n. \tag{4.5}$$

Summing (4.5) over i gives

$$\sum_{i=1}^{n}\frac{f_i(b)}{F_i(b)}=\frac{n}{u(b)}+O(1).$$

Hence, by (3.5b),

$$\frac{1}{u}=\frac{1}{n}\sum_{i=1}^{n}\frac{f_i(b)}{F_i(b)}=\frac{f_G(b)}{F_G(b)}. \tag{4.6}$$

Furthermore, (4.5) and (4.6) imply that

$$w_i-\frac{1}{n-1}\sum_{j=1}^{n}w_j=u^2\frac{f_i}{F_i}-u+O\left(\frac{1}{n}\right).$$

The $O\left(\frac{1}{n}\right)$ error can be absorbed into the next-order terms in the expansion. Therefore, the equations for $\{w_i\}_{i=1}^n$ read

$$\mathbf{B}_0\mathbf{w}=\mathbf{d}, \qquad d_i=u^2\frac{f_i}{F_i}-u. \tag{4.7}$$

Since $\sum_{i=1}^{n} d_i = 0$ (see (3.5b) and (4.2)), the general solution of (4.7) is (see Lemma 3.2)

$$\mathbf{w} = \frac{n-1}{n}\mathbf{d} + \bar{w}(b)\mathbf{1},$$

where $\bar{w}(b)$ is an arbitrary function. Hence,

$$w_i = u^2 \frac{f_i}{F_i} - u(b) + \bar{w}(b) + O\left(\frac{1}{n}\right). \tag{4.8}$$

Carrying the expansion to the next order yields $\bar{w}(b) \equiv 0$ (see the supplementary materials for this paper). Therefore, the result follows. ◻

We now compute $\bar{b}$ by imposing the right boundary condition, as follows.

COROLLARY 4.2. *Assume that* $v_i = v_i^{\text{outer}} + O\left(\frac{1}{n^3}\right)$*; see* (4.2). *Then*

$$\bar{b} = 1 - \frac{1}{n-1}\frac{1}{f_G(1)} + O\left(\frac{1}{n^2}\right). \tag{4.9}$$

Proof. Substituting $v_i^{\text{outer}}(\bar{b}) = 1$ gives $1 = \bar{b} + \frac{1}{n-1}\frac{F_G(\bar{b})}{f_G(\bar{b})} + O\left(\frac{1}{n^2}\right)$. Therefore, $\bar{b} = 1 + O\left(\frac{1}{n}\right)$. Hence, $\frac{F_G(\bar{b})}{f_G(\bar{b})} = \frac{1}{f_G(1)}\left(1 + O\left(\frac{1}{n}\right)\right)$, from which the result follows. ◻

4.1. Inconsistencies. If we carry the expansion of $v_i^{\text{outer}}(\bar{b}) = 1 + O\left(\frac{1}{n^3}\right)$ to the next order, we get

$$1 = v_i(\bar{b}) = \bar{b} + \frac{1}{n-1}u(\bar{b}) + \frac{1}{(n-1)^2}w_i(\bar{b}) + O\left(\frac{1}{n^3}\right).$$

This yields

$$\bar{b} = 1 - \frac{1}{n-1}\frac{1}{f_G(1)} - \frac{1}{(n-1)^2}\left[\frac{f_i(1)}{f_G^2(1)} - \frac{1}{f_G(1)}\right] + O\left(\frac{1}{n^3}\right). \tag{4.10}$$

Since $\bar{b}$ depends on i, v_i^{outer} does not satisfy the right boundary condition (1.1c). Note, however, that this inconsistency comes only from the $O\left(\frac{1}{n^2}\right)$ terms. This suggests, therefore, that the accuracy of v_i^{outer} is $O\left(\frac{1}{n^2}\right)$. In fact, the situation is "much worse," since the approximation $v \approx v_i^{\text{outer}}$ leads to $O(1)$ inconsistencies at $\bar{b}$, as shown next.

LEMMA 4.3. *If there exist* $1 \le i, j \le n$ *such that* $f_i(1) \neq f_j(1)$*, then* v_i^{outer} *does not satisfy* (1.1) *at* $b = \bar{b}$*, even to leading order.*

Proof. Substituting $b = \bar{b}$ into (1.1a) gives

$$v_i'(\bar{b}) = \frac{F_i(v_i(\bar{b}))}{f_i(v_i(\bar{b}))}\left[\left(\frac{1}{n-1}\sum_{j=1}^{n}\frac{1}{v_j(\bar{b}) - \bar{b}}\right) - \frac{1}{v_i(\bar{b}) - \bar{b}}\right], \qquad i = 1, \ldots, n. \tag{4.11}$$

Substituting $v_i(\bar{b}) = 1$, one obtains

$$v_i'(\bar{b}) = \frac{c}{f_i(1)}, \qquad c = \frac{1}{(n-1)}\frac{1}{1-\bar{b}}. \tag{4.12}$$

On the other hand, by (4.2),

$$(v_i^{\text{outer}})'(\bar{b}) = 1 + O\left(\frac{1}{n}\right). \tag{4.13}$$

Therefore, if v_i^{outer} satisfies (1.1) at $b = \bar{b}$ to leading order, then by (4.12) and (4.13),

$$f_i(1)\left(1 + O\left(\frac{1}{n}\right)\right) = \frac{1}{(n-1)(1-\bar{b})}.$$

Hence, $f_i(1) - f_j(1) = O\left(\frac{1}{n}\right)(f_i(1) - f_j(1))$, which is impossible. ∎

This result is surprising, since v_i^{outer} does satisfy (1.1a) to leading order. Note, however, that the inconsistency arises only after we impose the boundary condition at $\bar{b}$; i.e., v_i^{outer} satisfies (4.11) to leading order, but it does not satisfy (4.12) to leading order.[3]

5. A nonstandard boundary-layer solution. We thus have a nonstandard situation, where a straightforward expansion leads to the approximate solution v_i^{outer} (see (4.2)), which satisfies the equation and each of the boundary conditions by itself to $O\left(\frac{1}{n^2}\right)$. However, surprisingly, if one first substitutes the boundary conditions (1.1c) into the ODEs (1.1a) and then substitutes $v = v_i^{\text{outer}}$, then v_i^{outer} does not satisfy these equations at $\bar{b}$, even to leading order. Since the inconsistencies arise only at $\bar{b}$, this suggests that v_i' has a boundary layer near $\bar{b}$ where it undergoes fast changes from $v_i'(b) \approx 1$ (see (4.2)) to $v_i'(\bar{b}) = \frac{1}{(n-1)}\frac{1}{f_i(1)}\frac{1}{1-\bar{b}} \approx \frac{f_G(1)}{f_i(1)}$ (see (4.9) and (4.12)).

In standard boundary-layer problems, there is a small parameter that multiplies the highest-order derivative in the equation, and the outer solution is found by setting this parameter to zero. This is not the case in (1.1), where the small parameter $\frac{1}{n-1}$ does not multiply the highest-order derivative in (1.1a). Indeed, it is not obvious at all from inspection of (1.1a) that the solution should have a boundary layer! In addition, in standard boundary-layer problems, the outer solution satisfies the ODE and one boundary condition but does not satisfy the second boundary condition even to leading order, which is not the case here.

5.1. Boundary-layer thickness. The solution in the boundary-layer region is usually found by rescaling the independent variable by the boundary-layer thickness. In the case of (1.1), this means looking for a solution of the form $v_i \sim v_i^{\text{inner}}\left(\frac{\bar{b}-b}{\delta}\right)$, where $\delta = \delta(n)$ is the boundary-layer thickness. Because of the nonstandard form of (1.1), however, substitution of this form does not lead to a consistent solution of (1.1). Indeed, the most difficult part of this research has been to find the solution form in the boundary layer. It turns out that the solution in the boundary layer is the *sum* of the outer solution in the original variable and a boundary-layer component in the rescaled variable, i.e.,

$$v_i(b) = v_i^{\text{bl}}(b) + O\left(\frac{1}{n^3}\right), \quad v_i^{\text{bl}}(b) = v_i^{\text{outer}}(b) + \gamma(n) v_i^{\text{inner}}(\xi), \quad \xi = \frac{\bar{b}-b}{\delta(n)}, \tag{5.1}$$

where v_i^{outer} is given by (4.2), δ is the boundary-layer thickness, and γ is the magnitude of the boundary-layer component.

LEMMA 5.1. $\delta(n) = \gamma(n) = \frac{1}{(n-1)^2}$.

Proof. By (4.9) and (4.12),[4]

$$v_i'(\bar{b}) = \frac{1}{(n-1)}\frac{1}{f_i(1)}\frac{1}{1-\bar{b}} = \frac{f_G(1)}{f_i(1)} + O\left(\frac{1}{n}\right).$$

[3] These inconsistencies will be resolved in section 5.1.

