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Asymmetric First-Price Auctions—A Dynamical-Systems Approach

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We introduce a new approach for analysis and numerical simulations of asymmetric first-price auctions, which is based on dynamical systems. We apply this approach to asymmetric auctions in which players' valuations are power-law distributed. We utilize a dynamical-systems formulation to provide a proof of the existence and uniqueness of the equilibrium strategies in the cases of two coalitions and of two types of players. In the case of n different players, the singular point of the original system at $b=0$ corresponds to a saddle point of the dynamical system with $n-1$ admissible directions. This insight enables us to use forward solutions in the analysis and in the numerical simulations, in contrast with previous analytic and numerical studies that used backward solutions. The dynamical-systems approach provides an intuitive explanation for why the standard backward-shooting method for computing the equilibrium strategies is inherently unstable, and enables us to devise a stable forward-shooting method. In particular, in the case of two types of players, this method is extremely simple, as it does not require any shooting.

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1. Introduction. Auctions provide an important tool for selling and buying goods and have become central to the backbone of the modern economy.[1] One of the most common auction mechanisms is the first-price auction, in which each bidder submits a sealed bid, the bidder with the highest bid wins the object and pays his bid, and all other bidders do not pay anything. The theory of auctions as games of incomplete information originated in 1961 in the work of Vickrey [20]. Vickrey studied symmetric auctions, in which the valuations of all bidders are drawn according to the same distribution function $F(v)$. In this case, the equation for the inverse equilibrium bidding strategy $v(b)$ in a first-price auction is

$$v'(b) = \frac{1}{n-1}\frac{F(v(b))}{F'(v(b))}\frac{1}{v(b)-b}, \qquad v(0)=0,$$

where $v = b^{-1}$, and $b(v)$ is the equilibrium strategy. This equation can be solved explicitly, yielding

$$b(v) = v - \frac{\int_0^v F^{n-1}(s)\,ds}{F^{n-1}(v)}. \tag{1}$$

In practice, it often happens that bidders are asymmetric, i.e., their valuations are drawn according to different distribution functions $\{F_i(v)\}_{i=1}^n$. In that case, the equations for the inverse equilibrium bids $\{v_i(b)\}_{i=1}^n$ are given by

$$v_i'(b) = \frac{F_i(v_i(b))}{F_i'(v_i(b))}\left[\left(\frac{1}{n-1}\sum_{j=1}^{n}\frac{1}{(v_j(b)-b)}\right) - \frac{1}{(v_i(b)-b)}\right], \quad i=1,\ldots,n, \tag{2a}$$

with the initial conditions

$$v_i(b=0)=0, \quad i=1,\ldots,n, \tag{2b}$$

where $v_i = b_i^{-1}$ and $b_i(v)$ is the equilibrium strategy of player i.

One cannot apply the standard local existence and uniqueness theorem to the initial value problem (2a), (2b), because by (2b), the right-hand-side terms $1/(v_i(b)-b)$ are unbounded at $b=0$. Indeed, in this paper we provide examples in which (2a), (2b) admits nonunique solutions. The equilibrium bids, however, satisfy the additional condition that the maximal bids $\{b_i(v=1)\}_{i=1}^n$ of all bidders are the same. Therefore,

$$v_i(\bar{b}) = 1, \quad i=1,\ldots,n, \tag{2c}$$

for some a priori unknown $\bar{b}$ between 0 and 1.

Unlike the symmetric case, explicit solutions of (2) are not available. In addition, the nonlinear boundary value problem (2) is nonstandard, because it is singular at the left boundary, and the location of the right boundary ($\bar{b}$)

[1] For more details on auction theory and applications, see, e.g., Klemperer [7], Krishna [8], and Milgrom [16].

is unknown. Therefore, relatively little is known at present on asymmetric auctions, and almost all the theory concerns symmetric auctions.

In a series of breakthrough papers, Lebrun [9, 11, 12] proved the existence and uniqueness of a solution to (2). As noted by Lebrun [11, p. 127], "the difficulty of this proof stems from the singularity of the differential system at $b=0$." Lebrun overcame this difficulty by considering backward solutions of (2a) for $b \le \bar{b}_\varepsilon$, with the "initial" condition

$$v_i(\bar{b} = \bar{b}_\varepsilon) = 1, \quad i = 1, \ldots, n. \tag{3}$$

Lebrun considered the backward solutions as $\bar{b}_\varepsilon$ decreases from 1 ("type I"), and as $\bar{b}_\varepsilon$ increases from 0 ("type II"). Then, he used a continuity argument to show that there exists a unique value of $\bar{b}$ between 0 and 1, such that the backward solution with $\bar{b}_\varepsilon = \bar{b}$ satisfies the left boundary conditions (2b). Therefore, existence and uniqueness to (2) follow. Because backward solutions with $\bar{b}_\varepsilon \ne \bar{b}$ do not satisfy the left boundary condition (2b), Lebrun's proof did not "deal directly" with the singularity at $b=0$.

Numerical simulations can play an important role in the study of asymmetric auctions. Because of the nonstandard nature of the system (2), it cannot be solved using standard numerical methods. The first study of numerical methods for solving (2) is due to Marshall et al. [14]. In this work, Marshall et al. [14] initially tried a forward-shooting method, i.e., search for the solution of the initial-value problem (2a), (2b) that satisfies (2c). They concluded, however, that the solutions of the forward method are attracted to an explicit solution of (2a), (2b) which does not satisfy the right boundary condition (2c). Therefore, they opted for solving the boundary-value problem (7) using a backward-shooting approach, i.e., search for the value of $\bar{b}_\varepsilon$ for which the backward solution of (2a), (3) satisfies (2b). Although the backward-shooting method has become the standard method for computing the equilibrium strategies in asymmetric first-price auctions, it is far from optimal. Indeed, already Marshall et al. [14, p. 195] observed that "backward solutions are well behaved except in neighborhoods of the origin where they become (highly) unstable." Recently, we showed analytically that the backward-shooting method is unstable in the symmetric case, and that the instability increases with n, see [3]. In that study, however, we did not analyze the backward-shooting method in the asymmetric case.

In this study we address the following questions:

(i) What is the nature of singularity/nonuniqueness of the initial-value problem (2a), (2b)?

(ii) Why does the addition of condition (2c), which involves $n-1$ constraints, lead to uniqueness of the solution of (2)?

(iii) Why did the forward-shooting method of Marshall et al. [14] not converge to the solution of (2)?

(iv) Why do numerical backward solutions become unstable near $b=0$?

(v) Is it possible to devise a stable shooting method for solving (2)?

In this work, we consider the special case of power-law distributions $F_i = v^{\alpha_i}$. In this case, the system (2) can be transformed into an autonomous dynamical system. Therefore, we can address these questions using dynamical-systems tools.[2] In particular, the dynamical-systems formulation enables us to "deal directly" with the singularity at $b=0$, and to use forward solutions, both in analysis and in the numerical simulations, in contrast to previous analytic and numerical studies that only used backward solutions. It is reasonable to believe that the results for power-law distributions can be extended to distributions that satisfy

$$F_i(v) \sim c_i v^{\alpha_i}, \quad 0 \le v \ll 1, \tag{4}$$

because for these distributions, Equations (2) are likely to have a similar structure near the origin.

The paper is organized as follows: in §2 we derive the general model (2) for asymmetric first-price auctions. Then, we present several special cases of (2):

(i) Two players with $F_1 = v^\alpha$ and $F_2 = v^\beta$. This case also arises when players form two coalitions (or cartels) with α and β players (§2.2).

(ii) Two types of players: n_1 players with $F_1 = v^\alpha$, and n_2 players with $F_2 = v^\beta$ (§2.3).

(iii) The case of n players with n different distributions $\{F_i = v^{\alpha_i}\}_{i=1}^n$. This case also arises when players form n coalitions, with α_i players in coalition i (§2.4).

In §3 we analyze the three cases by transforming Equations (2) with $F_i = v^{\alpha_i}$ into an autonomous dynamical system. In §3.1 we use the dynamical-systems formulation to prove existence, uniqueness, and differentiability of the equilibrium in the case of two players with power-law distributions (or two coalitions). In §3.2 we extend these results to the case of two types of players. Unlike previous studies, we use forward solutions in the analysis. In §3.1.3 we show, however, that one can prove existence and uniqueness using backward solutions of the dynamical systems. In that case, we recover Lebrun's characterization of type I and type II solutions.

[2] For books on dynamical systems, see, e.g., Perko [17], Strogatz [19], and Wiggins [21].

In §3.3 we consider the case of n players with n different power-law distributions. In this case, the dynamical-systems formulation "reveals the nature of nonuniqueness" by showing that the initial-value problem (2a), (2b) does not have a unique solution, but rather an $n-1$ parameter family of solutions, which can be approximated with

$$\mathbf{v}(b) \sim \mathbf{V}^{\text{saddle}} \cdot b + c_1 \mathbf{U}_1 b^{1+\lambda_1} + \cdots + c_{n-1} \mathbf{U}_{n-1} b^{1+\lambda_{n-1}}, \quad 0 \le b \ll 1. \tag{5}$$

Here, $\mathbf{V}^{\text{saddle}}$ is the saddle point of the corresponding dynamical system, $\{\lambda_i\}_{i=1}^{n-1}$ are the $n-1$ positive eigenvalues of the Jacobian matrix of the dynamical system at $\mathbf{V}^{\text{saddle}}$, and $\{\mathbf{U}_i\}_{i=1}^{n-1}$ are the corresponding eigenvectors. Because condition (2c) adds $n-1$ constraints to the solution, the number of degrees of freedom of the solution of the initial-value problem (2a), (2b) is equal to the number of constraints in the boundary condition (2c). We also prove existence (and "local uniqueness") for the special case of a weak asymmetry, i.e., $\alpha_i = \alpha + O(\varepsilon)$, $i = 1, \ldots, n$, where $0 < \varepsilon \ll 1$.

In §4 we analyze numerical methods for computing the bidding strategies in asymmetric first-price auctions. In §4.1 we provide an intuitive explanation for why the standard backward-shooting method is inherently unstable, and why the forward-shooting method of Marshall et al. [14] did not converge to the solution of (2). Then, exploiting the understanding of the nature of nonuniqueness, in §4.2 we devise a stable forward-shooting method for computing the equilibrium strategies of asymmetric first-price auctions. In the case of n different distributions, the forward-shooting method requires an $n-2$ dimensional search. Hence, the forward-shooting method becomes impractical for auctions with many different distributions. In this case, the boundary-value method, which we recently developed in Fibich and Gavish [3], is preferable. In contrast, in the case of two different power-law distributions, the forward method does not involve any shooting, and can be easily applied to auctions with hundreds of players. In this case, the instability of the backward-shooting method is so severe that it fails completely (Fibich and Gavish [3]). The boundary-value method can be applied, but it requires a special attention to the grid resolution in the boundary layer region near $\bar{b}$.

In summary, the dynamical-systems formulation provides a new approach for analyzing asymmetric auctions. In particular, the insight gained on the nature of nonuniqueness enables us to use forward solutions in the analysis and numerics, in contrast to previous analytic and numerical studies that only used backward solutions.

2. Mathematical model.

2.1. General case. Consider n risk-neutral players bidding for a single object. The value v_i of the object for the ith player ($i = 1, \ldots, n$) is private information to i, and is drawn independently from the interval $[0, 1]$ according to a monotonically increasing distribution function $F_i(v)$, such that $F_i(0) = 0$ and $F_i(1) = 1$. The functions $\{F_i\}_{i=1}^n$ are known to all players. We assume that $\{F_i(v)\}_{i=1}^n$ have continuous densities $f_i(v) = F_i'(v)$ that are strictly positive for $v > 0$.

In a first-price auction, each bidder submits a sealed bid, the highest bidder wins the object and pays his bid, and all other bidders do not pay anything. We denote by $b_i = b_i(v_i)$ the bid of player i whose value is v_i. We assume that $b_i(v)$ is an increasing and smooth function, and we denote by $v_i = v_i(b_i)$ the inverse of $b_i(v)$. The utility of the ith player is $v_i - b_i$ when he wins and zero when he loses. Therefore, the expected utility of player i when he submits a bid b_i and all players bid according to their bidding strategies $\{b_j(v_j)\}_{j\neq i}$ is

$$U_i(v_i, b_i) = (v_i - b_i)\,\text{Prob}\Big(\max_{i\neq j} b_j < b_i\Big) = (v_i - b_i)\left(\prod_{\substack{j=1\\ j\neq i}}^{n} F_j(v_j(b_i))\right).$$

Hence, b_i is the solution of the optimization problem $\max_{b_i} U_i(v_i, b_i)$, where v_i is given and fixed. Therefore,

$$0 = \frac{\partial}{\partial b_i} U_i(v_i, b_i) = (v_i - b_i) \sum_{\substack{j=1\\ j\neq i}} \left(\prod_{\substack{k=1\\ k\neq i,j}}^{n} F_k(v_k(b_i))\right) f_j(v_j(b_i))v_j'(b_i) - \left(\prod_{\substack{j=1\\ j\neq i}}^{n} F_j(v_j(b_i))\right).$$

Since at equilibrium $v_i = v_i(b_i)$,

$$(v_i(b_i) - b_i) \sum_{\substack{j=1\\ j\neq i}} \left(\prod_{\substack{k=1\\ k\neq i,\,j}}^{n} F_k(v_k(b_i))\right) f_j(v_j(b_i))v_j'(b_i) - \left(\prod_{\substack{j=1\\ j\neq i}}^{n} F_j(v_j(b_i))\right) = 0. \tag{6}$$

Denoting b_i by b and rearranging Equations (6) gives

$$v_i'(b) = \frac{F_i(v_i(b))}{f_i(v_i(b))}\left[\left(\frac{1}{n-1}\sum_{j=1}^{n} \frac{1}{v_j(b) - b}\right) - \frac{1}{v_i(b) - b}\right], \quad i = 1, \ldots, n. \tag{7a}$$

The inverse equilibrium strategies satisfy the condition that all bidders with a zero value submit a zero bid. Therefore, the initial condition for the system (7a) is given by

$$v_i(0) = 0, \quad i = 1, \ldots, n. \tag{7b}$$

In equilibrium, there exists an additional condition that the maximal bid of all bidders is the same (Lebrun [11], Maskin and Riley [15]), i.e., there exists some $\bar{b} > 0$ such that

$$v_i(\bar{b}) = 1, \quad i = 1, \ldots, n. \tag{7c}$$

The value of $\bar{b}$ is a priori unknown. Finally, we note that in equilibrium we have that

$$v_i(b) \geq b, \quad 0 \leq b \leq \bar{b}, \ \ i = 1, \ldots, n, \tag{8}$$

since otherwise the bidder will lose by winning, and would therefore be better off submitting a zero bid.