[4] Here we assume that we can use estimate (4.9) for $\bar{b}$, which was derived by approximating v_i with v_i^{outer}. An a posteriori justification for this approximation is that because $\delta(n) = (n-1)^{-2}$, the effect of the boundary-layer component on $\bar{b}$ is only $O(n^{-2})$.

On the other hand, by (4.2) and (5.1),

$$v_i'(\bar{b}) = \frac{d}{db} v_i^{\text{outer}}\bigg|_{b=\bar{b}} - \frac{\gamma(n)}{\delta(n)} \frac{d}{d\xi} v_i^{\text{inner}}\bigg|_{\xi=0} = 1 - \frac{\gamma(n)}{\delta(n)} \frac{d}{d\xi} v_i^{\text{inner}}\bigg|_{\xi=0} + O\left(\frac{1}{n}\right).$$

Therefore,

$$1 - \frac{f_G(1)}{f_i(1)} = \frac{\gamma(n)}{\delta(n)} \frac{d}{d\xi} v_i^{\text{inner}}\bigg|_{\xi=0} + O\left(\frac{1}{n}\right). \tag{5.2}$$

Since $1 - \frac{f_G(1)}{f_i(1)}$ and $\frac{d}{d\xi} v_i^{\text{inner}}\big|_{\xi=0}$ are both $O(1)$, it follows that $\gamma(n) = \delta(n)$.

Substituting this into (5.1) gives

$$v_i^{\text{bl}}(b) = v_i^{\text{outer}}(b) + \delta(n) v_i^{\text{inner}}(\xi), \qquad \xi = \frac{\bar{b} - b}{\delta(n)}.$$

We now show that $\delta(n) = (n-1)^{-2}$ by ruling out all other possibilities.

- Assume that $\delta(n) \ll \frac{1}{(n-1)^2}$. Then $v_i = v_i^{\text{outer}}(b) + O\left(\frac{1}{n^2}\right)$. Therefore, $\bar{b}$ depends on i; see (4.10). This is a contradiction.
- Assume that $\frac{1}{(n-1)^2} \ll \delta(n) \ll 1$. By (4.2),

$$v_i(b) - b = \frac{u(b)}{n-1} + \delta(n) v_i^{\text{inner}}(\xi) + \frac{w_i}{n^2} + o\left(\frac{1}{n^2}\right). \tag{5.3}$$

Therefore, as in the derivation of (4.3), substituting (5.3) into (1.1a) yields

$$\begin{aligned} &1 - \frac{d}{d\xi} v_i^{\text{inner}} + O\left(\frac{1}{n}\right) \\ &= \frac{F_i(b)}{f_i(b)} \left[(n-1) \left[\frac{1}{n-1} \sum_{j=1}^{n} \frac{1}{u + \frac{\delta(n)}{n-1} v_j^{\text{inner}}(\xi) + \frac{w_j}{n}} - \frac{1}{u + \frac{\delta(n)}{n-1} v_i^{\text{inner}} + \frac{w_i}{n}} \right] + O(1) \right]. \end{aligned}$$

As in the proof of Proposition 4.1, the requirement that both sides be of $O(1)$ yields

$$\frac{1}{u + (n-1)\delta(n) v_i^{\text{inner}} + \frac{w_i}{n^2}} - \frac{1}{n-1} \sum_{j=1}^{n} \frac{1}{u + (n-1)\delta(n) v_j^{\text{inner}}(\xi) + \frac{w_i}{n}} = O\left(\frac{1}{n}\right).$$

Thus, for $j \neq i$,

$$\frac{1}{u + (n-1)\delta(n) v_j^{\text{inner}} + \frac{w_i}{n}} - \frac{1}{u + (n-1)\delta(n) v_i^{\text{inner}} + \frac{w_i}{n}} = O\left(\frac{1}{n}\right).$$

Now, since $(n-1)\delta(n) \gg 1/n$,

$$\begin{aligned} &\frac{1}{u + (n-1)\delta(n) v_j^{\text{inner}} + \frac{w_i}{n}} - \frac{1}{u + (n-1)\delta(n) v_i^{\text{inner}} + \frac{w_i}{n}} \\ &\sim \frac{1}{u + (n-1)\delta(n) v_j^{\text{inner}}} - \frac{1}{u + (n-1)\delta(n) v_i^{\text{inner}}} \\ &= \frac{(n-1)\delta(n)[v_i^{\text{inner}} - v_j^{\text{inner}}]}{(u + (n-1)\delta(n) v_i^{\text{inner}})(u + (n-1)\delta(n) v_j^{\text{inner}})} = O\left(\frac{1}{n}\right). \end{aligned}$$

Since $v_i^{\text{inner}} - v_j^{\text{inner}} = O(1)$ in the boundary layer,[5] we have that

$$\frac{(n-1)\delta(n)}{1+(n-1)^2\delta^2(n)} = O\left(\frac{1}{n}\right).$$

This equation is satisfied only for $\delta(n) = 1$ or $\delta(n) = \frac{1}{(n-1)^2}$, a contradiction. ∎

By Lemma 5.1, the solution of (1.1) is of the form

$$v_i(b) = v_i^{\text{bl}}(b) + O\left(\frac{1}{n^3}\right), \tag{5.4a}$$

where

$$v_i^{\text{bl}}(b) = v_i^{\text{outer}}(b) + \frac{v_i^{\text{inner}}(\xi)}{(n-1)^2} = b + \frac{1}{n-1}u(b) + \frac{w_i(b) + v_i^{\text{inner}}(\xi)}{(n-1)^2}, \tag{5.4b}$$

$u = F_G/f_G$, $w_i = u^2 \frac{f_i}{F_i} - u$, and $\xi = (n-1)^2(\bar{b} - b)$ is the boundary-layer variable. Note that $\bar{b} = \bar{b}(n)$.

Expression (5.4) resolves the inconsistencies encountered in section 4.1. Indeed, since

$$v_i^{\text{bl}}(b) - v_i^{\text{outer}}(b) \sim \frac{v_i^{\text{inner}}(\xi)}{(n-1)^2}, \qquad (v_i^{\text{bl}})'(b) - (v_i^{\text{outer}})'(b) \sim -\frac{d}{d\xi}v_i^{\text{inner}},$$

in the boundary-layer region (i.e., for $\xi = O(1)$ or $0 \le \bar{b} - b = O(n^{-2})$), we have that

$$v_i - v_i^{\text{outer}} = O\left(\frac{1}{n^2}\right), \qquad v_i' - (v_i^{\text{outer}})' = O(1).$$

5.2. Computing the inner solution. We now compute v_i^{inner} by requiring that v_i^{bl} (see (5.4)) satisfies (1.1) to leading order at and near $\bar{b}$, as given next.

LEMMA 5.2. *Let* $\mathbf{v}^{\text{inner}} := [v_1^{\text{inner}}, \dots, v_n^{\text{inner}}]^T$. *Then*

$$\mathbf{v}^{\text{inner}}(\xi) = \mathbf{D}_{\frac{1}{\sqrt{f_i}}} \sum_{j=1}^{n} c_j e^{\lambda_j \xi} \mathbf{u}_j(\xi), \tag{5.5}$$

where

$$\mathbf{D}_{\frac{1}{\sqrt{f_i}}} := \operatorname{diag}\left(\frac{1}{\sqrt{f_1(1)}}, \dots, \frac{1}{\sqrt{f_n(1)}}\right),$$

$\{c_j\}_{j=1}^n$ *are constants, and* $\{\lambda_j\}_{j=1}^n$ *and* $\{\mathbf{u}_j\}_{j=1}^n$ *are the eigenvalues and normalized eigenvectors of the symmetric matrix* $\mathbf{S}$*; i.e.,*

$$\mathbf{S}\mathbf{u}_j = \lambda_j \mathbf{u}_j, \qquad \langle \mathbf{u}_j, \mathbf{u}_j \rangle = 1, \qquad j = 1, \dots, n, \tag{5.6}$$

where

$$\mathbf{S} := \mathbf{D}_{\frac{1}{\sqrt{f_i}}} \left(-f_G^2(1)\mathbf{B}\right) \mathbf{D}_{\frac{1}{\sqrt{f_i}}}, \tag{5.7}$$

[5]Since $\gamma = \delta$, from (5.2) we have that $1 - \frac{f_G(1)}{f_i(1)} = \frac{d}{d\xi}v_i^{\text{inner}}(\xi = 0) + O\left(\frac{1}{n}\right)$. Consequently, if $f_i(1) \neq f_j(1)$, then $\frac{dv_i^{\text{inner}}}{d\xi}(0) - \frac{dv_j^{\text{inner}}}{d\xi}(0) = O(1)$. Hence, $v_i^{\text{inner}}(\xi) - v_j^{\text{inner}}(\xi) = O(1)$ for $\xi = O(1)$.

and $\mathbf{B}$ *is given by* (3.4).