We now present several special cases of (7), which will be analyzed in this study.

2.2. Two coalitions. Consider a group of k players with valuations $\{v_1, \ldots, v_k\}$ that forms a coalition (or cartel), whose goal is to win the object for the coalition player with the maximal value. Such a coalition effectively acts as a single bidder that draws his valuation according to the distribution

$$F^{\text{coalition}}(v) = \text{Prob}\Big(\max_{i=1,\ldots,k} v_i \leq v\Big) = \prod_{i=1}^{k} F_i(v). \tag{9}$$

Let us assume that α bidders form a coalition, the other $\beta = n - \alpha$ bidders form a counter coalition, and all players draw their value according to the uniform distribution $F_i(v) = v$. Then, the first coalition acts as a single bidder with $F_1^{\text{coalition}} = v^\alpha$, see (9), and the counter coalition acts as a single bidder with $F_2^{\text{coalition}} = v^\beta$. In this case, the system (7) reduces to

$$v_1'(b) = \frac{v_1(b)}{\alpha} \frac{1}{v_2(b) - b}, \qquad v_2'(b) = \frac{v_2(b)}{\beta} \frac{1}{v_1(b) - b}, \tag{10a}$$

with the boundary conditions

$$v_1(0) = v_2(0) = 0, \tag{10b}$$

and

$$v_1(\bar{b}) = v_2(\bar{b}) = 1. \tag{10c}$$

The two-coalition problem had drawn considerable attention in the literature. For example, Marshall et al. [14] showed that the maximal equilibrium bid for the system (10) is given by[3]

$$\bar{b} = 1 - \left[\frac{\beta^{\alpha\beta}(1+\alpha)^{\beta(1+\alpha)}}{\alpha^{\alpha\beta}(1+\beta)^{\alpha(1+\beta)}}\right]^{1/(\alpha-\beta)}. \tag{11}$$

Remark 2.1. The system (10) also corresponds to the case of two players with $F_1 = v^\alpha$ and $F_2 = v^\beta$. In this case, α and β can be nonintegers.

2.3. Two types of players. Consider the case of n_1 players with distribution $F_1 = v^\alpha$, and n_2 players with distribution $F_2 = v^\beta$. In contrast to coalitions, each player bids to maximize his own utility. Because the bidding strategies of all bidders with the same distribution are the same (Lebrun [11]), the system (7) reduces to

$$\begin{aligned} v_1'(b) &= \tfrac{v_1(b)}{\alpha} \frac{1}{n_1 + n_2 - 1}\left[\frac{n_2}{v_2(b) - b} - \frac{n_2 - 1}{v_1(b) - b}\right], \\ v_2'(b) &= \tfrac{v_2(b)}{\beta} \frac{1}{n_1 + n_2 - 1}\left[\frac{n_1}{v_1(b) - b} - \frac{n_1 - 1}{v_2(b) - b}\right], \end{aligned} \tag{12a}$$

with the boundary conditions

$$v_1(0) = v_2(0) = 0, \tag{12b}$$

and

$$v_1(\bar{b}) = v_2(\bar{b}) = 1, \tag{12c}$$

where $v_i(b)$ is the inverse bidding strategy of players with distribution F_i. Unlike Equation (10), in the case of Equation (12) there is no explicit expression for $\bar{b}$.

[3] See Remark 3.5 for a proof of Equation (11).

2.4. n coalitions. Consider the case where bidders form n coalitions, such that there are α_i bidders in coalition i, and all players draw their value according to the uniform distribution $F(v) = v$. Then, coalition i acts as a single bidder with $F_i^{\text{coalition}} = v^{\alpha_i}$, see (9). In this case, the inverse bid functions are the solutions of

$$v_i'(b) = \frac{v_i}{\alpha_i}\left[\left(\frac{1}{n-1}\sum_{j=1}^{n}\frac{1}{v_j(b)-b}\right) - \frac{1}{v_i(b)-b}\right], \quad i = 1, \ldots, n, \tag{13a}$$

subject to boundary conditions

$$v_i(0) = 0, \quad i = 1, \ldots, n, \tag{13b}$$

and

$$v_i(\bar{b}) = 1, \quad i = 1, \ldots, n. \tag{13c}$$

Unlike Equation (10), in the case of Equation (13) there is no explicit expression for $\bar{b}$.

REMARK 2.2. The system (13) also corresponds to the case of n players with $F_i = v^{\alpha_i}$, $i = 1, \ldots, n$.

3. A dynamical-systems approach.

We now analyze the three problems of two coalitions, two types of players, and n coalitions, using a dynamical-systems approach.

3.1. Two coalitions. To study the system (10), we first transform it into an autonomous dynamical system as follows. Let

$$v_1 = bV_1(b), \quad v_2 = bV_2(b). \tag{14}$$

Then, Equation (10a) becomes

$$bV_1'(b) = V_1(b)\frac{1+1/\alpha - V_2(b)}{V_2(b)-1}, \qquad bV_2'(b) = V_2(b)\frac{1+1/\beta - V_1(b)}{V_1(b)-1}. \tag{15}$$

Substituting

$$b = e^w, \tag{16}$$

gives the autonomous system

$$V_1'(w) = V_1(w)\frac{1+1/\alpha - V_2(w)}{V_2(w)-1}, \qquad V_2'(w) = V_2(w)\frac{1+1/\beta - V_1(w)}{V_1(w)-1}. \tag{17a}$$

The boundary conditions for solutions of (17a) are

$$V_1(w = -\infty) = 1 + \frac{1}{\beta}, \qquad V_2(-\infty) = 1 + \frac{1}{\alpha}, \tag{17b}$$

(see Lemma 3.2 below) and

$$V_1(\bar{w}) = V_2(\bar{w}) = e^{-\bar{w}}, \tag{17c}$$

where $e^{\bar{w}} = \bar{b}$ (see Equation (10c)).

The dynamical system (17a) has a fixed point at

$$\mathbf{V}^{\text{saddle}} = \left(1 + \frac{1}{\beta}, 1 + \frac{1}{\alpha}\right), \tag{18}$$

which is a saddle point. Indeed, the linearization $\mathbf{V}(w) \sim \mathbf{V}^{\text{saddle}} + \mathbf{V}^{\text{lin}}(w)$ of (17a) about $\mathbf{V}^{\text{saddle}}$ gives

$$\frac{d}{dw}\mathbf{V}^{\text{lin}} = \mathbf{A}\,\mathbf{V}^{\text{lin}}, \tag{19a}$$

where $\mathbf{A}$ is the Jacobian matrix of (17a) at $\mathbf{V} = \mathbf{V}^{\text{saddle}}$, i.e.,

$$\mathbf{A} = \left.\begin{pmatrix} \dfrac{1+1/\alpha - V_2}{V_2 - 1} & -\dfrac{V_1}{\alpha(V_2-1)^2} \\ -\dfrac{V_2}{\beta(V_1-1)^2} & \dfrac{1+1/\beta - V_1}{V_1 - 1} \end{pmatrix}\right|_{\mathbf{V}=\mathbf{V}^{\text{saddle}}} = \begin{pmatrix} 0 & -\dfrac{\alpha}{\beta}(1+\beta) \\ -\dfrac{\beta}{\alpha}(1+\alpha) & 0 \end{pmatrix}. \tag{19b}$$

The eigenvalues and corresponding eigenvectors of $\mathbf{A}$ are

$$\lambda_\pm = \pm\sqrt{(1+\alpha)(1+\beta)}, \qquad \mathbf{U}_\pm = \begin{pmatrix} \sqrt{\alpha\left(1+\dfrac{1}{\beta}\right)} \\ \mp\sqrt{\beta\left(1+\dfrac{1}{\alpha}\right)} \end{pmatrix}. \tag{20}$$

The following two Lemmas show that solutions of the initial value problem (10a), (10b) correspond to trajectories in the phase plane that start at $\mathbf{V}(w=-\infty)=\mathbf{V}^{\text{saddle}}$:

Lemma 3.1. *Let $\{v_1, v_2\}$ be a solution of (10a), (10b) that satisfies (8). Then, the corresponding solution $\{V_1, V_2\}$ of (17a) is bounded as $w \longrightarrow -\infty$.*

Proof. By (8), (14),

$$V_i(w) \ge 1, \quad i=1,2. \tag{21}$$

Therefore, V_1 and V_2 are bounded from below as $w \longrightarrow -\infty$.

By negation, assume that V_1 and V_2 are not both bounded from above as $w \longrightarrow -\infty$. Without loss of generality, assume that $\limsup_{w\to-\infty} V_2(w) = \infty$. If $\liminf_{w\to-\infty} V_2(w) < \infty$, then $V_2(w)$ oscillates between increasingly larger values and a bounded region. Therefore, there is a series $\{w_k\} \longrightarrow -\infty$ of local maxima points, such that $V_2'(w_k)=0$ and $\lim_{k\to\infty} V_2(w_k)=\infty$. By (17a), $V_1(w_k)=V_1^{\text{saddle}}$, and

$$\frac{d^2V_2}{dw^2}(w_k) = V_2(w_k)\frac{-V_1'(w_k)}{V_1^{\text{saddle}}-1} = \frac{V_2(w_k)}{V_1^{\text{saddle}}-1}V_1^{\text{saddle}}\frac{V_2(w_k)-1-1/\alpha}{V_2(w_k)-1} \xrightarrow[k\to\infty]{} +\infty.$$

Hence, for some $K>0$, $\{V_2(w_k)\}_{k\ge K}$ are all local minima points. Contradiction. Therefore, $\limsup_{w\to-\infty} V_2(w)=\infty$ implies that $\lim_{w\to-\infty} V_2=\infty$.

We now show that $\lim_{w\to-\infty} V_2=\infty$ implies that $\lim_{w\to-\infty} V_1=\infty$. Indeed, by (17a), (21), as $w\to-\infty$,

$$V_1' = -V_1\left(1-\frac{1}{\alpha(V_2-1)}\right) \sim -V_1 \le -1. \tag{22}$$

Therefore, $\lim_{w\to-\infty} V_1=\infty$.

So far, we proved that if V_1 and V_2 are not both bounded from above as $w \longrightarrow -\infty$, then both $V_1\to\infty$ and $V_2\to\infty$ as $w\to-\infty$. Hence, by (17a), as $w\to-\infty$,

$$V_1'+V_1 = \frac{V_1}{\alpha(V_2-1)} \sim \frac{V_1}{\alpha V_2}, \qquad V_2'+V_2 \sim \frac{V_2}{\beta V_1}. \tag{23}$$

We claim that either $V_1\le V_2$ or $V_2\le V_1$ as $w\to-\infty$. Indeed, along the diagonal $V_1=V_2$,

$$\frac{d}{dw}(V_1-V_2)\bigg|_{V_1=V_2} = \left[\frac{V_1}{V_2-1}\frac{1}{\alpha} - V_1 - \left(\frac{V_2}{V_1-1}\frac{1}{\beta}-V_2\right)\right]_{V_1=V_2} = \frac{V_1}{V_1-1}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right), \tag{24}$$

does not change its sign. Hence, any trajectory can cross the diagonal at most once. Without loss of generality, $V_1\le V_2$ as $w\to-\infty$. Therefore, by (23), there exists $\tilde{w}$, such that

$$V_1'(w)+V_1(w) \le \frac{2}{\alpha}, \quad -\infty < w < \tilde{w}.$$

Hence, for all $-\infty<w<\tilde{w}$, $\int_{-\infty}^{w}(e^wV_1)' \le (2/\alpha)\int_{-\infty}^{w} e^w$ and $e^w(V_1-2/\alpha) \le \lim_{w\to-\infty} e^wV_1$. Since $\lim_{w\to-\infty} V_1=\infty$, there exists $-\infty<w_1\ll -1$ such that $e^{w_1}(V_1(w_1)-2/\alpha)>0$. Therefore, $\lim_{w\to-\infty} e^wV_1>0$.[4] This implies that $v_1(0)>0$, in contradiction with (10b). ☐

Remark 3.1. The proof of Lemma 3.1 can be simplified by making use of the conserved quantity $E(V_1, V_2)$ (see §3.1.1). We do not take that approach, however, because we will later extend this proof to the two-types-of-players case (see Lemma 3.9), for which there is no known conserved quantity.

Lemma 3.2. *Let $\{v_1, v_2\}$ be a solution of (10a), (10b) that satisfies (8). Then, the corresponding solution $\{V_1, V_2\}$ of (17a) satisfies (17b).*

[4] This limit exists, since (23) shows that e^wV_1 is monotone.

PROOF. By Lemma 3.1, as $w \to -\infty$, V_1 and V_2 remain in a bounded region in the quarter plane $V_1, V_2 \geq 1$. Inside this quarter plane, there are no center points, sources, or sinks. Hence, by the index theorem, there are no periodic orbits. Therefore, trajectories can only start at $w = -\infty$ from the singular lines $V_1 = 1$ or $V_2 = 1$, or from the saddle point $\mathbf{V}^{\text{saddle}}$.

We now rule out the possibility that trajectory starts at $w = -\infty$ from one or several accumulations points that lie on the singular lines $V_1 = 1$ or $V_2 = 1$, which will conclude the proof. Assume by negation that the trajectory starts from the point $(1, V_2^0)$, where $1 < V_2^0 < \infty$. Near $(1, V_2^0)$, the equation for V_1 in the dynamical system (17a) reduces to

$$V_1' \sim \frac{V_2^{\text{saddle}} - V_2^0}{V_2^0 - 1} \sim c, \qquad V_2' \sim \frac{V_2^0}{\beta} \frac{1}{V_1 - 1} \longrightarrow +\infty.$$

Therefore, if $1 < V_2^0 < V_2^{\text{saddle}}$, then $c > 0$. Hence, all the trajectories flow back to the singular point $(1, 1)$. On the line $V_2 = V_2^{\text{saddle}}$, $V_1' = 0$ and $V_2' > 0$. Therefore, the trajectories flow downward, and hence by the previous case, also end at $(1, 1)$. Finally, if $V_2^0 > V_2^{\text{saddle}}$, then $c < 0$. Therefore, in this case, all the trajectories flow back away from the singular line $V_1 = 1$.[5]

A similar argument shows that the trajectory cannot start on a point $(V_1^0, 1)$ where $1 < V_1^0 < \infty$. Finally, we show that the trajectory cannot start at $w = -\infty$ from the point $(1, 1)$. Indeed, near $(1, 1)$ the dynamical system (17a) reduces to

$$(V_2 - 1)V_1'(w) \sim \frac{1}{\alpha}, \qquad (V_1 - 1)V_2'(w) \sim \frac{1}{\beta}.$$

Summing the above two equations gives

$$\frac{d[(V_1 - 1)(V_2 - 1)]}{dw} \sim c, \quad c = \frac{1}{\alpha} + \frac{1}{\beta}.$$

Integration gives $w \sim w_0 + c^{-1}(V_1(w) - 1)(V_2(w) - 1)$ near $(1, 1)$, where $w_0 \in \mathbb{R}$. Therefore, the trajectory starts from $(1, 1)$ at a finite w, rather than at $w = -\infty$. Contradiction. □

Since $\mathbf{V}^{\text{saddle}}$ is a saddle point, there are three different trajectories that start at $\mathbf{V}^{\text{saddle}}$ (see Figure 1(A)).