Proof. Substitution of (5.4) into (1.1a) gives (4.5) plus new terms due to v_i^{inner}:

$$1-\frac{d}{d\xi}v_i^{\text{inner}} = \frac{F_i(b)}{f_i(b)}\left(\frac{1}{u(b)} - \frac{1}{u^2(b)}\left[\frac{1}{n-1}\sum_{j=1}^{n}(w_j + v_j^{\text{inner}}) - (w_i + v_i^{\text{inner}})\right]\right)+O\left(\frac{1}{n}\right).$$

Taking the difference of this equation and (4.5) yields

$$\frac{d}{d\xi}v_i^{\text{inner}} = \frac{F_i(b)}{f_i(b)}\frac{1}{u^2(b)}\left[\frac{1}{n-1}\sum_{j=1}^{n}v_j^{\text{inner}} - v_i^{\text{inner}}\right] + O\left(\frac{1}{n}\right).$$

Since in the boundary-layer region $b = \bar{b} + O\left(\frac{1}{n^2}\right) = 1 + O\left(\frac{1}{n}\right)$, then

$$\frac{F_i(b)}{f_i(b)}\frac{1}{u^2(b)} = \frac{f_G^2(1)}{f_i(1)}\left(1 + O\left(\frac{1}{n}\right)\right).$$

Hence, to leading order, $v_i^{\text{inner}}(\xi)$ satisfies

$$\frac{d}{d\xi}v_i^{\text{inner}} = \frac{f_G^2(1)}{f_i(1)}\left[\frac{1}{n-1}\sum_{j=1}^{n}v_j^{\text{inner}} - v_i^{\text{inner}}\right], \qquad i = 1,\dots,n.$$

These equations can be written in matrix form as

(5.8)
$$\frac{d}{d\xi}\mathbf{v}^{\text{inner}} = \mathbf{A}\mathbf{v}^{\text{inner}}, \quad \mathbf{A} = \mathbf{D}_{\frac{1}{f_i}}\left(-f_G^2(1)\mathbf{B}\right), \quad \mathbf{D}_{\frac{1}{f_i}} := \operatorname{diag}\left(\frac{1}{f_1(1)},\dots,\frac{1}{f_n(1)}\right).$$

In order to work with a symmetric matrix, let

(5.9)
$$\mathbf{u}^{\text{inner}} := \mathbf{D}_{\sqrt{f_i}}\mathbf{v}^{\text{inner}}, \qquad \mathbf{D}_{\sqrt{f_i}} := \operatorname{diag}\left(\sqrt{f_1(1)},\dots,\sqrt{f_n(1)}\right).$$

Then

(5.10)
$$\frac{d}{d\xi}\mathbf{u}^{\text{inner}} = \mathbf{S}\,\mathbf{u}^{\text{inner}}.$$

Since $\mathbf{S}$ is symmetric, it has n real eigenvalues and corresponding orthonormal eigenvectors. Therefore, the general solution of (5.10) is $\mathbf{u}^{\text{inner}} = \sum_{j=1}^{n} c_j e^{\lambda_j \xi}\mathbf{u}_j$. Hence, by (5.9), $\mathbf{v}^{\text{inner}}$ is given by (5.5). ∎

5.3. Matching. Expression (5.5) for $\mathbf{v}^{\text{inner}}$ contains the n undetermined coefficient $\{c_j\}_{j=1}^{n}$. These coefficients will be determined from the requirements that the boundary-layer solution $v_i^{\text{bl}}(b)$ (see (5.4)) satisfy the right boundary conditions (1.1c), and that it match with v_i^{outer}. Since $v_i^{\text{bl}}(b)$ is the sum of the outer and inner solutions, the appropriate matching condition is that $v_i^{\text{bl}}(b)$ reduces to $v_i^{\text{outer}}(b)$ as $\xi \to \infty$.[6]

In order to perform the matching, we first prove the following.

LEMMA 5.3. *The matrix* $\mathbf{S}$ *(see* (5.7)*) has* $n-1$ *negative eigenvalues* $\{\lambda_i\}_{i=1}^{n-1}$ *and one positive eigenvalue* $\lambda_n > 0$.

[6]This is different from the standard matching condition $\lim_{\xi\to\infty} v_i^{\text{inner}}(\xi) = \lim_{b\to\bar{b}} v_i^{\text{outer}}(b)$ in boundary-layer theory.

Proof. By Lemma 3.3, the matrix $\mathbf{B}$ has $n-1$ positive eigenvalues and one negative eigenvalue. Since $\mathbf{B}$ is symmetric and

$$\mathbf{S} = \mathbf{P}^T(-\mathbf{B})\mathbf{P}, \qquad \mathbf{P} := f_G(1)\mathbf{D}_{\frac{1}{\sqrt{f_i}}}, \tag{5.11}$$

it follows from Sylvester's law of inertia that $\mathbf{S}$ and $-\mathbf{B}$ have the same number of positive and negative eigenvalues. ∎

LEMMA 5.4. *The matching condition yields* $c_n = 0$.

Proof. Since $\lambda_n > 0$ and $\lambda_j < 0$ for $j = 1, \dots, n-1$, $\lim_{\xi\to\infty} v_i^{\text{bl}}(b,\xi) = v_i^{\text{outer}}(b)$ if and only if $c_n = 0$. ∎

Therefore,

$$\mathbf{v}^{\text{inner}} = \mathbf{D}_{\frac{1}{\sqrt{f_i}}} \sum_{j=1}^{n-1} c_j e^{\lambda_j \xi}\mathbf{u}_j. \tag{5.12}$$

To find $\{c_j\}_{j=1}^{n-1}$, we impose the boundary conditions (1.1c), as follows.

LEMMA 5.5. *Let* v_i *be given by* (5.4). *Then*

$$\bar{b} = 1 - \frac{1}{n-1}\frac{1}{f_G(1)} + \frac{\bar{b}_2}{(n-1)^2} + O\left(\frac{1}{n^3}\right), \tag{5.13a}$$

where

$$\bar{b}_2 = \frac{1}{f_G(1)} - \frac{f_G'(1)}{f_G^3(1)} - w_i(1) - v_i^{\text{inner}}(0), \qquad i = 1, \dots, n. \tag{5.13b}$$

Proof. Substituting $v_i(\bar{b}) = 1$ into (5.4) gives

$$1 = \bar{b} + \frac{1}{n-1}\frac{F_G(\bar{b})}{f_G(\bar{b})} + \frac{w_i(\bar{b}) + v_i^{\text{inner}}(0)}{(n-1)^2} + O\left(\frac{1}{n^3}\right). \tag{5.14}$$

Let $\bar{b} = 1 + \frac{\bar{b}_1}{n-1} + \frac{\bar{b}_2}{(n-1)^2} + O(n^{-3})$. As in Corollary 4.2, the balance of the $O(n^{-1})$ terms yields $b_1 = -1/f_G(1)$. Since

$$\begin{aligned} F_G(\bar{b}) &= F_G(1) + (\bar{b}-1)f_G(1) + O((\bar{b}-1)^2) = 1 + \frac{\bar{b}_1}{n-1}f_G(1) + O\left(\frac{1}{n^2}\right) \\ &= 1 - \frac{1}{n-1} + O\left(\frac{1}{n^2}\right) \end{aligned}$$

and

$$\begin{aligned} f_G(\bar{b}) &= f_G(1) + (\bar{b}-1)f_G'(1) + O((\bar{b}-1)^2) = f_G(1) + \frac{\bar{b}_1}{n-1}f_G'(1) + O\left(\frac{1}{n^2}\right) \\ &= f_G(1)\left(1 - \frac{1}{n-1}\frac{f_G'(1)}{f_G^2(1)} + O\left(\frac{1}{n^2}\right)\right), \end{aligned}$$

then

$$\frac{F_G(\bar{b})}{f_G(\bar{b})} = \frac{1}{f_G(1)}\left(1 - \frac{1}{n-1} + \frac{1}{n-1}\frac{f_G'(1)}{f_G^2(1)}\right) + O\left(\frac{1}{n^2}\right).$$

The balance of the $O((n-1)^{-2})$ terms in (5.14) gives (5.13b). ∎

Before we proceed, we prove an auxiliary result.

LEMMA 5.6. *Let $\mathbf{u}_n$ be the eigenvector of $\mathbf{S}$ that corresponds to the positive eigenvalue λ_n. Then*

$$\mathbf{u}_n = \frac{\sqrt{\mathbf{f_i}}}{\|\sqrt{\mathbf{f_i}}\|}\left(1 + O\left(\frac{1}{n}\right)\right), \qquad \sqrt{\mathbf{f_i}} := (\sqrt{f_i(1)}, \dots, \sqrt{f_n(1)})^T, \tag{5.15}$$

and $\lambda_n = f_G(1)/n + O(n^{-2}) > 0$.