LEMMA 3.3. *The unstable manifold of* $\mathbf{V}^{\text{saddle}}$ *consists of*
(i) *the trajectory* $\mathbf{V}(w) \equiv \mathbf{V}^{\text{saddle}}$;
(ii) *a trajectory* Γ *that leaves from* $\mathbf{V}^{\text{saddle}}$ *in the direction of* $\mathbf{U}_+$ *and intersects with the diagonal* $V_1 = V_2$;
(iii) *a trajectory* Γ' *that leaves from* $\mathbf{V}^{\text{saddle}}$ *in the direction of* $-\mathbf{U}_+$. *This trajectory does not intersect with the diagonal* $V_1 = V_2$.

PROOF. Without loss of generality, $\alpha < \beta$. Since (17a), (17b) is an autonomous dynamical system, there are precisely two trajectories Γ and Γ' that approach $\mathbf{V}^{\text{saddle}}$ as $w \longrightarrow -\infty$. These trajectories exit $\mathbf{V}^{\text{saddle}}$ in the directions of $\mathbf{U}_+$ and $-\mathbf{U}_+$, respectively, where $\mathbf{U}_+$ is given by (20).

We now prove that Γ intersects with the diagonal $V_1 = V_2$. Let S be the triangular region bounded by the lines $V_1 = V_1^{\text{saddle}}$, $V_2 = V_2^{\text{saddle}}$, and $V_1 = V_2$, see Figure 2. For any $0 < \varepsilon \ll 1$, $\mathbf{V}^{\text{saddle}} + \varepsilon \mathbf{U}_+$ resides in S. In the region S, $V_1 > V_1^{\text{saddle}}$ and $V_2 < V_2^{\text{saddle}}$. Hence, by (17a), $V_1'(w) > 0$ and $V_2'(w) < 0$ in S. Therefore Γ cannot intersect with the lines $V_1 = V_1^{\text{saddle}}$ and $V_2 = V_2^{\text{saddle}}$, and, in particular cannot return to $\mathbf{V}^{\text{saddle}}$. Since $\mathbf{V}^{\text{saddle}}$ is the only fixed point in S, Γ must exit the region S in finite time, and it can only do so by intersecting with the diagonal $V_1 = V_2$.

Similarly, for any $0 < \varepsilon \ll 1$, $\mathbf{V}^{\text{saddle}} - \varepsilon \mathbf{U}_+$ resides in the quarter plane $S' = \{V_1 < V_1^{\text{saddle}}, V_2 > V_2^{\text{saddle}}\}$. By (17a), $V_1'(w) < 0$ and $V_2'(w) > 0$ in S'. Hence for any $w > -\infty$, Γ' remains in S'. In particular, $V_1 \neq V_2$ for any $-\infty < w < \infty$. □

The trajectories $\mathbf{V} \equiv \mathbf{V}^{\text{saddle}}$ and Γ' are not solutions of (17), because they do not intersect with the diagonal $V_1 = V_2$ and, therefore, cannot satisfy (17c).

COROLLARY 3.1. *The only trajectory that corresponds to a solution of* (17a), (17b) *and intersects with the diagonal* $V_1 = V_2$ *is* Γ.

Let $\{V_1(w), V_2(w)\}$ be a solution of (17a), (17b). Then, for any $c \in \mathbb{R}$,

$$\{V_1(w + c), V_2(w + c)\}, \tag{25}$$

[5] These trajectories, however, flow back eventually to the point $(1, 1)$, see Figure 1(A) and Figure 3.

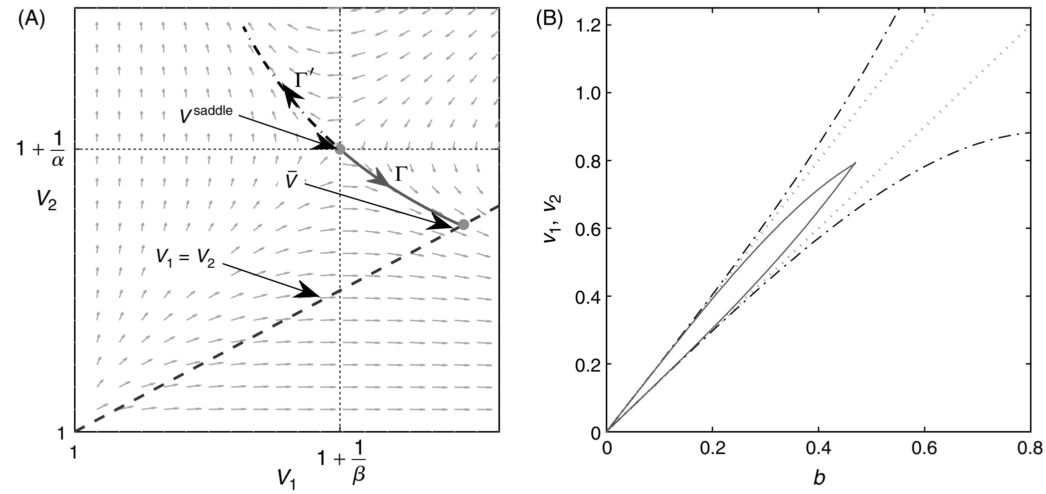

FIGURE 1. (A) The trajectories Γ (solid) and Γ' (dash dotted) of (17a), (17b) that descend from the saddle point $\mathbf{V}^{\text{saddle}} = (1+1/\beta, 1+1/\alpha)$, illustrated for the case of $\alpha < \beta$. The dashed curve is the diagonal $V_1 = V_2$. The intersection point of Γ with the diagonal $V_1 = V_2$ is marked by $\bar{\mathbf{V}}$. (B) The solutions $\{v_1(b), v_2(b)\}$ of (10a), (10b) that correspond to the trajectory Γ (solid), to the trajectory Γ' (dash dotted), and to the trajectory $\mathbf{V} \equiv \mathbf{V}^{\text{saddle}}$ (dotted). The bottom curve in each pair of curves is v_1.

is also a solution of (17a), (17b). Hence, the trajectory Γ represents a one-parameter family of solutions. Let $\{V_1(w), V_2(w)\}$ be a solution of (17a), (17b) that belongs to the one-parameter family (25) of Γ, and denote by $\tilde{w}$ the point where the trajectory of $\{V_1(w), V_2(w)\}$ intersects with the diagonal $V_1 = V_2$, i.e.,

$$V_1(\tilde{w}) = V_2(\tilde{w}) = \bar{V}.$$

In general, $\bar{V} \neq e^{-\tilde{w}}$. Hence, $\{V_1(w), V_2(w)\}$ is a solution of (17a), (17b), but it does not satisfy the right-boundary condition (17c). If, however, $c = \tilde{w} + \log \bar{V}$, then $\{V_1(w+c), V_2(w+c)\}$ satisfies (17c), and is therefore the unique solution of (17). We therefore proved that

THEOREM 3.1. *There exists a unique solution to* (17).

For future reference, we note that the analysis in this section shows that near $\mathbf{V}^{\text{saddle}}$, the solution of (17) can be approximated with

$$\mathbf{V}(w) \approx \mathbf{V}^{\text{saddle}} + c_+ \mathbf{U}_+ e^{\lambda_+ w}, \quad w \ll -1. \tag{26}$$

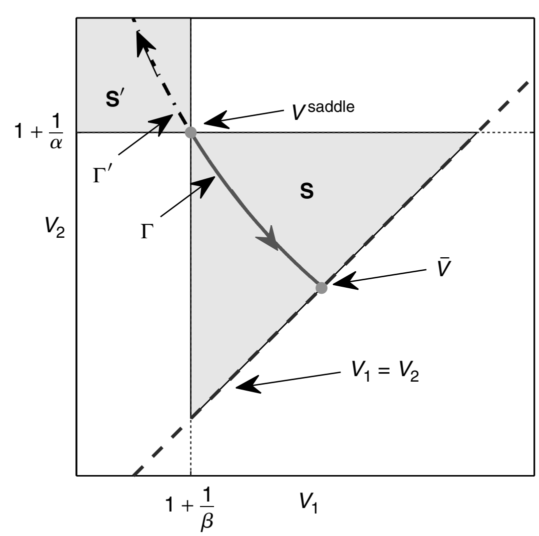

FIGURE 2. The same data as in Figure 1(A). Zoom-in on the regions S and S'.

3.1.1. Conservative system. Equation (17a) is a conservative dynamical system:

LEMMA 3.4. *The dynamical system* (17a) *admits the first integral*

$$E(V_1, V_2) = \frac{(V_1-1)^{\alpha}}{(V_2-1)^{\beta}} \frac{V_2^{\beta(1+\alpha)}}{V_1^{\alpha(1+\beta)}} \equiv E_0, \tag{27}$$

where E_0 is a constant.

PROOF. See Appendix A. □

In Figure 3 we plot the level sets of $E(V_1, V_2)$. The curve that starts at the saddle point V^{saddle} and intersects with the diagonal $V_1 = V_2$ is the trajectory Γ, which corresponds to the solution of (17). The intersection point $\bar{\mathbf{V}} = (\bar{V}, \bar{V})$ of Γ with the diagonal $V_1 = V_2$ can be found explicitly using the conserved quantity $E(V_1, V_2)$. Indeed, since Γ starts at $\mathbf{V}^{\text{saddle}}$, it follows that

$$E(\bar{V}, \bar{V}) = E(V_1^{\text{saddle}}, V_2^{\text{saddle}}) = \frac{\beta^{\alpha\beta}(1+\alpha)^{\beta(1+\alpha)}}{\alpha^{\alpha\beta}(1+\beta)^{\alpha(1+\beta)}}.$$

COROLLARY 3.2. *The intersection point of Γ with the diagonal $V_1 = V_2$ is given by $\bar{\mathbf{V}} = (\bar{V}, \bar{V})$, where*

$$\bar{V} = \left(1 - \left[\frac{\beta^{\alpha\beta}(1+\alpha)^{\beta(1+\alpha)}}{\alpha^{\alpha\beta}(1+\beta)^{\alpha(1+\beta)}}\right]^{1/(\alpha-\beta)}\right)^{-1}. \tag{28}$$

3.1.2. Going back to the system (10). We now express the results that were obtained with the dynamical-systems approach, in terms of the original system (10). In particular, we show that the trajectory Γ corresponds to the equilibrium strategies of the first-price auction, i.e., that the corresponding functions $(v_1(b), v_2(b))$ are monotonically increasing and satisfy (10).

Lemma 3.2 shows that trajectories of the dynamical system (17a) that start at $\mathbf{V}(w = -\infty) = \mathbf{V}^{\text{saddle}}$ correspond to solutions of (10a) that satisfy the initial condition (10b). Lemma 3.3 shows that there are three families of solutions of (10a) that satisfy the initial condition (10b) (see Figure 1(B)):

(i) The explicit linear solution

$$v_1^{\text{explicit}} = \left(1 + \frac{1}{\beta}\right) b, \qquad v_2^{\text{explicit}} = \left(1 + \frac{1}{\alpha}\right) b, \tag{29}$$

which corresponds to the trajectory $\mathbf{V}(w) \equiv \mathbf{V}^{\text{saddle}}$.

(ii) Solutions $\{v_1(b), v_2(b)\}$ that intersect at some $\tilde{b} > 0$, i.e., $v_1(\tilde{b}) = v_2(\tilde{b}) = \tilde{b}\bar{V}$, where $\bar{V}$ is given by (28). These solutions correspond to the trajectory Γ.

(iii) Solutions $\{v_1(b), v_2(b)\}$ that do not intersect at any $b > 0$. These solutions correspond to the trajectory Γ'.

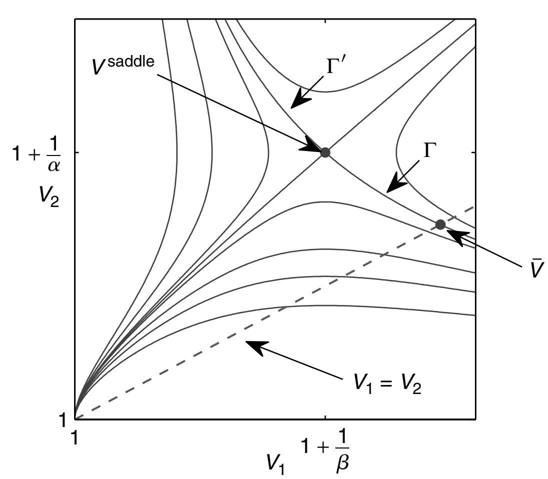

FIGURE 3. Level sets of $E(V_1, V_2)$, see (27), for $\alpha = 1$ and $\beta = 2$. The dashed line is $V_1 = V_2$.

To write the one-parameter family (25) in terms of the system (10a), note that $b = e^w$. Hence,

$$w \to w + c \Rightarrow b = e^w \to e^{w+c} = \tilde{c}e^w = \tilde{c}b, \qquad \tilde{c} = e^c.$$

Dropping the tilde, and replacing b with $c \cdot b$ in (14) gives the one-parameter family of solutions of (10a), (10b)

$$\left\{\frac{1}{c}v_1(cb), \frac{1}{c}v_2(cb)\right\}, \quad c \in \mathbb{R}^+. \tag{30}$$

Each solution of the one-parameter family (30) satisfies (10a), (10b) for $0 \le b \le \tilde{b} = e^{\tilde{w}}$, and the boundary condition

$$v_1(\tilde{b}) = v_2(\tilde{b}) = \bar{v}^c, \quad \tilde{b} > 0. \tag{31}$$

In general, $\bar{v}^c \neq 1$. Therefore, the solution does not satisfy (10c). Nevertheless, these solutions do satisfy (10), albeit for a different first-price auction system:[6]

LEMMA 3.5. *The solutions* (30) *of* (10a), (10b) *that satisfy* (31) *are the solutions of* (10) *with the distributions*

$$F_i^c(v) = \frac{F_i(v)}{F_i(\bar{v}^c)}, \quad i = 1, 2.$$

PROOF. The proof is immediate. □

In particular, when $c = 1/\bar{v}^c$, $F_i^c(v) = F_i(v)$, hence (30) is the solution of (10a), (10b), (10c). Therefore,

THEOREM 3.2. *There exists a unique solution to* (10).

We now show that the unique solution of (10) is monotonically increasing:

LEMMA 3.6. *The solutions* $v_1(b)$ *and* $v_2(b)$ *of* (10) *are strictly monotonically increasing.*

PROOF. By Equation (15),

$$v_1'(b) = (bV_1)' = \frac{V_1(b)}{\alpha}\frac{1}{V_2(b)-1}, \qquad v_2'(b) = \frac{V_2(b)}{\beta}\frac{1}{V_1(b)-1}.$$

From the proof of Lemma 3.3 it follows that in the region S, $V_2(b) > V_1(b) > V_1^{\text{saddle}} > 1$. Therefore, $v_1' > 0$ and $v_2' > 0$. □

REMARK 3.2. Lemma 3.2 implies that the solution of (10) is differentiable at $b = 0$.[7] Indeed, by Lemma 3.2,

$$V_i^{\text{saddle}} = \lim_{b\to 0} V_i(b) = \lim_{b\to 0}\frac{v_i(b)}{b} = v'(0), \quad i = 1, 2.$$

We note that the differentiability at $b = 0$ is not obvious. For example, in the case of a reserve price $r > 0$ the solution satisfies the left boundary condition $v_1(r) = v_2(r) = r$ and therefore it is not differentiable at the lower end $b = r$. In this case the proof of Lemma 3.2 fails, since $w \not\longrightarrow -\infty$ as $b \longrightarrow r > 0$. Indeed, in this case Γ starts from $(1, 1)$, and not from $\mathbf{V}^{\text{saddle}}$.

REMARK 3.3. From (26) it follows that near $b = 0$, the solution of (10) can be approximated with

$$\mathbf{v}(b) \approx \mathbf{V}^{\text{saddle}} \cdot b + c_+ \mathbf{U}_+ b^{1+\lambda_+}, \quad 0 \le b \ll 1.$$

In general, λ_+ is not an integer (see (20)). Therefore, the solution of (10) is not in C^∞. Note, however, that $\lambda_+ > 1$, and therefore $\mathbf{v}(b) \in C^2$.

REMARK 3.4. In terms of the original system (10), the first integral (27) becomes

$$e(v_1, v_2) = \frac{(v_1 - b)^\alpha v_2^{\beta(1+\alpha)}}{(v_2 - b)^\beta v_1^{\alpha(1+\beta)}}.$$

This first integral was found by Marshall et al. [14, Appendix A].

REMARK 3.5. The conserved quantity can be used to recover the explicit expression for $\bar{b}$ of Marshall et al. [14]. Indeed, since $\bar{b} = 1/\bar{V}$ and $\bar{V}$ is given by Equation (28), we recover Equation (11).

[6] When $\bar{v}^c > 1$, the result holds for $\{F_i\}$ that are extended monotonically and smoothly to $[0, \bar{v}^c]$.

[7] This result was proved by Lebrun [10].

3.1.3. Backward solutions. As noted, Lebrun [9, 11, 12] proved existence and uniqueness using backward solutions of (2a), (2c). The dynamical system (17) can also be analyzed using backward solutions, i.e., solutions $\{V_1, V_2\}$ of (17a) that start at the diagonal $V_1 = V_2$, see (17c), and flow backward. This approach leads to an alternative uniqueness proof, and also to Lebrun's characterization of type I and type II backward solutions.

Remark 3.6. Without loss of generality, we assume that $\alpha < \beta$ throughout this section.

By Lemma 3.3, there exists a trajectory Γ that starts at $\mathbf{V}^{\text{saddle}}$ and intersects with the diagonal at a point, which we denote by $(\bar{V}, \bar{V})$. We first consider backward solutions that cross the diagonal above $(\bar{V}, \bar{V})$.

Lemma 3.7. *All trajectories that cross the diagonal at $V_1 = V_2 > \bar{V}$, must start at $w = -\infty$ from $(+\infty, +\infty)$.*

Proof. Denote by T_{I} the open region bounded from below by $V_1 = V_1^{\text{saddle}}$, the trajectory Γ, and the diagonal $V_1 = V_2$, see Figure 4(A). Let us consider trajectories that cross the diagonal at $V_1(w_0) = V_2(w_0) > \bar{V}$. We first show that as we move backward along these trajectories, they do not exit the region T_{I}. Indeed, by (24), along the diagonal $V_1 = V_2$, $(d/dw)(V_1 - V_2) > 0$. On the nullcline $V_1 = V_1^{\text{saddle}}$ and $V_2 > V_2^{\text{saddle}}$, the trajectories point in the direction $(-1, 0)$. Hence, trajectories that cross the diagonal and the line $V_1 = V_1^{\text{saddle}}$ point outward of T_{I}. In addition, the trajectories cannot cross the trajectory Γ. Therefore, for $w < w_0$, they have to remain in the region T_{I}.

In the region T_{I}, $V_1', V_2' < 0$. Hence, V_1 and V_2 have (finite of infinite) limits as $w \longrightarrow -\infty$. Obviously, they cannot both have finite limits, because there are no singular points in T_{I}, and the backward trajectories that reach the fixed point $\mathbf{V}^{\text{saddle}}$ are not in T_{I}, see Lemma 3.3. Therefore, since $V_1 \le V_2$ in T_{I}, then $\lim_{w\to-\infty} V_2 = \infty$. Hence, by (17a), as $w \longrightarrow -\infty$,

$$V_1' \sim -V_1 < -\tfrac{1}{2} V_1.$$

Hence, $V_1 \ge c \cdot e^{-w/2} \longrightarrow \infty$. □

Because the trajectories are above the diagonal, then, as in the proof of Lemma 3.1, we have that $\lim_{w\to-\infty} e^w V_1 > 0$. Since $V_1 \le V_2$, then $\lim_{w\to-\infty} e^w V_2 > 0$. Therefore, the corresponding type I backward solutions of (10a), (10c) satisfy $\lim_{b\to 0} v_1 > 0$ and $\lim_{b\to 0} v_2 > 0$. In particular, they do not correspond to solutions of (10).

Lemma 3.8. *All trajectories that cross the diagonal at $1 < V_1 = V_2 < \bar{V}$, must start from $(1, 1)$.*

Proof. We denote by T_{II} the region that is bounded from below by $V_1 = 1$ and the diagonal $V_1 = V_2$, and from above by $V_2 = V_2^{\text{saddle}}$ and the trajectory Γ (see Figure 4(B)). Let us consider trajectories that cross the diagonal at $1 < V_1(w_0) = V_2(w_0) < \bar{V}$. We first show that as we move backward along these trajectories, they do not exit the region T_{II}.

Indeed, as in the proof of Lemma 3.7, along the diagonal all trajectories point outward of T_{II} (see (24)). On the nullcline $V_2 = V_2^{\text{saddle}}$, the trajectories point in the direction $(0, 1)$, outward of T_{II}. In addition, the trajectories cannot cross the singular line $V_1 = 1$ or the trajectory Γ. Therefore, for $w < w_0$, they have to remain in the region T_{II}.

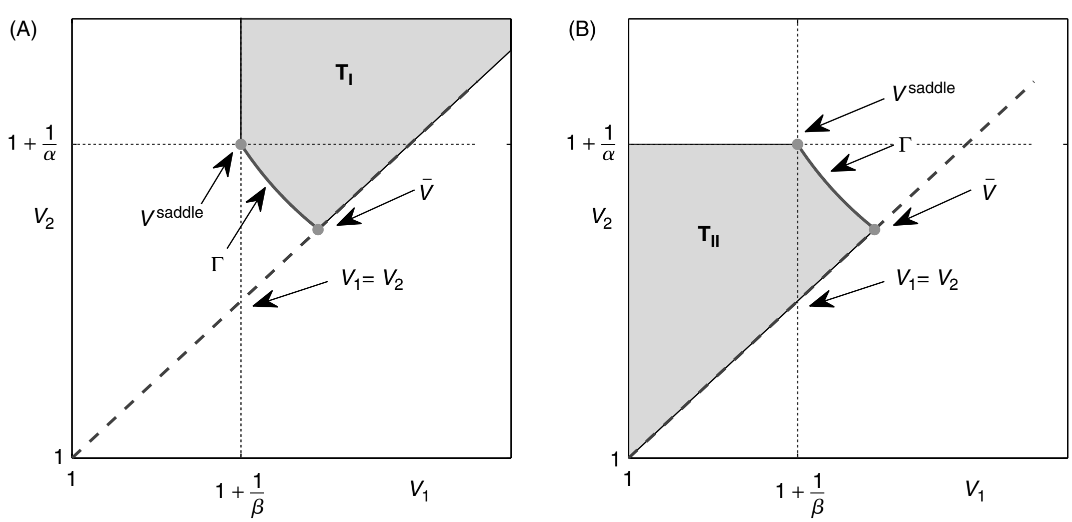

Figure 4. The same data as in Figure 1(A). Zoom-in on the regions T_{I} and T_{II}.

Because T_{II} is a bounded region, all trajectories within it must start from the fixed-point V^{saddle} or from the singular point $(1,1)$. The trajectories cannot start from $\mathbf{V}^{\text{saddle}}$, however, because the unstable manifold of $\mathbf{V}^{\text{saddle}}$ consists of Γ that defines the boundary of T_{II}, and Γ' that does not lie in T_{II} but rather in the region S', see proof of Lemma 3.3. □

In the proof of Lemma 3.2 we saw that trajectories start from $(1,1)$ at a finite w. Moreover, by (15), the corresponding solutions satisfy

$$v_1'(b) = V_1 + bV_1' = V_1 + V_1\frac{V_2^{\text{saddle}} - V_2}{V_2 - 1} \longrightarrow +\infty, \quad (V_1, V_2) \longrightarrow (1,1),$$

and similarly for $v_2'(b)$. Therefore, they are type II backward solutions of (10a), (10c), which have an infinite derivative at some $0 < b_1$. In particular, they do not correspond to solutions of (10).

Finally, we note that the results of this section provide a alternative proof that Γ is the unique trajectory that corresponds to a solution of (10), hence to the uniqueness Theorem 3.2.

3.2. Two types of players. To study the system (12), we transform it into an autonomous dynamical system using the change of variables (14), (16). This gives

$$\begin{aligned} V_1' &= \frac{V_1}{(n_1+n_2-1)\alpha}\left[\frac{n_2}{V_2^{\text{saddle}}-1}\frac{V_2^{\text{saddle}}-V_2}{V_2-1} - \frac{n_2-1}{V_1^{\text{saddle}}-1}\frac{V_1^{\text{saddle}}-V_1}{V_1-1}\right],\\ V_2' &= \frac{V_2}{(n_1+n_2-1)\beta}\left[\frac{n_1}{V_1^{\text{saddle}}-1}\frac{V_1^{\text{saddle}}-V_1}{V_1-1} - \frac{n_1-1}{V_2^{\text{saddle}}-1}\frac{V_2^{\text{saddle}}-V_2}{V_2-1}\right].\end{aligned} \tag{32a}$$

As in the two coalitions case, the boundary conditions are given by

$$V_1(-\infty) = V_1^{\text{saddle}}, \qquad V_2(-\infty) = V_2^{\text{saddle}}, \tag{32b}$$

(see Lemma 3.9) and

$$V_1(\bar{w}) = V_2(\bar{w}) = e^{-\bar{w}}, \tag{32c}$$

where $\bar{w} = \log \bar{b}$, and

$$V_1^{\text{saddle}} = 1 + \frac{1}{(n_1-1)\alpha + n_2\beta}, \qquad V_2^{\text{saddle}} = 1 + \frac{1}{n_1\alpha + (n_2-1)\beta}. \tag{32d}$$

As in the two coalitions case, the dynamical system (32a) has a fixed point at $\mathbf{V}^{\text{saddle}}$, see (32d), which is a saddle point. Indeed, the linearization $\mathbf{V}(w) \sim \mathbf{V}^{\text{saddle}} + \mathbf{V}^{\text{lin}}(w)$ of (17a) about $\mathbf{V}^{\text{saddle}}$ gives

$$\frac{d}{dw}\mathbf{V}^{\text{lin}} = \mathbf{A}\,\mathbf{V}^{\text{lin}}, \tag{33a}$$

where $\mathbf{A}$ is the Jacobian matrix of (32a) at $\mathbf{V} = \mathbf{V}^{\text{saddle}}$, i.e.,

$$\mathbf{A} = \begin{pmatrix} (n_2-1)\dfrac{V_1^{\text{saddle}}}{(V_1^{\text{saddle}}-1)^2\alpha(n_1+n_2-1)} & -n_1\dfrac{V_2^{\text{saddle}}}{(V_1^{\text{saddle}}-1)^2\beta(n_1+n_2-1)} \\ -n_2\dfrac{V_1^{\text{saddle}}}{(V_2^{\text{saddle}}-1)^2\alpha(n_1+n_2-1)} & (n_1-1)\dfrac{V_2^{\text{saddle}}}{(V_2^{\text{saddle}}-1)^2\beta(n_1+n_2-1)} \end{pmatrix}. \tag{33b}$$

The eigenvalues and corresponding eigenvectors of $\mathbf{A}$ are

$$\lambda_\pm = \frac{a_{11}+a_{22} \pm \sqrt{(a_{11}-a_{22})^2 + 4a_{12}a_{21}}}{2}, \qquad \mathbf{U}_\pm = \begin{pmatrix} \dfrac{-2a_{21}}{(a_{11}-a_{22}) \mp \sqrt{(a_{11}-a_{22})^2+4a_{12}a_{21}}} \\ 1 \end{pmatrix}. \tag{34}$$

Equation (34) implies that $\lambda_+ > 0$. By Vieta's formula,

$$\lambda_-\lambda_+ = \det \mathbf{A} = \frac{(n_1-1)(n_2-1) - n_1n_2}{4}\,\frac{V_1^{\text{saddle}}V_2^{\text{saddle}}}{\alpha\beta(n-1)^2(V_1^{\text{saddle}}-1)^2(V_2^{\text{saddle}}-1)^2} < 0.$$

Hence $\lambda_- < 0$. Therefore, $\mathbf{V}^{\text{saddle}}$ is a saddle point.