Proof. It is easy to see that the matrix $\mathbf{S}_0 := \mathbf{P}^T(-\mathbf{B}_0)\mathbf{P}$ has the eigenvalue $\lambda_n = 0$ with normalized eigenvector $\mathbf{u}_n^0 = \sqrt{\mathbf{f_i}}/\|\sqrt{\mathbf{f_i}}\|$. Let

$$\mathbf{S}^\varepsilon := \mathbf{P}^T(-(\mathbf{B}_0 - \varepsilon \mathbf{I}))\mathbf{P} = \mathbf{S}_0 + \varepsilon \mathbf{E}, \qquad \mathbf{E} := \mathbf{P}^T\mathbf{P} = f_G^2(1)\mathbf{D}_{\frac{1}{f_i}}.$$

By Rellich's theory for perturbation of eigenvalues of Hermitian matrices [14], the eigenvalue λ_n^ε and eigenvector $\mathbf{u}_n^\varepsilon$ of $\mathbf{S}^\varepsilon$ are analytic in ε. Therefore,

$$\lambda_n^\varepsilon = \varepsilon\lambda_n^1 + O(\varepsilon^2), \qquad \mathbf{u}_n(\varepsilon) = \mathbf{u}_n^0 + \varepsilon \mathbf{u}_n^1 + O(\varepsilon^2).$$

Substituting this expansion into $\mathbf{S}^\varepsilon \mathbf{u}_n(\varepsilon) = \lambda_n(\varepsilon)\mathbf{u}_n(\varepsilon)$ and balancing the $O(\varepsilon)$ terms gives $\mathbf{S}_0\mathbf{u}_n^1 + \mathbf{E}\mathbf{u}_n^0 = \lambda_n^1\mathbf{u}_n^0$. Taking the inner product with $\mathbf{u}_n^0$, recalling that $\mathbf{S}_0\mathbf{u}_n^0 = 0$, and rearranging gives

$$\lambda_n^1 = \langle \mathbf{u}_n^0, \mathbf{E}\mathbf{u}_n^0\rangle = \frac{f_G^2(1)}{\|\sqrt{\mathbf{f_i}}\|^2}\langle \sqrt{\mathbf{f_i}}, \mathbf{D}_{\frac{1}{f_i}}\sqrt{\mathbf{f_i}}\rangle = f_G(1) > 0.$$

Since $\mathbf{S} = \mathbf{S}_{\varepsilon=\frac{1}{n}}$, the result follows. ∎

LEMMA 5.7. *Let $\mathbf{f_i^{3/2}} := (f_1^{3/2}(1), \dots, f_n^{3/2}(1))^T$. Then*

$$c_j = -\frac{1}{f_G^2(1)}\langle \mathbf{f_i^{3/2}}, \mathbf{u}_j\rangle, \qquad j = 1, \dots, n-1.$$

Proof. By (4.2),

$$w_i(1) = \frac{f_i(1)}{f_G^2(1)} - \frac{1}{f_G(1)}.$$

Since $\bar{b}$ is independent of i, so is $w_i(1) + v_i^{\text{inner}}(0)$; see (5.13b). Therefore,

$$v_i^{\text{inner}}(0) = -\frac{f_i(1)}{f_G^2(1)} + \kappa, \qquad i = 1, \dots, n. \tag{5.16a}$$

Multiplying this equation by $\sqrt{f_i(1)}$ and using (5.12), one obtains $\sum_{j=1}^{n-1} c_j(\mathbf{u}_j)_i = -\frac{f_i^{3/2}(1)}{f_G^2(1)} + \kappa\sqrt{f_i(1)}$, or in vectorial form,

$$\sum_{j=1}^{n-1} c_j \mathbf{u}_j = -\frac{\mathbf{f_i^{3/2}}}{f_G^2(1)} + \kappa \mathbf{u}_{\sqrt{f_i}}.$$

Taking the inner product with $\mathbf{u}_n$ and using (5.15) gives

$$\kappa := \frac{1}{f_G^3(1)}\frac{1}{n}\sum_{i=1}^{n} f_i^2(1). \tag{5.16b}$$

Taking the inner product with $\mathbf{u}_j$ and using (5.15) gives c_j. ∎

LEMMA 5.8.

$$\bar{b}_2 = \frac{2}{f_G(1)} - \frac{f_G'(1)}{f_G^3(1)} - \frac{1}{f_G^3(1)}\frac{1}{n}\sum_{i=1}^{n} f_i^2(1) = \frac{2}{f_G(1)} - \frac{1}{f_G^3(1)}\frac{1}{n}\sum_{i=1}^{n} f_i'(1).$$

Proof. By Lemma 5.5,

$$\bar{b}_2 = \frac{1}{f_G(1)} - \frac{f_G'(1)}{f_G^3(1)} - \frac{1}{n}\sum_{i=1}^{n} v_i^{\text{inner}}(0).$$

In addition, by (5.16),

$$\frac{1}{n}\sum_{i=1}^{n} v_i^{\text{inner}}(0) = -\frac{1}{f_G(1)} + \kappa = -\frac{1}{f_G(1)} + \frac{1}{f_G^3(1)}\frac{1}{n}\sum_{i=1}^{n} f_i^2(1).$$

The second expression for $\bar{b}_2$ follows from (3.5c). ∎

5.4. Summary. The results of Proposition 4.1 and section 5 are summarized in the following proposition.

PROPOSITION 5.9. *Let $n \gg 1$. Then the solution of* (1.1) *is given by*

$$v_i(b) = v_i^{\text{bl}}(b) + O\left(\frac{1}{n^3}\right), \qquad 0 \le b \le \bar{b}, \tag{5.17a}$$

where

$$v_i^{\text{bl}}(b) = v_i^{\text{outer}}(b) + \frac{1}{(n-1)^2} v_i^{\text{inner}}(\xi), \tag{5.17b}$$

$$v_i^{\text{outer}}(b) = b + \frac{1}{n-1}\frac{F_G(b)}{f_G(b)} + \frac{1}{(n-1)^2}\left(\frac{F_G^2(b)}{f_G^2(b)}\frac{f_i(b)}{F_i(b)} - \frac{F_G(b)}{f_G(b)}\right); \tag{5.17c}$$

$\xi = (n-1)^2(\bar{b} - b)$*; the maximal bid is given by*

$$\bar{b} = 1 - \frac{1}{n-1}\frac{1}{f_G(1)} + \frac{1}{(n-1)^2}\left(\frac{2}{f_G(1)} - \frac{1}{f_G^3(1)}\frac{1}{n}\sum_{i=1}^{n} f_i'(1)\right) + O\left(\frac{1}{n^3}\right), \tag{5.17d}$$

$$v_i^{\text{inner}}(\xi) = \frac{1}{f_i^{1/2}(1)}\sum_{j=1}^{n-1} c_j e^{\lambda_j \xi}(\mathbf{u}_j)_i; \tag{5.17e}$$

$\{\lambda_j\}_{j=1}^{n-1}$ *and* $\{\mathbf{u}_j\}_{j=1}^{n-1}$ *are the positive eigenvalues and corresponding eigenvectors of* $\mathbf{S}$ *(see* (5.7)*); and*

$$c_j = -\frac{1}{f_G^2(1)}\langle \mathbf{f_i^{3/2}}, \mathbf{u}_j\rangle, \qquad j = 1, \dots, n-1. \tag{5.17f}$$

As noted, the boundary value problem (1.1) consists of n first-order ODEs, $2n$ boundary conditions, and a free boundary. The n ODEs, coupled with the n boundary conditions at $b = 0$, determine a unique outer solution. This suggests that the inner solution is overdetermined, since it has $n + 1$ degrees of freedom (n from the n first-order ODEs and one from the free boundary) and $2n$ constraints (n matching conditions and n boundary conditions at $\bar{b}$). The boundary-layer analysis reveals, however, that the n matching conditions reduce to a single condition ($c_n = 0$).[7]

6. Simulations. We compute solutions of (1.1) using the *boundary value method* (section 8.2). We first consider the power-law distributions

$$F_i = v^{\alpha_i}, \qquad \alpha_i = \frac{1}{2} + 3\frac{i-1}{n-1}, \qquad i = 1, \ldots, n. \tag{6.1}$$

For example, when $n = 5$, then $\{F_i\}_{i=1}^5 = \{v^{1/2}, v^{5/4}, v^2, v^{11/4}, v^{7/2}\}$ (see Figure 1(A)), and when $n = 10$, then $\{F_i\}_{i=1}^{10} = \{v^{1/2}, v^{5/6}, \ldots, v^{19/6}, v^{7/2}\}$. Since α_i are equidistributed in $[0.5, 3.5]$, then $\bar{\alpha} := \frac{1}{n}\sum_{j=1}^n \alpha_j = 2$. Hence, the CDFs vary with n, but their geometric average $F_G = v^{\bar{\alpha}} = v^2$ does not.

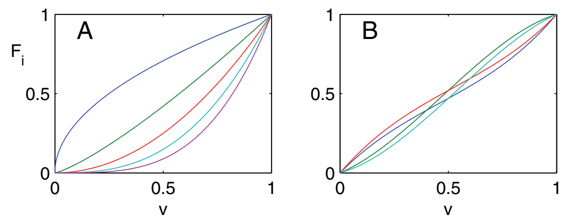

FIG. 1. (A) *The distributions* (6.1) *for* $n = 5$. (B) *The distributions* (6.3).

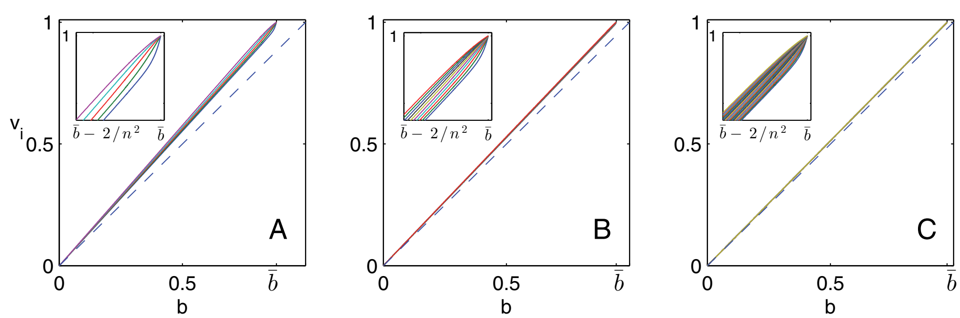

FIG. 2. *Solution of* (1.1)*, where* $\{F_i\}$ *are given by* (6.1). *The dashed line is* $v_i = b$. (A) $n = 5$. (B) $n = 10$. (C) $n = 20$. *Inserts: Zoom-in near* $b = \bar{b}$ *and* $v_i = 1$.

In the case of $n = 5$ players, the five computed curves $\{v_i(b)\}_{i=1}^5$ are nearly indistinguishable (Figure 2(A)). Indeed, by (5.17),

$$v_i(b) - v_k(b) = O\left(\frac{1}{n^2}\right), \qquad 0 \le b \le \bar{b}. \tag{6.2a}$$

The five curves are closer to each other than to $\lim_{n\to\infty} v_i(b) = b$. Indeed, by (5.17),

$$v_i(b) - b = O\left(\frac{1}{n}\right). \tag{6.2b}$$

[7] A similar conclusion follows from the dynamical systems analysis in [5]; see section 7. Indeed, a priori, matching the trajectory that ends at the diagonal (which corresponds to the inner solution) with the saddle-point solution (which corresponds to the outer solution) amounts to n matching conditions. Since the dimension of the unstable manifold at the saddle point is $n - 1$, however, the n matching conditions reduce to a single condition.

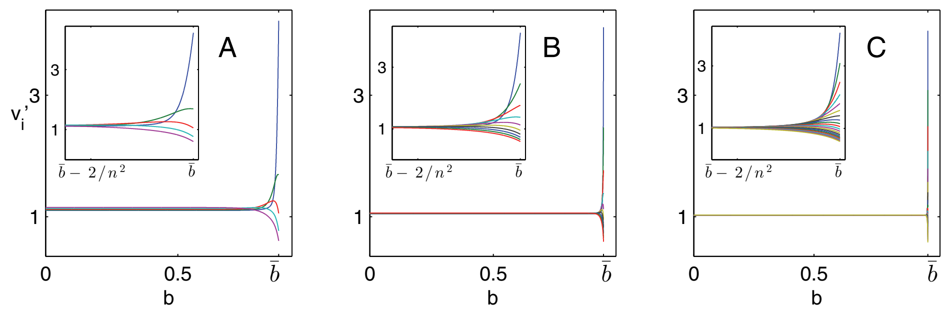

FIG. 3. $\{v_i'\}$ *for the solution of Figure* 2.

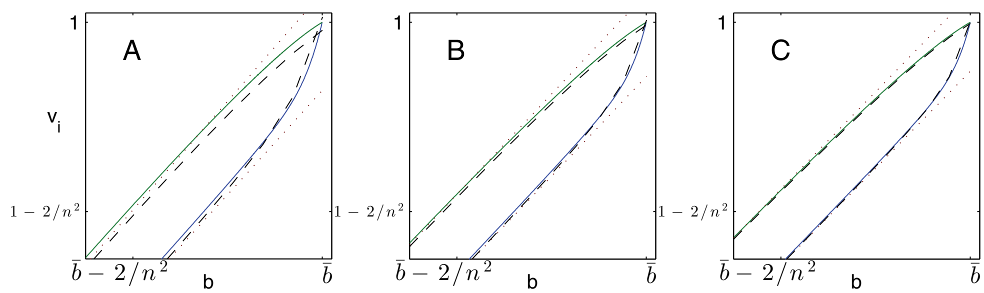

FIG. 4. *The curves* v_1 *(solid, bottom) and* v_n *(solid, top) from Figure* 2*. Also shown are the approximations* v_i^{outer} *(dots; see* (4.2)*) and* v_i^{bl} *(dashes; see* (5.17)*), where* $i = 1, n$*.*

In the inner graph of Figure 2(A) we zoom in near $b = \bar{b}$ and $v_i = 1$ and observe that the five curves are different and well resolved. As we increase the number of bidders to $n = 10$ and then to $n = 20$, the curves get closer to each other, and to a lesser extent to $v_i = b$ (Figure 2(B) and (C)), in agreement with (6.2). The ten (or twenty) curves are well resolved; see the inner graphs of Figure 2(B) and (C).

Figure 2 may give the false impression that "all" the differences among $\{v_i\}_{i=1}^n$ disappear as $n \to \infty$. To see that this is true everywhere but near $\bar{b}$, in Figure 3 we plot the derivatives of the solutions plotted in Figure 2. In all three cases, except near $\bar{b}$, $\{v_i'\}_{i=1}^n$ are nearly indistinguishable from each other and approach 1 as $n \to \infty$. The $\{v_i'\}$ have a *boundary layer* near $\bar{b}$; i.e., they undergo fast changes in a narrow region near $\bar{b}$, whose width decreases to zero as $n \to \infty$. In the inner graphs of Figure 3 we zoom in on an $O(n^{-2})$ region near $\bar{b}$. In all three simulations, $\{v_i'\}$ are still close to each other and to $v_i' \approx 1$ for $b = \bar{b} - \frac{2}{n^2}$, but they have $O(1)$ differences among themselves for $b = \bar{b} - \frac{1}{2n^2}$. This confirms that the boundary-layer width scales as n^{-2}.[8]

In order to confirm numerically the validity of the boundary-layer approximation v_i^{bl} (see (5.17)), in Figure 4 we plot v_1 and v_n from Figure 2 and observe that in the boundary-layer region, v_i^{bl} provides a considerably more accurate approximation than v_i^{outer} (see (4.2)). The advantage of v_i^{bl} over v_i^{outer} in the boundary-layer region is even more dramatic when we compare v_i' with $(v_i^{\text{bl}})'$ and $(v_i^{\text{outer}})'$ in Figure 5.

[8]Figures 2 and 3 show that v_i^{outer} is a uniform asymptotic approximation of v_i, but $(v_i^{\text{outer}})'$ is not a uniform asymptotic approximation of v_i'. See Figures 8 and 9 for a systematic study of this issue.

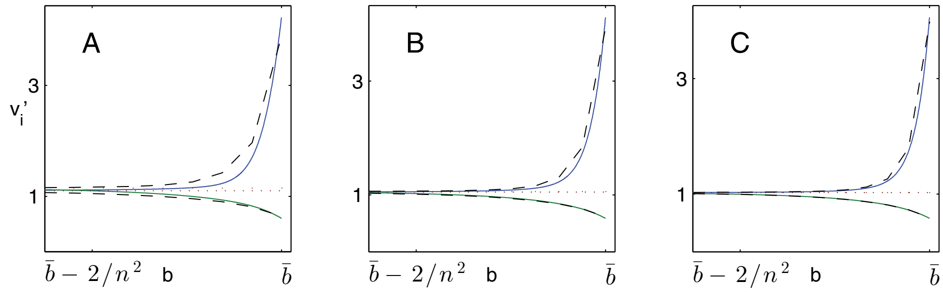

FIG. 5. *Same as Figure 4 for v_1' (solid, top) and v_n' (solid, bottom). Also shown are $(v_i^{\text{outer}})'$ (dots) and $(v_i^{\text{bl}})'$ (dashes).*

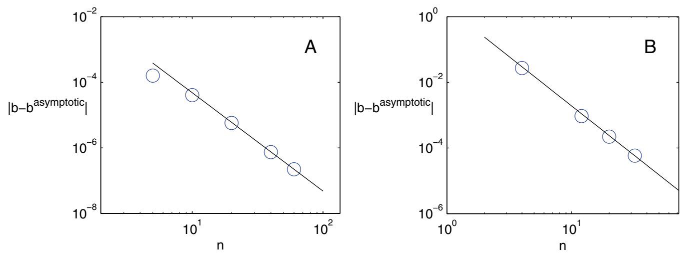

FIG. 6. *The difference between the asymptotic approximation* (5.17d) *and the computed value of $\bar{b}$ (○).* (A) $\{F_i\}$ *given by* (6.1)*, and $n = 5, 10, 20, 40,$ and* 60*. Solid line is* $0.0475/n^3$*.* (B) $\{F_i\}$ *given by* (6.3)*, and $n = 4, 12, 20,$ and* 32*. Solid line is* $2/n^3$*.*

In Figure 6(A) we plot the difference between $\bar{b}^{\text{asymptotic}}$ (see (5.17d)) and the computed value of $\bar{b}$. This difference scales as $|\bar{b} - \bar{b}^{\text{asymptotic}}| \sim 0.0475/n^3$, thus confirming the $O(n^{-3})$ accuracy of $\bar{b}^{\text{asymptotic}}$ (see (5.17d)).