The following Lemma shows that solutions of the initial value problem (12a), (12b) correspond to trajectories in the phase plane that start at $\mathbf{V}(w=-\infty) = \mathbf{V}^{\text{saddle}}$:

Lemma 3.9. *Let $\{v_1, v_2\}$ be a solution of (12a), (12b) that satisfies (8). Then, the corresponding solution $\{V_1, V_2\}$ of (32a) satisfies (32b).*

Proof. The proof is similar to the proofs of Lemma 3.1 and Lemma 3.2 (see Appendix B). □

Lemma 3.10. *the unstable manifold of $\mathbf{V}^{\text{saddle}}$ consists of*
(i) *the trajectory $\mathbf{V}(w) \equiv \mathbf{V}^{\text{saddle}}$;*
(ii) *a trajectory Γ that descends from $\mathbf{V}^{\text{saddle}}$ in the direction of $\mathbf{U}_+$, and intersects with the diagonal $V_1 = V_2$;*
(iii) *a trajectory Γ' that descends from $\mathbf{V}^{\text{saddle}}$ in the direction of $-\mathbf{U}_+$. This trajectory does not intersect with the diagonal $V_1 = V_2$.*

Proof. The proof is identical to the proof of Lemma 3.3. □

As in the two-coalition case, the trajectories $\mathbf{V}^{\text{saddle}}$ and Γ' cannot be solutions of (32), because they do not intersect with the diagonal $V_1 = V_2$. Therefore, in analogy to Corollary 3.1, the only trajectory that corresponds to a solution of (32a), (32b) that intersects with the diagonal $V_1 = V_2$ is Γ.

As in the two-coalition case, the trajectory Γ represents a one-parameter family of solutions (see (25)).

Theorem 3.3. *There exists a unique solution to (32).*

Proof. The proof is identical to the proof of Theorem 3.1. □

Theorem 3.3 implies that the boundary-value problem (12) has a unique solution:

Theorem 3.4. *There exists a unique solution to (12).*

We now show that the unique solution of (12) is monotonically increasing.

Lemma 3.11. *The solutions $v_1(b)$ and $v_2(b)$ of (12) are strictly monotonically increasing.*

Proof. Without loss of generality, we assume that $\alpha < \beta$. Substituting $v_i(b) = bV_i(b)$ in Equation (12a) yields

$$v_1'(b) = \frac{V_1}{\alpha}\frac{1}{n_1+n_2-1}\left[\frac{n_2}{V_2-1} - \frac{n_2-1}{V_1-1}\right], \qquad v_2'(b) = \frac{V_2}{\beta}\frac{1}{n_1+n_2-1}\left[\frac{n_1}{V_1-1} - \frac{n_1-1}{V_2-1}\right].$$

Monotonicity of v_1 and v_2 requires that

$$\frac{n_2}{V_2-1} - \frac{n_2-1}{V_1-1} > 0, \tag{35a}$$

$$\frac{n_1}{V_1-1} - \frac{n_1-1}{V_2-1} > 0. \tag{35b}$$

From the proof of Lemma 3.10 it follows that in the region S,

$$1 < V_1^{\text{saddle}} \le V_1 \le V_2 \le V_2^{\text{saddle}}.$$

Hence, by (32d),

$$(n_1-1)\alpha + n_2\beta \ge \frac{1}{V_1-1} \ge \frac{1}{V_2-1} \ge n_1\alpha + (n_2-1)\beta,$$

from which inequality (35a) follows. In addition, since $1 < V_1 \le V_2$, inequality (35b) is also satisfied. □

Theorem 3.5. *The solution of (12) is differentiable at $b = 0$.*
Proof. *By Lemma 3.9,*

$$\lim_{w\to-\infty} V_i(w) = V_i^{\text{saddle}} = \lim_{b\to 0}\frac{v_i(b)}{b} = v_i'(0). \quad \square$$

The differentiability at $b = 0$ is not obvious (see Remark 3.2). To the best of our knowledge, the result of Theorem 3.5 is new.

It is not known whether (32) admits a first integral. In particular, an explicit expression for $\bar{V}$ and $\bar{b}$ is not available. Finally, we note that the backward-solutions analysis of §3.1.3 can be extended to the two-player case.

3.3. ***n*** **coalitions.** Following the analysis in §§3.1 and 3.2, we can transform the n-coalition system (13a) into an autonomous system by substituting $v_i(b) = b\,V_i(b)$ and $b = e^w$ in (13a). This gives

$$V_i'(w) = G_i(V_1, \dots, V_n), \quad i = 1, \dots, n, \tag{36a}$$

where

$$G_i(V_1, \dots, V_n) = \frac{V_i}{\alpha_i}\left[\left(\frac{1}{n-1}\sum_{j=1}^{n}\frac{1}{V_j - 1}\right) - \frac{1}{V_i - 1}\right] - V_i, \quad i = 1, \dots, n, \tag{36b}$$

with the boundary conditions

$$V_i(w = -\infty) = V_i^{\text{saddle}}, \quad i = 1, \dots, n, \tag{36c}$$

and

$$V_i(\bar{w}) = \bar{V}, \quad i = 1, \dots, n, \tag{36d}$$

where $\bar{V} = e^{-\bar{w}}$, $\bar{w} = \log \bar{b}$ and

$$V_i^{\text{saddle}} = 1 + \frac{1}{\sum_{j=1}^{n}\alpha_j - \alpha_i}, \quad i = 1, \dots, n. \tag{36e}$$

Remark 3.7. By imposing the boundary condition (36c), we implicitly assume that the solution of (13) is differentiable at $b = 0$. In Lemma 3.2 and Lemma 3.9 we justified this assumption for the case of two coalitions or two types of players. The proofs of these Lemmas, however, relied on the relatively simple structure of planar dynamical systems. Whether the solutions in the general n-dimensional case are also differentiable is currently an open problem.

The dynamical system (36) has a fixed point at $\mathbf{V}^{\text{saddle}} = (V_1^{\text{saddle}}, \dots, V_n^{\text{saddle}})$. This fixed point is a saddle point with a one-dimensional stable manifold and an $n-1$ dimensional unstable manifold:

Lemma 3.12. *Let* $\mathbf{A}$ *be the* $n \times n$ *Jacobian matrix*

$$\mathbf{A} = \left.\frac{\partial G_i(V_1, V_2, \dots, V_n)}{\partial(V_1, V_2, \dots, V_n)}\right|_{\mathbf{V}=\mathbf{V}^{\text{saddle}}}, \tag{37}$$

where $G_i(V_1, V_2, \dots, V_n)$ *is given by* (36b). *Then,* $\mathbf{A}$ *has* $n-1$ *positive eigenvalues and one negative eigenvalue.*

Proof. Let us first compute the matrix elements of $\mathbf{A} = [a_{ij}]$. For $i \neq j$,

$$a_{ij} = \left.\frac{\partial G_i}{\partial V_j}\right|_{\mathbf{V}=\mathbf{V}^{\text{saddle}}} = -\frac{V_i^{\text{saddle}}}{(n-1)\alpha_i(V_j^{\text{saddle}} - 1)^2},$$

and for $i = j$,

$$a_{ii} = \left.\frac{\partial G_i}{\partial V_i}\right|_{\mathbf{V}=\mathbf{V}^{\text{saddle}}} = \frac{(n-2)V_i^{\text{saddle}}}{(n-1)\alpha_i(V_i^{\text{saddle}} - 1)^2},$$

where in the calculation of a_{ii} we used the fact that $G_i(\mathbf{V}^{\text{saddle}}) = 0$. The Jacobian matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{D}_L \mathbf{B} \mathbf{D}_R,$$

where

$$\mathbf{B} = \begin{bmatrix} n-2 & -1 & \cdots & -1 \\ -1 & n-2 & \cdots & -1 \\ \vdots & -1 & \ddots & \vdots \\ -1 & \cdots & -1 & n-2 \end{bmatrix},$$

and $\mathbf{D}_L$ and $\mathbf{D}_R$ the diagonal matrices

$$\mathbf{D}_L = \operatorname{diag}\left(\frac{1}{(V_1^{\text{saddle}} - 1)^2}, \dots, \frac{1}{(V_n^{\text{saddle}} - 1)^2}\right), \qquad \mathbf{D}_R = \operatorname{diag}\left(\frac{V_1^{\text{saddle}}}{(n-1)\alpha_1}, \dots, \frac{V_n^{\text{saddle}}}{(n-1)\alpha_n}\right). \tag{38}$$

Since $|\mathbf{A} - \lambda I| = 0$ implies that $|\mathbf{B} - \lambda \mathbf{D}_L^{-1}\mathbf{D}_R^{-1}| = 0$, the eigenvalues of $\mathbf{A}$ are the same as those of the generalized eigenvalue problem

$$\mathbf{B}v = \lambda \mathbf{D}v, \qquad \mathbf{D} = \mathbf{D}_L^{-1}\mathbf{D}_R^{-1}. \tag{39}$$

Since **B** is real and symmetric and **D** is diagonal and strictly positive, **B** and **D** can be simultaneously reduced to a diagonal form, i.e., there exists a matrix **M** such that

$$\mathbf{M}^t\mathbf{B}\mathbf{M} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), \qquad \mathbf{M}^t\mathbf{D}\mathbf{M} = \mathbf{I}.$$

Hence, $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of the generalized eigenvalue problem (39), and therefore also the eigenvalues of **A**. Since **B** is real and symmetric, it follows from Sylvester's law of inertia that $\mathbf{M}^t\mathbf{B}\mathbf{M}$ and **B** have the same number of positive and negative eigenvalues. Therefore, it is sufficient to show that **B** has $n-1$ positive eigenvalues and one negative eigenvalue.

The matrix **B** has the positive eigenvalue $\delta_i = n-1$ with multiplicity $n-1$, whose corresponding eigenvectors $\mathbf{u}_i$ are spanned by the $n-1$-dimensional space of vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ for which $\sum_{i=1}^n u_i = 0$. In addition, it has the simple negative eigenvalue $\delta_n = -1$ with the corresponding eigenvector $\mathbf{u}_n = (1, 1, \ldots, 1)^t$. Therefore, the result is proved. □

Let us consider a solution of (36a) near the saddle point $\mathbf{V}^{\text{saddle}}$, i.e.,

$$\mathbf{V}(w) \sim \mathbf{V}^{\text{saddle}} + \mathbf{V}^{\text{lin}}(w).$$

By Lemma 3.12,

$$\mathbf{V}^{\text{lin}}(w) = c_1\mathbf{U_1}e^{\lambda_1 w} + \cdots + c_{n-1}\mathbf{U_{n-1}}e^{\lambda_{n-1} w} + c_n\mathbf{U}_n e^{\lambda_n w}, \tag{40}$$

where $\{\lambda_i\}_{i=1}^{n-1}$ are the positive eigenvalues of **A** and λ_n is the negative eigenvalue of **A**. If in addition, the solution satisfies the initial condition (36c), then $c_n = 0$. Hence, the solutions of (36a), (36c) constitute an $n-1$ parameter family.

Condition (36d) implies that the trajectory Γ should intersect with the diagonal

$$V_1 = V_2 = \cdots = V_n. \tag{41}$$

This condition adds $n-1$ constraints to the solution. Therefore, the number of degrees of freedom equals the number of constraints. When $n = 2$, there is, indeed, a unique trajectory that satisfies (41), see Corollary 3.1. Proving the existence and uniqueness of the trajectory for $n > 2$, amounts to showing that the $(n-1)$-dimensional unstable manifold intersects with the diagonal (41) at a single point. The following theorem proves existence in the special case of a weak asymmetry, i.e., $\alpha_i = \alpha + O(\varepsilon)$.

Theorem 3.6. *Let $\alpha > 0$, and let $\{h_i\}_{i=1}^n$ be constants. There exists $\varepsilon_0 > 0$ such that for any $0 < |\varepsilon| < \varepsilon_0$, there exists a differentiable solution of* (36) *with*

$$\alpha_i(\varepsilon) = \alpha + \varepsilon h_i. \tag{42}$$

Proof. Let $\mathbf{V}^{\text{saddle}}(\varepsilon)$ be given by (36e), where $\{\alpha_i(\varepsilon)\}$ are given by (42). By Lemma 3.12, the linearized system around $\mathbf{V}^{\text{saddle}}(\varepsilon)$ has $n-1$ positive eigenvalues with corresponding eigenvectors $\{\psi_i(\varepsilon)\}_{i=1}^{n-1}$ and one negative eigenvalue with corresponding eigenvector $\psi_n(\varepsilon)$.

According to the stable manifold theorem, the dynamical system (36a) has an unstable manifold $\mathbf{U}_\varepsilon$ near the saddle point. Namely, there exists a smooth manifold $\mathbf{U}_\varepsilon$ such that if the dynamical system (36a) is integrated backward from any point on $\mathbf{U}_\varepsilon$, it will reach $\mathbf{V}^{\text{saddle}}(\varepsilon)$ at $w = -\infty$. Moreover, the stable manifold theorem shows that $\mathbf{U}_\varepsilon$ is tangent to the unstable subspace $E^s = \operatorname{span}\{\psi_1(\varepsilon), \ldots, \psi_{n-1}(\varepsilon)\}$ at $\mathbf{V}^{\text{saddle}}(\varepsilon)$. Therefore, $\mathbf{U}_\varepsilon$ can be parameterized as

$$\mathbf{U}_\varepsilon(c_1, \ldots, c_{n-1}) = \mathbf{V}^{\text{saddle}}(\varepsilon) + \sum_{i=1}^{n-1} c_i\psi_i(\varepsilon) + u_\varepsilon(c_1, \ldots, c_{n-1})\psi_n(\varepsilon),$$

where $c_1, \ldots, c_{n-1}$ are small enough, $u_\varepsilon(0, \ldots, 0) = 0$, and

$$\left.\frac{\partial u_\varepsilon}{\partial c_i}\right|_{c_1 = \cdots = c_{n-1} = 0} = 0. \tag{43}$$

Let

$$\mathbf{1} = \underbrace{(1, \ldots, 1)}_{n \text{ times}}.$$

To prove that (36) has a solution, it suffices to show that $\mathbf{U}_\varepsilon$ intersects with the diagonal $V_1 = \cdots = V_n$ at some point $\mathbf{1}\cdot\bar{V} = (\bar{V}, \ldots, \bar{V})$, since this implies that there exists a trajectory leaving $\mathbf{V}^{\text{saddle}}$ and ending at $\mathbf{1}\cdot\bar{V}$. Let us define

$$\mathbf{H}(\varepsilon, \bar{c}_1, \ldots, \bar{c}_{n-1}, \bar{V}) = \mathbf{U}_\varepsilon(\bar{c}_1, \ldots, \bar{c}_{n-1}) - \mathbf{1}\cdot\bar{V}.$$

When $\varepsilon = 0$, then $\mathbf{V}^{\text{saddle}}(0) = \mathbf{1}\cdot\bar{V}(0)$, where $\bar{V}(0) = 1 + 1/((n-1)\alpha)$. Hence,

$$\mathbf{H}\left(0, \underbrace{0, \ldots, 0}_{n-1}, 1 + \frac{1}{(n-1)\alpha}\right) = 0. \tag{44}$$

We now show that there exists a neighborhood $0 < |\varepsilon| \ll 1$ for which $\mathbf{U}_\varepsilon$ intersects with the line $\mathbf{1}\cdot\bar{V}$, by using the implicit function theorem to show that there exist parameters $\{\bar{c}_1(\varepsilon), \ldots, \bar{c}_{n-1}(\varepsilon), \bar{V}(\varepsilon)\}$ such that

$$\mathbf{H}(\varepsilon, \bar{c}_1(\varepsilon), \ldots, \bar{c}_{n-1}(\varepsilon), \bar{V}(\varepsilon)) = 0, \quad 0 \le \varepsilon < \varepsilon_0. \tag{45}$$

To do that, we first show that $\mathbf{U}_\varepsilon$ is smooth with respect to changes in ε. The differentiability in ε follows from, e.g., Robbin [18, section 4]. Alternatively, let $\mathbf{P}(w) := (V_1(w), \ldots, V_n(w), \varepsilon(w))$. Then,

$$\mathbf{P}'(w) = \tilde{\mathbf{G}} = (G_1, \ldots, G_n, 0)^T, \tag{46}$$

where $\{G_i\}$ are given by (36b). Note that $\varepsilon'(w) = 0$ since ε does not depend on w. For any small enough ε, the point $(\mathbf{V}^{\text{saddle}}(\varepsilon), \varepsilon)$ is a fixed point of (46). Therefore, the curve $\{(\mathbf{V}^{\text{saddle}}(\varepsilon), \varepsilon)\}$ is the center manifold of (46), and accordingly $\{(U_\varepsilon, \varepsilon)\}$ is the center-unstable manifold of (46). By Chow and Hale [1, Theorem 2.11, p. 319], the center-unstable manifold is smooth, since $\tilde{\mathbf{G}}$ is smooth in a neighborhood of $(\mathbf{V}^{\text{saddle}}, \varepsilon)$. Hence, $\mathbf{U}_\varepsilon$ is smooth with respect to ε.