Power-law distributions have some unique properties, such as stochastic dominance and zero crossings. In addition, for the specific choice (6.1) we have that $F_G = v^2$. Therefore, the high-order derivatives, $F_G^{(n)}$, which might affect the high-order terms in the asymptotic expansion, vanish identically. To repeat the simulations with CDFs that do not have these properties, we consider players that are divided into four groups such that each group has $n/4$ players, and each player in the ith group independently draws his value according to the distribution

$$F_i = v + (-1)^{i+1} v(v-1)(v-a_i), \qquad \mathbf{a} = [0.38, 0.42, 0.58, 0.62], \tag{6.3}$$

where $i = 1, \ldots, 4$. These distributions have multiple crossings (see Figure 1(B)), and their geometric average is $F_G = (F_1F_2F_3F_4)^{1/4}$ for all n.

In Figure 7(A) we plot $\{v_i(b)\}_{i=1}^n$ for the case of $n = 4$ players. As predicted by Kirkegaard [7], since the CDFs cross each other, so do the four bidding strategies. As the number of players increases to $n = 12$ and $n = 20$, $\{v_i(b)\}_{i=1}^n$ get closer to each other and to a lesser extent to $\lim_{n\to\infty} v_i(b) = b$ (see Figure 7(B) and (C)), in agreement with (6.2).

Comparison of $\{v_i\}_{i=1}^n$ from Figure 7 with v_i^{bl} and v_i^{outer} yields results similar to those in Figures 4 and 5 (data not shown). To further consider the approximation error, in Figure 8(A) we plot the difference $v_1(b) - v_1^{\text{bl}}(b)$ and observe that it decreases with n. To confirm that $v_i - v_i^{\text{bl}} = O(n^{-3})$ uniformly in $[0, \bar{b}]$ (see (5.17a)), we plot

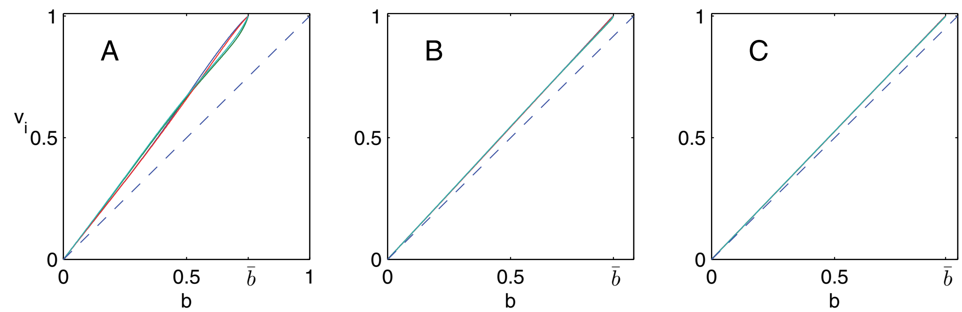

Fig. 7. *Solution of* (1.1) *with* n *players that are divided into four equal groups such that the CDFs of players in each group are given by* (6.3). *The dashed line is* $v_i = b$. (A) $n = 4$. (B) $n = 12$. (C) $n = 20$.

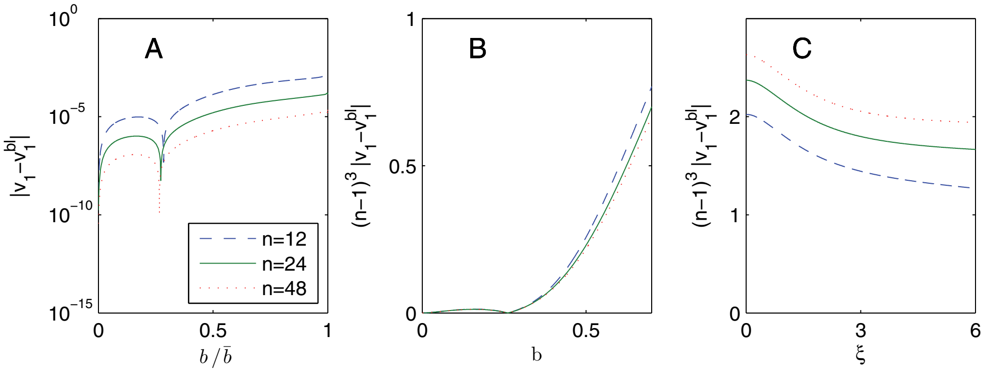

Fig. 8. *Solution of* (1.1) *with* n *players that are divided into four equal groups such that the CDFs of players in each group are given by* (6.3) *and* $n = 12$, 24, *and* 48. (A) *The error* $|v_1 - v_1^{\mathrm{bl}}|$. (B) *The scaled error* $(n-1)^3|v_1 - (v_1)^{\mathrm{bl}}|$ *in the outer region* $0 \le b \le 0.7$. (C) *The scaled error* $(n-1)^3|v_1 - (v_1)^{\mathrm{bl}}|$ *in the boundary layer region* $0 \le \xi \le 6$ *(*$\bar{b} - 6(n-1)^{-2} \le b \le \bar{b}$*).*

$(n-1)^3|v_1 - v_1^{\mathrm{bl}}|$ as a function of b and observe that the scaled difference is almost independent of n, both in the outer region (Figure 8(B)), and in the boundary layer region (Figure 8(C)). In Figure 9 we repeat this procedure for $v_1' - (v_1^{\mathrm{bl}})'$ and observe that $v_i' - (v_i^{\mathrm{bl}})' = O(n^{-3})$ in the outer region (Figure 9(B)), but $v_i' - (v_i^{\mathrm{bl}})' = O(1/n)$ in the boundary-layer region (Figure 9(C)). Indeed, v_i^{inner} is computed so that v_i^{bl} satisfies (1.1) to *leading order* at and near $\bar{b}$. Therefore, the error of $(v_i')^{\mathrm{bl}}$ is expected to be $O(1/n)$ in the boundary-layer region (see proof of Lemma 5.2). In the outer region, however, the inner solution component becomes exponentially small, and so $v_i' - (v_i^{\mathrm{bl}})' \sim v_i' - (v_i^{\mathrm{outer}})' = O(n^{-3})$. Finally, Figure 6(B) shows that $|\bar{b} - \bar{b}^{\mathrm{asymptotic}}| \sim 2/n^3$, thus confirming the $O(n^{-3})$ accuracy of $\bar{b}^{\mathrm{asymptotic}}$.

7. Dynamical systems perspective. In [5], we considered the special case of power-law distributions,

$$F_i = v^{\alpha_i}, \qquad i = 1, \dots, n. \tag{7.1}$$

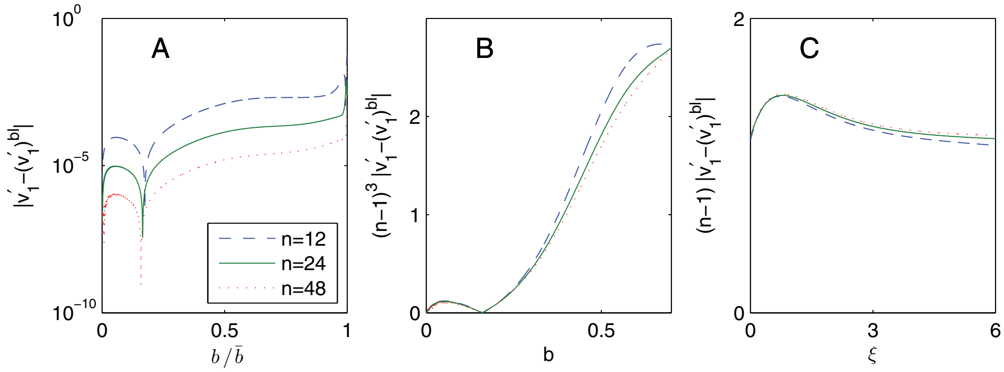

FIG. 9. *Same as Figure* 8 *for* $|v_1' - (v_1')^{\mathrm{bl}}|$. *In* C, *the scaled error is* $(n-1)\,|v_1' - (v_1')^{\mathrm{bl}}|$.