Next, we first show that the Jacobian matrix of $\mathbf{H}$ at $\varepsilon = 0$ is not singular. Indeed, the Jacobian matrix of $\mathbf{H}$ is given by

$$J_{\mathbf{H}} = \left[\frac{\partial \mathbf{H}}{\partial \bar{c}_1}, \ldots, \frac{\partial \mathbf{H}}{\partial \bar{c}_{n-1}}, \frac{\partial \mathbf{H}}{\partial \bar{V}}\right] = \left[\psi_1 + \frac{\partial u_\varepsilon}{\partial \bar{c}_1}\psi_n, \ldots, \psi_{n-1} + \frac{\partial u_\varepsilon}{\partial \bar{c}_{n-1}}\psi_n, -\mathbf{1}\right].$$

By (43),

$$J_{\mathbf{H}}(\varepsilon = 0) = [\psi_1(\varepsilon = 0), \ldots, \psi_{n-1}(\varepsilon = 0), -\mathbf{1}].$$

When $\varepsilon = 0$, the matrices $\mathbf{D}_L$ and $\mathbf{D}_R$ are, up to multiplication by a scalar, the identity matrix (see (38)). Therefore, the eigenvectors of $\mathbf{A}$ and of $\mathbf{B}$ are the same. In particular, the eigenvector ψ_n of $\mathbf{A}$ is given by the eigenvector $\underline{1}$ of $\mathbf{B}$, see proof of Lemma 3.12. Therefore,

$$J_{\mathbf{H}}(\varepsilon = 0) = c[\psi_1(0), \ldots, \psi_{n-1}(0), \psi_n(0)], \quad c \neq 0.$$

Since $\{\psi_i(0)\}_{i=1}^n$ are linearly independent, it follows $\det(J_{\mathbf{H}}(\varepsilon = 0)) \neq 0$. In addition, by the stable manifold theorem, the surface $\mathbf{U}_\varepsilon$, hence also the function $\mathbf{H}$, are smooth. Therefore, by the implicit function theorem, there exists a region $0 < |\varepsilon| < \varepsilon_0$ and smooth functions $\{\bar{c}_1(\varepsilon), \ldots, \bar{c}_{n-1}(\varepsilon), \bar{V}(\varepsilon)\}$ that satisfy (45).[8] □

As in the two-coalition problem, a trajectory in the phase plane of (36a) corresponds to a one-parameter family of solutions of (13a). Indeed, the invariance $w \to w + c$ ensures that if $\{V_i\}_{i=1}^n$ is a solution of (36a), (36c) then $\{V_i(w + c)\}_{i=1,\ldots,n}$ is also a solution of (36a), (36c). Therefore, if $\{V_i\}$ satisfies the right boundary conditions

$$V_i(\tilde{w}) = \bar{V}, \quad i = 1, \ldots, n, \tag{47}$$

for some $\bar{V} > 0$, then $\{V_i(w + \tilde{c})\}_{i=1}^n$, where $\tilde{c} = -\tilde{w} - \log\bar{V}$, is a solution of (36a), (36c) that also satisfies (36).

Theorem 3.6 implies the existence of solutions of (13) when $0 < |\varepsilon| < \varepsilon_0$. A proof of existence (and uniqueness) in the general case is more complex, and we shall not pursue it here. Note that this result follows from Lebrun's proof (Lebrun [9, 11, 12]) of the existence and uniqueness of a solution to (7).

We now show that the corresponding solutions of (13) in the weakly asymmetric case (42) are strictly monotonically increasing:

Lemma 3.13. *Let $\alpha > 0$, and let $\{h_i\}_{i=1}^n$ be constants. There exists $\varepsilon_0 > 0$ such that for any $0 < |\varepsilon| < \varepsilon_0$, there exist solutions $v_i(b; \varepsilon)$ of* (13) *in the weakly asymmetric case* (42), *which are strictly monotonically increasing in b.*

[8] The implicit function theorem also implies "*local uniqueness*," i.e., that there exists a unique trajectory that connects $\bar{V}(\varepsilon)$ with the diagonal $\mathbf{1}\cdot\bar{V}$ for $\bar{V}$ near $\mathbf{V}^{\text{saddle}}$.

Proof. In light of Theorem 3.6, we only need to show that the corresponding solutions of (13) satisfy $v_i'(b;\varepsilon) > 0$ for $0 \le b \le \bar{b}(\varepsilon)$ and $-\varepsilon_0 \le \varepsilon \le \varepsilon_0$. By (1), the symmetric solution (i.e., the solution for $\varepsilon = 0$) is given by

$$b(v) = v - \frac{\int_0^v s^{\alpha(n-1)}\,ds}{v^{\alpha(n-1)}} = v\frac{\alpha(n-1)}{1+\alpha(n-1)}.$$

Therefore, $v_i'(b;\varepsilon = 0) = v'(b) = (1+\alpha(n-1))/(\alpha(n-1)) > 0$ for $0 \le b \le \bar{b}$ $(\varepsilon = 0)$. The proof of Theorem 3.6 implies that $\bar{V}(\varepsilon)$ is continuous in ε. Hence, $\bar{b}(\varepsilon) = 1/\bar{V}(\varepsilon)$ is continuous in ε. Therefore, it is enough to prove that $v_i'(b;\varepsilon) = V_i + bV_i'$ are continuous in ε for $0 \le b \le \bar{b}(\varepsilon)$ and $-\varepsilon_0 \le \varepsilon \le \varepsilon_0$.

The functions $V_i(w;\varepsilon)$ are continuous in ε uniformly in w for $-\infty < w < \bar{w}$, because they are the backward solutions of Equations (36a), (36b), which are continuous in ε, subject to the right boundary conditions (36d), which are continuous in ε since $V_i(\bar{w},\varepsilon) = \bar{V}(\varepsilon)$ is continuous in ε, and the left boundary conditions (36c), which is continuous in ε, since $V_i(w = -\infty, \varepsilon) = V_i^{\text{saddle}} = 1 + 1/((n-1)\alpha + \varepsilon(\sum_{j=1}^n h_j - h_i))$ is continuous in ε. Therefore, by (36a), (36b), $V_i'(w;\varepsilon)$ are continuous in ε. Hence, $v_i'(b;\varepsilon)$ are continuous in ε. Therefore, $v_i'(b;\varepsilon) > 0$ for a sufficiently small ε_0. □

4. Backward- and forward-shooting methods. We now use the dynamical-system formulation to analyze the standard backward-shooting method for computing the equilibrium strategies of asymmetric first-price auctions, to understand why the previous attempt to devise a forward-shooting method failed, and to devise a novel numerical method that is based on forward shooting.

4.1. The backward-shooting method.

4.1.1. Review. The differential system (7) for the equilibrium strategies of asymmetric first-price auctions is a boundary-value problem. From a numerical point of view, because the value of $\bar{b}$ is unknown, it seems natural to solve (7) using a forward-shooting approach, i.e., search for the solution of the initial value problem (7a), (7b) that satisfies (7c).

The right-hand side of (7a) at $b = 0$ is of the form $\frac{0}{0}$. Hence, a special treatment at $b = 0$ is required in order to numerically solve (7a), (7b). To overcome this problem, Marshall et al. [14] obtained an analytic approximation for the solution at $b = h \ll 1$, i.e., $v_i(b = h) \approx v_{i,h}$, and then solved (7a) for $b \ge h$ with the initial condition

$$v_i(b = h) = v_{i,h}. \tag{48}$$

Specifically, Marshall et al. [14] considered the two-coalition system (10). By assuming differentiability at $b = 0$[9] and applying l'Hospital's rule to (10), Marshall et al. [14] obtained that

$$v_1'(0) = 1 + \frac{1}{\beta}, \qquad v_2'(0) = 1 + \frac{1}{\alpha}.$$

An additional application of l'Hospital's rule gives that $v_1''(0) = v_2''(0) = 0$. Therefore,

$$v_1(h) = \left(1 + \frac{1}{\beta}\right)h + o(h^2), \qquad v_2(h) = \left(1 + \frac{1}{\alpha}\right)h + o(h^2).$$

Hence, it is reasonable to approximate $v_i(b = h)$ with

$$v_{1,h} = \left(1 + \frac{1}{\beta}\right)h, \qquad v_{2,h} = \left(1 + \frac{1}{\alpha}\right)h. \tag{49}$$

The solution of the Equation (10a) with the initial condition (48), (49) is given by

$$v_1^{\text{explicit}}(b) = \left(1 + \frac{1}{\beta}\right)b, \qquad v_2^{\text{explicit}}(b) = \left(1 + \frac{1}{\alpha}\right)b. \tag{50}$$

The explicit solution (50), however, does not satisfy the right boundary condition (10c).

Marshall et al. [14] stated that one cannot solve (10) using forward shooting, because the explicit solution (50) acts as an "attractor." Therefore, they opted for solving the boundary-value problem (7) using a backward

[9] See Remark 3.2.

shooting approach. In this approach, one searches for the value of $\bar{b}$ by solving Equation (7a) backward in b for $b \le \bar{b}_\varepsilon$, subject to the end condition

$$v_i(\bar{b}_\varepsilon) = 1, \qquad i = 1, \ldots, n, \tag{51}$$

and looking for the value of $\bar{b}_\varepsilon$ for which $v_i(0) = 0$, $i = 1, \ldots, n$.

The backward-shooting method was used in numerous subsequent studies (e.g., Fibich and Gavious [2], Fibich et al. [4, 5], Gayle and Richard [6], Li and Riley [13], Maskin and Riley [15]), and has been for many years the method of choice for computing the equilibrium bids in asymmetric first-price auctions. Nevertheless, this method is far from optimal. Indeed, Marshall et al. [14, p. 195] observed that "backward solutions are well behaved except in neighborhoods of the origin where they become (highly) unstable." Recently, we showed analytically that the backward-shooting method is unstable in the symmetric case (Fibich and Gavish [3]). In that study, however, we did not analyze the backward-shooting method in the asymmetric case.

4.1.2. Dynamical-systems view of backward solutions. The analysis in §3.1 implies that backward solutions of

$$v_1'(b) = \frac{v_1(b)}{\alpha}\frac{1}{v_2(b) - b}, \qquad v_2'(b) = \frac{v_2(b)}{\beta}\frac{1}{v_1(b) - b}, \quad b \le \bar{b}_\varepsilon,$$

with the initial condition

$$v_1(\bar{b}_\varepsilon) = v_2(\bar{b}_\varepsilon) = 1$$

(see (10a), (51)) correspond to solutions of

$$V_1'(w) = V_1(w)\frac{1 + 1/\alpha - V_2(w)}{V_2(w) - 1}, \qquad V_2'(w) = V_2(w)\frac{1 + 1/\beta - V_1(w)}{V_1(w) - 1}, \quad w \le \bar{w}_\varepsilon \tag{52a}$$

(see (17a)) with the initial condition

$$V_1(\bar{w}_\varepsilon) = V_2(\bar{w}_\varepsilon) = \bar{V}_\varepsilon, \tag{52b}$$

where $\bar{V}_\varepsilon = 1/\bar{b}_\varepsilon$ and $\bar{w}_\varepsilon = \ln \bar{b}_\varepsilon$.

Let us denote by Γ^ε the trajectory of the solution of the backward problem (52). This trajectory flows in the opposite direction to the vector field. If $\bar{V}_\varepsilon \ne \bar{V}$, Γ^ε does not end at $\mathbf{V}^{\text{saddle}}$, but rather escapes toward (∞, ∞) or $(1, 1)$ (see Figure 5(A)). Moreover, the distance between $\Gamma(w)$ and $\Gamma^\varepsilon(w)$ increases as w decreases from $\bar{w}$. Therefore, the dynamical system representation shows that backward solutions are inherently unstable. Indeed, in Fibich and Gavish [3] we conducted a systematic analysis of backward solutions of (7a) and found that the error $|v_\varepsilon(b) - v(b)|$ of backward solutions increases monotonically as b decreases.

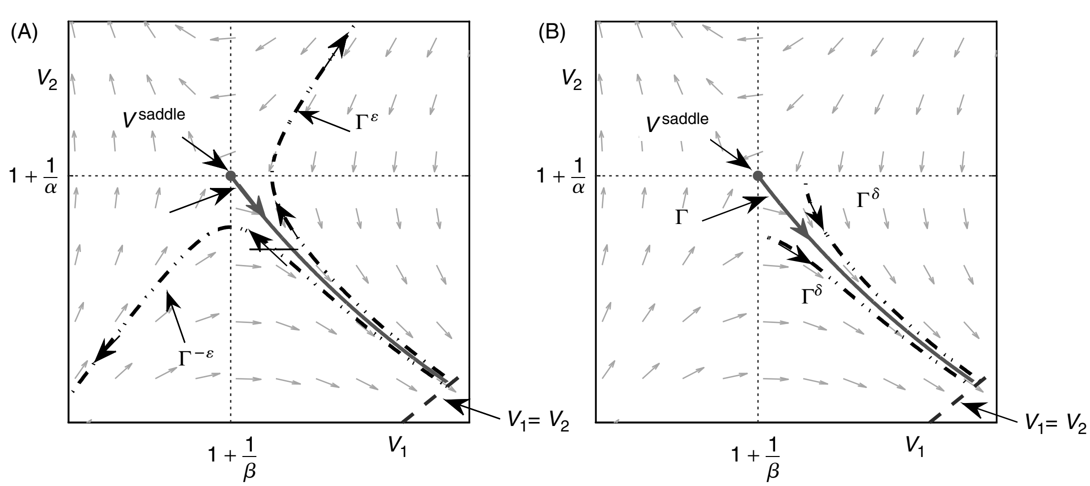

Figure 5. The trajectory Γ (solid) that corresponds to the solution of (10). Also plotted in (A), the trajectories $\Gamma^{\pm\varepsilon}$ (dash dotted) that corresponds to a backward solution of (52) with $\bar{V} = \bar{V} \pm \varepsilon$. (B) Trajectory Γ^δ (dash dotted) that corresponds to the forward solutions that start at $V^\delta := \mathbf{V}^{\text{saddle}} + \boldsymbol{\delta}$. The dashed curve in the lower-right corners of (A) and (B) is the diagonal $V_1 = V_2$.