In this case, the substitutions $v_i(b) = b\,V_i(b)$ and $b = e^w$ transform (1.1a) into the autonomous dynamical system

$$V_i'(w) = \frac{V_i}{\alpha_i}\left[\left(\frac{1}{n-1}\sum_{j=1}^{n}\frac{1}{V_j-1}\right) - \frac{1}{V_i-1}\right] - V_i, \qquad i = 1,\dots,n; \tag{7.2a}$$

the left boundary condition (1.1b) transforms into

$$V_i(w=-\infty) = V_i^{\mathrm{saddle}}, \qquad i = 1,\dots,n, \tag{7.2b}$$

where $V_i^{\mathrm{saddle}} := 1 + \frac{1}{n\bar{\alpha}-\alpha_i}$ and $\bar{\alpha} := \frac{1}{n}\sum_{j=1}^{n}\alpha_j$, and the right boundary condition (1.1c) transforms into

$$V_i(\bar{w}) = \bar{V}, \qquad i = 1,\dots,n, \tag{7.2c}$$

where $\bar{w} = \log\bar{b}$ and $\bar{V} = e^{-\bar{w}} = 1/\bar{b}$.

Equations (7.2a) have a fixed point at $\mathbf{V}^{\mathrm{saddle}} := (V_1^{\mathrm{saddle}},\dots,V_n^{\mathrm{saddle}})$, which is a saddle point. The solution of (7.2) is thus a trajectory that starts at $\mathbf{V}^{\mathrm{saddle}}$ and intersects with the diagonal $V_1 = \cdots = V_n$ at $\bar{\mathbf{V}} := (\bar{V},\dots,\bar{V})$.

We now relate the dynamical systems description to the boundary-layer analysis.

LEMMA 7.1. *Let* $\{F_i\}$ *be given by* (7.1). *Then* $v_i^{\mathrm{outer}}(b) = bV_i^{\mathrm{saddle}}$.

Proof. Transforming back the saddle-point trajectory $\mathbf{V}(w) \equiv \mathbf{V}^{\mathrm{saddle}}$ shows that $v_i = bV_i^{\mathrm{saddle}}$ is a solution of the initial value problem (1.1a)–(1.1b). This solution can be expanded as $v_i = b + \frac{1}{n-1}u_i(b) + \frac{1}{(n-1)^2}w_i(b) + O\big(\frac{1}{n^3}\big)$. Since v_i^{outer} is the unique solution of (1.1a)–(1.1b) that has this expansion (Proposition 4.1), the result follows. □

Thus, in the case of power-law distributions we have v_i^{outer} exactly, and not only to $O(n^{-3})$.

In [5] we showed that linearization of the solution of (7.2a) about $\mathbf{V}^{\mathrm{saddle}}$ gives

$$\mathbf{V}(w) \sim \mathbf{V}^{\mathrm{saddle}} + \sum_{j=1}^{n-1}\tilde{c}_j\tilde{\mathbf{U}}_j\,e^{\tilde{\lambda}_j w}, \qquad w \to -\infty, \tag{7.3}$$

where $\{\tilde{\lambda}_j\}_{j=1}^{n-1}$ and $\{\tilde{\mathbf{U}}_j\}_{j=1}^{n-1}$ are the positive eigenvalues and corresponding eigenvectors of the Jacobian matrix $\mathbf{J}$ of (7.2a) at $\mathbf{V}^{\text{saddle}}$, where

$$\mathbf{J} = n^2\bar{\alpha}^2\mathbf{B}\mathbf{D}_{\frac{1}{\alpha}i}, \qquad \mathbf{D}_{\frac{1}{\alpha}} = \text{diag}\left(\frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_n}\right),$$

and $\mathbf{B}$ is defined in (3.4). In terms of the original variables, (7.3) reads

$$\mathbf{v}(b) \sim \mathbf{V}^{\text{saddle}} b + \sum_{j=1}^{n-1} \tilde{c}_j \tilde{\mathbf{U}}_j b^{1+\tilde{\lambda}_j}, \qquad 0 \le b \ll 1. \tag{7.4}$$

We now show that (7.4) agrees with the boundary-layer approximation

$$\mathbf{v} \sim \mathbf{v}^{\text{outer}}(b) + \frac{\mathbf{v}^{\text{inner}}(\xi)}{(n-1)^2}, \quad \mathbf{v}^{\text{inner}} = \mathbf{D}_{\frac{1}{\sqrt{f_i}}} \sum_{j=1}^{n-1} c_j e^{\lambda_j \xi} \mathbf{u}_j, \quad b - \bar{b} = O\left(\frac{1}{n^2}\right); \tag{7.5}$$

see Proposition 5.9. Since $\mathbf{v}^{\text{outer}} = \mathbf{V}^{\text{saddle}} b$ (Lemma 7.1), we need to show that $b^{1+\tilde{\lambda}_j}$ identifies with $e^{\lambda_j \xi}$, and $\tilde{\mathbf{U}}_j$ identifies with $\mathbf{u}_j$. Indeed, substituting $b = \bar{b} - \frac{\xi}{(n-1)^2}$, expanding for $n \gg 1$, and using $\bar{b} = 1 + O(1/n)$ gives $b = \bar{b}(1 - \frac{\xi}{(n-1)^2\bar{b}}) \sim e^{-\frac{\xi}{(n-1)^2}}$. Hence, for $b^{1+\tilde{\lambda}_j}$ to identify with $e^{\lambda_j \xi}$, we should have that $\frac{1+\tilde{\lambda}_i}{n^2} \sim \lambda_i$, where λ_i is a positive eigenvalue of $\mathbf{A} := -\bar{\alpha}^2\mathbf{D}_{\frac{1}{\alpha}}\mathbf{B}$; see (5.8). This is indeed the case, since $\mathbf{J} = -n^2\left[\mathbf{A}^T + O\left(\frac{1}{n}\right)\right]$, and so $\tilde{\lambda}_i \sim -n^2\lambda_i$. This argument also shows that $\mathbf{u}_j \sim c\tilde{\mathbf{U}}_j$.

The dynamical systems expansion (7.4) is about $b = 0$ (or $\mathbf{V} = \mathbf{V}^{\text{saddle}}$), whereas the boundary-layer expansion (7.5) is about $b = \bar{b}$ (or $\mathbf{V} = \bar{\mathbf{V}}$). The above results indicate that these approximations overlap. Thus, *the $(n-1)$-dimensional unstable manifold at $b = 0$ identifies with the $(n-1)$-dimensional inner solution $\mathbf{v}^{\text{inner}}$ near $\bar{b}$.* Intuitively, this is because $\mathbf{V}(\bar{b}) - \mathbf{V}(b=0) = \bar{\mathbf{V}} - \mathbf{V}^{\text{saddle}} = O\left(\frac{1}{n^2}\right)$.[9]

Finally, we note that the dynamical systems formulation may give the false impression that "all the dynamics" occurs as the solution moves from the saddle point to the diagonal. In fact, the boundary-layer analysis reveals that "most of the dynamics" occurs near the saddle point, and that the transition from $\mathbf{V}^{\text{saddle}}$ to $\bar{\mathbf{V}}$ corresponds to the narrow boundary layer near $\bar{b}$.

7.1. Nonsmoothness at $b = 0$. The dynamical systems analysis showed that the solution can be approximated by (7.4). Since generically $\tilde{\lambda}_i$ is not an integer, the solution of (1.1) is generically not in C^∞ at $b = 0$. Since, however, $\tilde{\lambda}_i \sim -n^2\lambda_i$, the smoothness at $b = 0$ increases with the number of players; i.e., $v_i \in C^k$, where $k = O(n^2)$.

The boundary-layer analysis does not reveal the nonsmoothness at $b = 0$. This is because v^{inner} is expanded about $\bar{b}$, whereas the nonsmoothness occurs at $b = 0$.

8. Numerical methods. Because of the lack of explicit solutions, numerical simulations play an important role in the study of asymmetric auctions. Computing solutions of (1.1), however, cannot be done using standard numerical methods for boundary value problems, because of the nonuniqueness at $b = 0$ and because the location of the right boundary is unknown.

[9] Indeed, $\bar{V} = \frac{1}{\bar{b}} = 1 + \frac{1}{n}\frac{1}{\bar{\alpha}} + O\left(\frac{1}{n^2}\right) = V_i^{\text{saddle}}$.

8.1. Backward shooting. The first numerical method for solving (1.1) was developed in 1994 by Marshall et al. [11]. They used a backward shooting approach, in which they searched for the value of $\bar{b}$ for which the backward solution of (1.1a) subject to (1.1c) satisfies (1.1b). The backward shooting method became the standard method for asymmetric first-price auctions and was used in numerous studies. Nevertheless, its performance was far from optimal, and it quickly deteriorated as the number of bidders was increased. In fact, it could not even handle auctions with more than six players. Therefore, this method is ill-suited to studying large asymmetric auctions.[10]

8.2. Boundary value method. In [4] we developed a different method, the *boundary value method*, for computing solutions of (1.1). To overcome the difficulty that the location of the right boundary is unknown, we change the independent variable from b to v_n. As a result, the transformed system of ODEs resides on a fixed known domain and can thus be solved using standard methods such as fixed-point iterations or Newton's iterations. The boundary value method has proved to be much more robust than backward shooting, and it performs well even with hundreds of players. The method has no inherent bound on accuracy and can reach machine accuracy at reasonably large grids [4, section 4.3].

We briefly mention some implementation details. Because of the boundary-layer structure of the solution, we use a nonuniform grid that concentrates more grid points in the boundary layer. In addition, when implementing Newton's method, we used an appropriate staggered grid that leads to narrow-banded linear systems, which can be solved significantly faster than the linear systems that arise in naïve implementations. See [3] for more details, and [4, online supplement] for MATLAB codes.

8.3. Instability of backward shooting. In [4] we pointed out that the backward shooting method is inherently unstable. Because of the lack of explicit solutions in the asymmetric case, we used the explicit solution in the *symmetric* case (section 2.1) to show analytically that in the *symmetric* case the backward shooting method is unstable and that this instability increases with n. Then we showed numerically that the instability in the asymmetric case is "similar" to that in the symmetric case.

We can use boundary-layer analysis to gain further insight into the instability of the backward shooting method, as follows. Consider the backward solutions $\{v_i^\varepsilon(b)\}$ of (1.1a) for $b \le \bar{b}_\varepsilon := \bar{b}_\varepsilon$, with the "initial" conditions

$$v_i^\varepsilon(\bar{b}_\varepsilon) = 1, \qquad i = 1, \dots, n. \tag{8.1}$$

We first show (informally) the following claim.

LEMMA 8.1. *Backward solutions "always" have a boundary layer near* $\bar{b}_\varepsilon$.

Proof. Assume that $\{v_i^\varepsilon(b)\}_{i=1}^n$ does not have a boundary layer near $\bar{b}_\varepsilon$. In that case, since $\{v_i^\varepsilon\}_{i=1}^n$ is a solution of (1.1a), then by Proposition 4.1,

$$v_i^\varepsilon(b) = v_i^{\text{outer}}(b) + \mathcal{O}\left(\frac{1}{n^3}\right). \tag{8.2}$$

If (8.2) holds, however, then from Lemma 4.3 with $\bar{b}$ replaced by $\bar{b}_\varepsilon$ it follows that v_i^ε cannot satisfy the right boundary condition (8.1). ◻

To show that the instability of backward solutions increases with n, we now show that an $O(\varepsilon)$ error in $\bar{b}$ leads to an $O(n\varepsilon)$ error in the derivatives at the right boundary.

LEMMA 8.2. $v_i'(\bar{b}) - (v_i^\varepsilon)'(\bar{b}_\varepsilon) \sim \varepsilon(n-1)\frac{f_G^2(1)}{f_i(1)}$.

[10] None of the simulations presented in this study could have been done using backward shooting.

Proof. By (1.1a) and (8.1),

$$(v_i^\varepsilon)'(\bar{b}_\varepsilon) = \frac{1}{n-1}\frac{1}{f_i(1)}\frac{1}{1-\bar{b}_\varepsilon}.$$

Hence,

$$v_i'(\bar{b}) - (v_i^\varepsilon)'(\bar{b}_\varepsilon) = \frac{1}{n-1}\frac{1}{f_i(1)}\left[\frac{1}{1-\bar{b}} - \frac{1}{1-(\bar{b}+\varepsilon)}\right] \sim \frac{1}{n-1}\frac{1}{f_i(1)}\frac{\varepsilon}{(1-\bar{b})^2}.$$

Substituting $\bar{b}$ (see (4.9)) into the above expression yields the result. ∎

The instability of backward solutions is illustrated numerically in Figure 10(A), where an $\varepsilon = -0.01$ error in $\bar{b}$ leads to an $O(1)$ difference between v_i^ε and v_i. The existence of a boundary layer of width n^{-2} for the backward solution can be seen in Figure 10(B) and (C). Figure 10(A) and (B) also show that v_i^ε does not identify with v_i^{outer} outside of the boundary layer. Repeating this simulation with a larger n shows similar results and that the instability becomes more pronounced (data not shown).

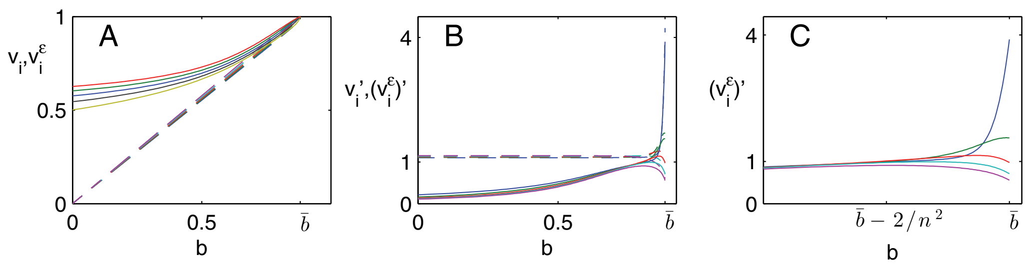

FIG. 10. (A) *Solutions* $\{v_i^\varepsilon(b)\}_{i=1}^n$ *of* (1.1a), (8.1) *with* $\varepsilon = -0.01$ *for* $\{F_i\}_{i=1}^n$ *given by* (6.1) *and* $n = 5$ *(solid curves). Also plotted are the solutions* $\{v_i(b)\}_{i=1}^n$ *for the case* $\varepsilon = 0$ *(dashed curves).* (B) *Derivatives of the solutions presented in* (A). (C) *Zoom-in on* $(v_i^\varepsilon)'$ *from* (B).

9. Concluding remarks. In this study we used boundary-layer theory to analyze asymmetric first-price auctions. While this methodology is standard in applied mathematics, to the best of our knowledge it has not been previously used in auction theory.

Asymmetric bidders have $O(1/n)$ differences in the competitions they face, since bidders i and k face the same $n-2$ bidders, but bidder i faces k whereas bidder k faces i. Consequently, one would expect to have $O(1/n)$ differences among their equilibrium strategies. Our analysis reveals, however, that $v_i(b) - v_k(b) = O(n^{-2})$, i.e., that these differences are much smaller. As $n \to \infty$, the $O(n^{-2})$ differences among $\{v_i\}_{i=1}^n$ vanish, and the boundary-layer thickness goes to zero. The boundary layer, however, does not disappear. In particular, the $O(1)$ differences among the derivatives $\{v_i'(\bar{b})\}_{i=1}^n$ persist as $n \to \infty$.

The analysis in this paper reveals that when n is large, the "correct" averaging of the CDFs is the geometric one. In the case of weakly asymmetric auctions, Fibich and Gavious used the arithmetic average of the asymmetric CDFs in their analysis [1]. Their results, however, did not imply that the correct averaging of the CDFs is the arithmetic one, since when the asymmetry among the CDFs is $O(\varepsilon)$, the arithmetic and geometric averages are $O(\varepsilon^2)$ equivalent. In contrast, as $n \to \infty$, the difference between the arithmetic and geometric averages does not go to zero. Therefore, it seems that the correct averaging in asymmetric first-price auctions is the geometric one.[11]

As n increases, the number of ODEs in (1.1) increases. This is different from standard boundary-layer problems, where the system itself does not change as the small parameter goes to zero. Therefore, we need to clarify what we mean by "letting n go to infinity." One possibility is to have $c_i = \beta_i n$ bidders with CDF $F_i(v)$, where $i = 1, \ldots, K$.[12]

Since the number of equations is fixed, it is clear what we mean by letting n go to infinity. A second approach is based on the observation that the geometrical average of the CDFs plays a key role in the limit. Therefore, we can consider a sequence of families of CDFs $\{F_1^{(n)}, \ldots, F_n^{(n)}\}$ whose geometrical average $(\prod_{i=1}^n F_i^{(n)})^{1/n}$ does not change with n.[13] Note that under this approach, some, or even all, of the CDFs change as n increases.

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[11] Typically, in homogenization theory, the homogenization result follows from the coupling of the fast and slow scale dynamics over the whole domain, which by proper averaging can be reduced to a homogeneous effective medium. This is not the case with our result that the "correct" averaging of the CDFs is the geometric one. Rather, the source of this homogenization-type result is that all solutions of the ODE system converge to the same asymptotic limit; i.e., $\lim_{n\to\infty} v_i = b$. This property holds for a wide class of auction mechanisms [16] and is, in fact, one of the reasons why the emergence of a boundary-layer structure in large asymmetric first-price auctions is surprising.

[12] This approach was used in the simulations of Figure 7.

[13] This approach was used in the simulations of Figure 2.