4.2. The forward-shooting approach rehabilitated. As noted, the dynamical-systems formulation shows that solving the equations for $\{v_i(b)\}$ backward is inherently unstable. Recall that Marshall et al. [14] resorted to the backward-solving approach, because they concluded that forward methods "attract" to the explicit solution (50) that does not satisfy the right boundary condition (10c). In §4.2.2 we will show that this conclusion is false. Indeed, we now use the dynamical systems representation to derive a stable forward-shooting method.

Consider the n-coalition system

$$V_i'(w) = G_i(V_1, \dots, V_n), \quad i = 1, \dots, n, \tag{53a}$$

see (36a), with the initial condition

$$V_i(w = -\infty) = V_i^{\text{saddle}}, \quad i = 1, \dots, n, \tag{53b}$$

see (36c). As shown in §3.3, solutions of the initial-value problem (53) can be approximated near the saddle point $\mathbf{V}^{\text{saddle}}$ with

$$\mathbf{V}(w) \sim \mathbf{V}^{\text{saddle}} + \mathbf{V}^{\text{lin}}(w), \tag{54a}$$

where $\mathbf{V}^{\text{lin}}(w)$ belongs to the $n-1$ dimensional subspace

$$\mathbf{V}^{\text{lin}}(w) = c_1 \mathbf{U_1}\, e^{\lambda_1 w} + \dots + c_{n-1} \mathbf{U_{n-1}}\, e^{\lambda_{n-1} w}. \tag{54b}$$

Therefore,

Observation 4.1. The initial-value problem (53) does not have a unique solution, but rather an $n-1$ parameter family of solutions.

We now develop a forward-marching method for solving the n-coalition problem (36). Obviously, it is not possible to solve the initial value problem (53) forward, because it does not have a unique solution. However, if the value of $\mathbf{V}(w)$ is known at some $-\infty < w_{-\infty} \ll -1$, then it is possible to solve the initial value problem (53a) for $w > w_{-\infty}$, because this problem has a unique solution. By (54),

$$\mathbf{V}(w_{-\infty}) \approx \mathbf{V}_{-\infty} := \mathbf{V}^{\text{saddle}} + \mathbf{V}^{\text{lin}}(w_{-\infty}; c_1, \dots, c_{n-1}), \tag{55}$$

where $\mathbf{V}^{\text{lin}}$ is given by (54b). Therefore, in the forward-shooting method we search for the values of $\{c_i\}_{i=1}^{n-1}$, such that the solution of (53a) for $w > w_{-\infty}$ with the initial condition $\mathbf{V}(w_{-\infty}) = V_{-\infty}$ (see (55)) satisfies

$$V_1(\tilde{w}) = V_2(\tilde{w}) = \dots = V_n(\tilde{w}) = \bar{V}, \tag{56}$$

for some $w_{-\infty} < \tilde{w} < \infty$ and for some $\bar{V} > 0$. Thus, computing the solution of the n-coalition problem (36) reduces to finding the $n-1$ values of $\{c_i\}_{i=1}^{n-1}$ with forward shooting.

The invariance $w \longrightarrow w + c$ can be used to reduce the search to $n-2$ parameters. Indeed, let $\mathbf{V}(w)$ be a solution of (53) that is of the form (54) for $w \ll -1$. Then, $\mathbf{V}(w+c)$ where $c = -\log c_1/\lambda_1$ is a solution of (53) that corresponds to the same trajectory as $\mathbf{V}(w)$ and can be approximated by (54a) for $w \ll -1$, where $\mathbf{V}^{\text{lin}}(w)$ is given by

$$\mathbf{V}^{\text{lin}}(w) = \pm \mathbf{U_1}\, e^{\lambda_1 w} + \tilde{c}_2 \mathbf{U_2}\, e^{\lambda_2 w} + \dots + \tilde{c}_{n-1} \mathbf{U_{n-1}}\, e^{\lambda_{n-1} w},$$

where $\tilde{c}_i = c_i \cdot c_1^{-\lambda_i/\lambda_1}$ for $i = 2, \dots n-1$, and the sign of $\pm\mathbf{U_1}$ is set such that $\pm\mathbf{U}_1$ points in the direction of the line (56).

Once the $n-2$ dimensional shooting process converges to a solution $\mathbf{V}(w)$ of (53) that satisfies (56), then, $\mathbf{V}(w + \tilde{c})$, where $\tilde{c} = -\tilde{w} - \log \bar{V}$ is the solution of (53) that satisfies

$$V_1(\bar{w}) = \dots = V_n(\bar{w}) = \bar{V}, \qquad \bar{V} = e^{-\bar{w}}. \tag{57}$$

Thus, the forward-shooting method for the n-coalition problem (36) is as follows:

Method 4.1. (i) Set the sign of $\pm U_1$ so that it points in the direction of the line (56), and choose $w_{-\infty} \ll -1$.

(ii) Search for the values of $\{c_2, \dots, c_{n-1}\}$ for which the solution of (53a) for $w \geq w_{-\infty}$ with the initial condition $\mathbf{V}(w = w_{-\infty}) = \mathbf{V}_{-\infty}$, where

$$\mathbf{V}_{-\infty} = \mathbf{V}^{\text{saddle}} \pm \mathbf{U_1}\, e^{\lambda_1 w} + c_2 \mathbf{U_2}\, e^{\lambda_2 w} + \dots + c_{n-1} \mathbf{U_{n-1}}\, e^{\lambda_{n-1} w},$$

satisfies the right boundary condition (56). Here, $\mathbf{V}^{\text{saddle}}$ is given by (36e), $\{\lambda_i\}_{i=1}^{n}$ are the positive eigenvalues of the matrix $\mathbf{A}$ (see (37)) and $\{\mathbf{U}_i\}_{i=1}^{n}$ are the corresponding eigenvectors.

(iii) The solution of (36) is given by $\mathbf{V}(w+\tilde{c})$, where $\tilde{c} = -\tilde{w} - \log \bar{V}$.

To recover the solution $\{v_i(b)\}_{i=1}^n$ of the original system (13), we substitute $v_i(b) = bV_i(e^w)$ for $i = 1, \ldots, n$. Alternatively, we can apply Method 4.1 directly to (13) (see §4.2.1).

4.2.1. A single-step forward method. We now apply the forward-shooting Method 4.1 to the two-coalition problem, formulated in the original variables.

Method 4.2 (i) Set

$$c_+ = \begin{cases} 1 & \alpha \le \beta \\ -1 & \alpha > \beta \end{cases}, \tag{58}$$

and

$$h = 10^{-11/(1+\lambda_+)}, \qquad \lambda_+ = \sqrt{(1+\alpha)(1+\beta)}. \tag{59}$$

(ii) Solve the initial value problem

$$v_1'(b) = \frac{v_1(b)}{\alpha}\frac{1}{v_2(b)-b}, \qquad v_2'(b) = \frac{v_2(b)}{\beta}\frac{1}{v_1(b)-b}, \tag{60}$$

for $b \ge h$, with the initial condition

$$\mathbf{v}(b=h) = \mathbf{V}^{\text{saddle}} \cdot h + c_+ \mathbf{U}_+ h^{1+\lambda_+}, \tag{61}$$

where

$$\mathbf{V}^{\text{saddle}} = \begin{pmatrix} 1+\dfrac{1}{\beta} \\ 1+\dfrac{1}{\alpha} \end{pmatrix}, \qquad \mathbf{U}_+ = \begin{pmatrix} \sqrt{\alpha\left(1+\dfrac{1}{\beta}\right)} \\ -\sqrt{\beta\left(1+\dfrac{1}{\alpha}\right)} \end{pmatrix}.$$

(iii) Denote by $\tilde{v}$ the value of $\{v_i(b)\}$ at the intersection point $\tilde{b} > 0$, i.e., $v_1(\tilde{b}) = v_2(\tilde{b}) = \tilde{v}$.

(iv) The solution of (10) is given by

$$\left\{\frac{1}{\tilde{v}}v_1(\tilde{v}b),\ \frac{1}{\tilde{v}}v_2(\tilde{v}b)\right\}.$$

A Matlab code for the forward-solving Method 4.2 is given in Appendix C.

In step 1, c_+ is chosen so that the solutions of (60) intersect at some $\tilde{b} > 0$ (i.e., the solution descends from the saddle point in the direction of Γ and not Γ'). Since generically $v_1(\tilde{b}) = v_2(\tilde{b}) \neq 1$, in step 4 we rescale the solution according to (30), in order to obtain the solution of (10) for which $v_1(\bar{b}) = v_2(\bar{b}) = 1$. The value of h in (59) is chosen so that the correction term $c_+\mathbf{U}_+h^{1+\lambda_+}$ in (61) is not too small (i.e., below machine accuracy), and also not too big so that the truncation error $v(h) \approx \mathbf{V}^{\text{saddle}} \cdot h + c_+\mathbf{U}_+h^{\lambda_+ +1}$ becomes too large. Typically, we choose h so that $h^{1+\lambda_+} = 10^{-11}$, and therefore h is given by (59).

The forward Method 4.2 can also be applied to the two-types-of-players problem (12). The only modifications are that $\mathbf{V}^{\text{saddle}}$ is given by (32d), λ_+ is given by (34), and $\mathbf{U}_+$ is given by (34). Note that the value of λ_+ can be quite large in this case. Indeed, from (34) it follows that $\lambda_+ \sim (n_1\alpha + n_2\beta)^2$. Therefore, already with four players, $\lambda_+ \gg 1$, in which case $h = \mathscr{O}(1)$.[10]

The key thing to note is that *Method 4.2 is a single-step method, i.e., it does not involve any shooting*, since the value of c_+ is given. In contrast, the forward-shooting Method 4.1 requires an $(n-2)$ dimensional search for the values for $\{c_2, \ldots, c_n\}$. Therefore, the forward-shooting method is extremely simple in the case of two coalitions or two types of players.

[10] When the initial step h is not small, this does not imply that the numerical solution is less accurate. Indeed, for $0 \le b \le h$, $V(b) = V^{\text{saddle}}b + c_+\mathbf{U}_+b^{1+\lambda_+} + o(b^{1+\lambda_+})$. Therefore, if we choose h so that $h^{1+\lambda_+} = \mathscr{O}(10^{-11})$, the approximation error in $[0, h]$ is $o(10^{-11})$, even if $h = O(1)$.

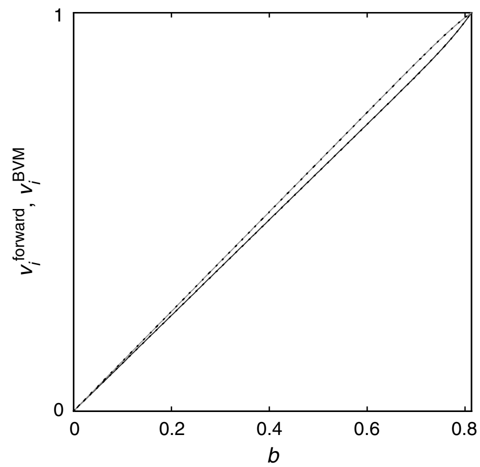

FIGURE 6. Solution of (12) with $n_1 = n_2 = 2$, $F_1(v) = v$ and $F_2(v) = v^2$ computed using the single-stage forward Method 4.2 (solid) and the boundary-value method (dots). The top two curves for v_2, and the bottom two curves for v_1 are indistinguishable.

4.2.2. Dynamical-systems view of forward solutions. The solution computed with the forward Methods 4.1 and 4.2 has an analytic truncation error that comes from the initial approximation

$$\mathbf{V}(w_{-\infty}) \approx \mathbf{V}_{-\infty} = \mathbf{V}^{\text{saddle}} + \mathbf{V}^{\text{lin}}(w_{-\infty}), \quad -\infty < w_{-\infty} \ll -1,$$

see (55). In terms of the dynamical system (53a), this approximation means that instead of starting from $\mathbf{V}(w_{-\infty})$, which lies on Γ, we start from $\mathbf{V}_{-\infty} = \mathbf{V}(w_{-\infty}) + \boldsymbol{\delta}$, which lies on a nearby trajectory, denoted by Γ^δ. As illustrated in Figure 5(B), the distance between Γ and Γ^δ decreases as w increases. Thus, in contrast to backward solutions, forward solutions are stable.

The dynamical-systems formulation also reveals why the forward method of Marshall et al. [14] (see §4.1.1) converged to the explicit solution (50). Indeed, this forward method is equivalent to solving (60) for $b \geq h$ with the initial condition (61) with $c_+ = 0$. In terms of the dynamical system (17a), this corresponds to the solution $\mathbf{V} \equiv \mathbf{V}^{\text{saddle}}$. To obtain a solution that corresponds to the trajectory Γ, however, one has to move away from the saddle point. In particular, because the explicit solution (50) corresponds to the solution of (17a) that stays at a saddle point, it follows that, contrary to the conclusion in Marshall et al. [14], the explicit solution is not an attractor.

4.2.3. Simulations. We first solve the two-coalition problem (10) with $F_1(v) = v$ and $F_2(v) = v^2$ using the single-stage forward-marching Method 4.2 with $h = 3.2 \cdot 10^{-5}$. In this case, the maximal bid $\bar{b}$ is given by $\bar{b} = 37/64$ (see (11)). The error $E = |\bar{b}^{\text{forward}} - \bar{b}| \approx 1.22 \cdot 10^{-15}$ is on the order of the machine roundoff error.

Next, we solve the two-types-of-players problem (12) with four players ($n_1 = n_2 = 2$), $F_1(v) = v$ and $F_2(v) = v^2$ with $h = 0.25$.[11] In this case, an analytic expression for the maximal bid $\bar{b}$ is unavailable. Therefore, we compare the results of the single-stage forward-marching method with simulation results obtained with the boundary-value method (BVM), which we recently developed in Fibich and Gavish [3]. Figure 6 shows that there is an excellent agreement between the results of these two methods.

In Fibich and Gavish [3] we considered the case of large auctions (i.e., when the number of players $n \gg 1$). We showed that the instability of backward solutions becomes more severe as the number of players increases. Therefore, the backward-shooting method fails for large auctions. We also showed that $\lim_{n\to\infty} v_i(b) = b$, i.e., the strategies tend to their true value as the number of players n goes to infinity. The derivatives $v_i'(b)$ converge to 1, except in the vicinity of the right boundary at $b = \bar{b}$ where they undergo abrupt changes.

We now use the forward-shooting method to solve the two-types-of-players problem (12) with 50 players ($n_1 = n_2 = 25$), $F_1(v) = v$, and $F_2(v) = v^2$ with $h = 0.982$.[12] In Figure 7(A) we plot the derivatives $\{v_i'(b)\}_{i=1,2}$ of the solution and observe that they are ≈ 1, except in the boundary layer near $\bar{b}$. The results of the single-stage forward-marching method and the BVM are in excellent agreement, both in the outer region, see Figure 7(A), and in the boundary layer region, see Figure 7(B). We stress that this problem could not be solved using backward shooting Fibich and Gavish [3].

[11] The value of h is chosen according to (59) with $\lambda_+ \approx 16.7$.

[12] The value of h is chosen according to (59) with $\lambda_+ \approx 4{,}115$ (see also footnote 10).

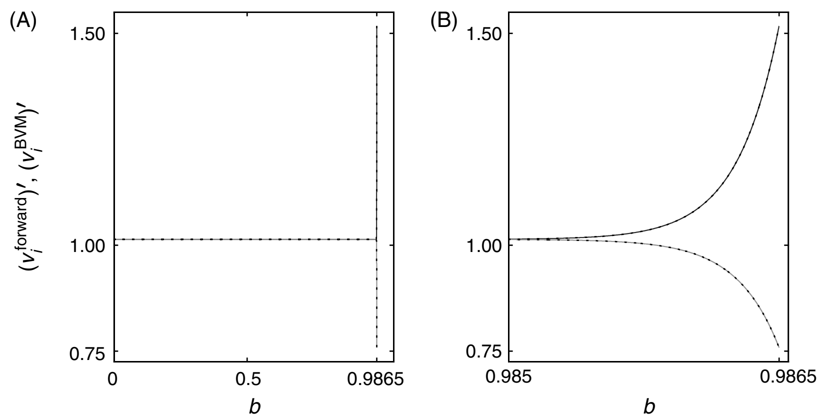

FIGURE 7. (A) v_1' and v_2' for the solution of (12) with $n_1 = n_2 = 25$, $F_1(v) = v$, and $F_2(v) = v^2$ computed using the single-stage Method 4.2 (solid) and the boundary-value method (dots). The top two curves for v_2 and the bottom two curves for v_1 are indistinguishable. (B) Zoom-in on the boundary-layer region.

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Appendix A. Proof of Lemma 3.4. By (17a),

$$\frac{dV_1}{dV_2} = \frac{(dV_1/dw)}{(dV_2/dw)} = \frac{V_1(V_1-1)(1+1/\alpha - V_2)}{V_2(V_2-1)(1+1/\beta - V_1)}.$$

This separable equation can be rewritten as

$$\frac{1+1/\beta - V_1}{V_1(V_1-1)}dV_1 = \frac{1+1/\alpha - V_2}{V_2(V_2-1)}dV_2.$$

Integrating this equation gives

$$\ln\frac{\sqrt[\beta]{V_1-1}}{\sqrt[\beta]{V_1}\,V_1} = C + \ln\frac{\sqrt[\alpha]{V_2-1}}{\sqrt[\alpha]{V_2}\,V_2}.$$

Taking the exponent of both sides of the equation yields (27).

Appendix B. Proof of Lemma 3.9. Following the proof of Lemma 3.1, we first prove that V_1 and V_2 are bounded as $w \longrightarrow -\infty$. By (8), (14), $V_i(w) \geq 1$ for $i = 1, 2$. Assume by negation that V_1 and V_2 are not both bounded from above as $w \longrightarrow -\infty$. Without loss of generality, assume that $\limsup_{w\to-\infty} V_2(w) = \infty$. If $\liminf_{w\to-\infty} V_2(w) < \infty$, then there is a series $\{w_k\} \longrightarrow -\infty$ of local maxima points, such that $V_2'(w_k) = 0$ and $\lim_{k\to\infty} V_2(w_k) = \infty$. By (32a),

$$\frac{n_1}{V_1^{\text{saddle}}-1}\frac{V_1^{\text{saddle}} - V_1(w_k)}{V_1(w_k)-1} = \frac{n_1-1}{V_2^{\text{saddle}}-1}\frac{V_2^{\text{saddle}} - V_2(w_k)}{V_2(w_k)-1}.$$

Therefore,

$$\frac{1}{V_1^{\text{saddle}}-1}\frac{V_1^{\text{saddle}} - V_1(w_k)}{V_1(w_k)-1} = \frac{1}{n_1}\frac{n_1-1}{V_2^{\text{saddle}}-1}\frac{V_2^{\text{saddle}} - V_2(w_k)}{V_2(w_k)-1} \sim -\frac{1}{n_1}\frac{n_1-1}{V_2^{\text{saddle}}-1}.$$

Hence, by this and (32a),

$$\begin{aligned} V_1'(w_k) &= \frac{V_1(w_k)}{(n_1+n_2-1)\alpha}\left[\frac{n_2}{V_2^{\text{saddle}}-1}\frac{V_2^{\text{saddle}} - V_2(w_k)}{V_2(w_k)-1} - \frac{n_2-1}{V_1^{\text{saddle}}-1}\frac{V_1^{\text{saddle}} - V_1(w_k)}{V_1(w_k)-1}\right] \\ &\sim \frac{V_1(w_k)}{(n_1+n_2-1)\alpha}\left[-\frac{n_2}{V_2^{\text{saddle}}-1} + \frac{n_2-1}{n_1}\frac{n_1-1}{V_2^{\text{saddle}}-1}\right] = -\frac{V_1(w_k)}{\alpha n_1(V_2^{\text{saddle}}-1)}. \end{aligned}$$

In addition, differentiating (32a) and using $V_2'(w_k)=0$ gives

$$V_2''(w_k) = \frac{V_2(w_k)}{(n_1+n_2-1)\beta}\frac{n_1}{V_1^{\text{saddle}}-1}\frac{d}{dw}\left(\frac{V_1^{\text{saddle}}-V_1}{V_1-1}\right) = \frac{V_2(w_k)}{(n_1+n_2-1)\beta}\frac{n_1}{V_1^{\text{saddle}}-1}\frac{1-V_1^{\text{saddle}}}{(V_1(w_k)-1)^2}V_1'(w_k)$$
$$= -\frac{V_2(w_k)}{(n_1+n_2-1)\beta}\frac{n_1}{(V_1(w_k)-1)^2}V_1'(w_k).$$

Therefore,

$$V_2''(w_k) \sim \frac{V_2(w_k)}{(n_1+n_2-1)\beta}\frac{n_1}{(V_1(w_k)-1)^2}\frac{V_1(w_k)}{\alpha n_1(V_2^{\text{saddle}}-1)} > 0.$$

Thus, for some $K>0$, $\{V_2(w_k)\}_{k\geq K}$ are all local minima points. Contradiction. Therefore, $\limsup_{w\to-\infty} V_2(w)=\infty$ implies that $\lim_{w\to-\infty} V_2=\infty$.

We now show that $\lim_{w\to-\infty} V_2=\infty$ implies that $\lim_{w\to-\infty} V_1=\infty$. Indeed, assume by negation that $\lim_{w\to-\infty} V_1<\infty$, then by (8), (32a), as $w\to-\infty$,

$$V_1' = -V_1\left[1-\frac{1}{(n_1+n_2-1)\alpha}\left(\frac{n_2}{V_2-1}-\frac{n_2-1}{V_1-1}\right)\right] \sim -V_1\left[1+\frac{1}{(n_1+n_2-1)\alpha}\frac{n_2-1}{V_1-1}\right] \leq -V_1 \leq -1. \quad \text{(B.1)}$$

Hence, $\lim_{w\to-\infty} V_1=\infty$. Contradiction.

So far, we proved that if V_1 and V_2 are not both bounded from above as $w\longrightarrow-\infty$, then both $V_1\to\infty$ and $V_2\to\infty$ as $w\to-\infty$. We now claim that either $V_1\leq V_2$ or $V_2\leq V_1$ as $w\to-\infty$. Indeed, by (B.1), along the diagonal $V_1=V_2$,

$$\frac{d}{dw}(V_1-V_2) = \frac{1}{n_1+n_2-1}\frac{V_1}{V_1-1}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right),$$

does not change its sign. Hence, any trajectory can cross the diagonal at most once. Without loss of generality, $V_1\leq V_2$ as $w\to-\infty$. As in the proof of Lemma 3.1, this implies that $v_1(0)=\lim_{w\to-\infty} e^w V_1>0$, in contradiction with (12b).

So far, we proved that V_1 and V_2 are bounded as $w\longrightarrow-\infty$. As in the proof of Lemma 3.2, we need to rule out the possibility that the trajectory starts at $w=-\infty$ from one or several accumulations points that lie on the singular lines $V_1=1$ or $V_2=1$.

Assume, by negation, that the trajectory starts from the point $(1, V_2^0)$, where $1<V_2^0<\infty$. Near $(1, V_2^0)$, the equation for V_1 in the dynamical system (32a) reduces to

$$V_1' \sim \frac{1}{(n_1+n_2-1)\alpha}\left[\frac{n_2(V_2^{\text{saddle}}-V_2^0)}{(V_2^{\text{saddle}}-1)(V_2^0-1)}-\frac{n_2-1}{V_1-1}\right].$$

Therefore, if $n_2>1$ then

$$V_1' \sim -\frac{1}{(n_1+n_2-1)\alpha}\frac{n_2-1}{V_1-1} < 0,$$

i.e., V_1 increases as we move backward along the trajectories near $(1, V_2^0)$. Hence, the trajectories flow back away from the singular line $V_1=1$. In particular, they cannot start from the point $(1, V_2^0)$. If $n_2=1$, then near $(1, V_2^0)$,

$$V_1' \sim \frac{V_2^{\text{saddle}}-V_2^0}{V_2^0-1} \sim c, \qquad V_2' \sim \frac{V_2^0}{\beta}\frac{1}{V_1-1} \longrightarrow +\infty.$$

Therefore, if $1<V_2^0<V_2^{\text{saddle}}$, then $c>0$. Hence, all the trajectories flow back to the singular point $(1, 1)$. On the line $V_2=V_2^{\text{saddle}}$, $V_1'=0$ and $V_2'>0$. Therefore, the trajectories flow downward, and hence by the previous case, also end at $(1, 1)$. Finally, if $V_2^0>V_2^{\text{saddle}}$, then $c<0$. Therefore, in this case, all the trajectories flow back away from the singular line $V_1=1$.

A similar argument rules out the possibility that there the trajectory starts from the point $(V_1^0, 1)$, where $1<V_1^0<\infty$. Finally, we show that the trajectory cannot start at $w=-\infty$ from the point $(1, 1)$. Indeed, near $(1, 1)$, the dynamical system (32a) reduces to

$$V_1' \sim \frac{1}{(n_1+n_2-1)\alpha}\left[\frac{n_2}{V_2-1}-\frac{n_2-1}{V_1-1}\right], \qquad V_2' \sim \frac{1}{(n_1+n_2-1)\beta}\left[\frac{n_1}{V_1-1}-\frac{n_1-1}{V_2-1}\right].$$

Therefore,

$$V_1'(V_2-1) \sim \frac{1}{n_1\alpha}-\frac{n_2-1}{n_1}\frac{\beta}{\alpha}V_2'(V_2-1), \qquad V_2'(V_1-1) \sim \frac{1}{n_2\beta}-\frac{n_1-1}{n_2}\frac{\alpha}{\beta}V_1'(V_1-1).$$

Summing these equations gives

$$[(V_1-1)(V_2-1)]' \sim \frac{1}{n_1\alpha}+\frac{1}{n_2\beta}-\frac{n_2-1}{2n_1}\frac{\beta}{\alpha}[(V_2-1)^2]'-\frac{n_1-1}{2n_2}\frac{\alpha}{\beta}[(V_1-1)^2]'.$$

Thus,

$$(V_1(w)-1)(V_2(w)-1) \approx w_0+\left(\frac{1}{n_1\alpha}+\frac{1}{n_2\beta}\right)w-\frac{n_2-1}{2n_1}\frac{\beta}{\alpha}(V_2-1)^2-\frac{n_1-1}{2n_2}\frac{\alpha}{\beta}(V_1-1)^2.$$

Therefore, the trajectory starts from $(1, 1)$ at a finite w, rather than at $w=-\infty$. Contradiction.

Appendix C. Matlab Code for Method 4.2. The Matlab code for the forward-solving method 4.2 with $\alpha=1$ and $\beta=2$ is as follows:

```
1  %% Parameters
2  global alpha beta
3  alpha=1;beta=2;
4
5  %% Step 1 - compute h according to (44)
6  lambda=sqrt((1+alpha)*(1+beta));
7  h= nthroot(1e-11,1+lambda);
8
9  %% Step 2 - solve initial-value problem (45)
10 % Construct initial condition vh according to (46)
11 Vsaddle=[1+1/beta;1+1/alpha];
12 U=[sqrt(alpha*(1+1/beta));-sqrt(beta*(1+1/alpha))];
13 vh=Vsaddle*h+U*h.^(1+lambda);
14 % Solve (45)
15 [b,v]=ode45(@dvi,[h 1],vh,odeset('Events',@detectIntersection));
16
17 %% Step 3 - denote vtilde
18 vtilde=v(end);
19
20 %% Step 4 - Recover the solution of (5)
21 b=b/vtilde;
22 v=v/vtilde;
23
24 %% Plot solution
25 plot(b,v);axis tight;
```

The Matlab function `dvi` "implements" equation (60).

```
function dv = dvi(b,v)
global alpha beta
dv=[v(1)/alpha/(v(2)-b);
    v(2)/beta/(v(1)-b)];
return
```

The Matlab function `detectIntersection` is passed as a parameter to the differential solver, see line 15, in order to stop the integration of (60) when $v_1(\tilde{b})=v_2(\tilde{b})$ for some $\tilde{b}>0$.

```
function [value,isterminal,direction] = detectIntersection(b,v)
value = v(2)-v(1);     % Detect intersection of the solutions
isterminal = 1;        % Stop the integration
direction = 0;
return
```

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