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SINGULAR SOLUTIONS OF THE BIHARMONIC NONLINEAR SCHRÖDINGER EQUATION*

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Abstract. We consider singular solutions of the L^2-critical biharmonic nonlinear Schrödinger equation. We prove that the blowup rate is bounded by a quartic-root, the solution approaches a quasi–self-similar profile, and a finite amount of L^2-norm, which is no less than the critical power, concentrates into the singularity. We also prove the existence of a ground-state solution. We use asymptotic analysis to show that the blowup rate of peak-type singular solutions is slightly faster than that of a quartic-root, and the self-similar profile is given by the ground-state standing wave. These findings are verified numerically (up to focusing levels of 10^8) using an adaptive grid method. We also use the spectral renormalization method to compute the ground state of the standing-wave equation, and the critical power for collapse, in one, two, and three dimensions.

Key words. blowup, nonlinear Schrödinger, NLS, self-similar solutions, biharmonic, high-order dispersion

AMS subject classifications. 35Q55, 35G25, 35B44, 35B40

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1. Introduction.

The focusing nonlinear Schrödinger equation (NLS)

$$i\psi_t(t,\mathbf{x}) + \Delta\psi + |\psi|^{2\sigma}\psi = 0, \qquad \psi(0,\mathbf{x}) = \psi_0(\mathbf{x}) \in H^1(\mathbb{R}^d), \tag{1.1}$$

where $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, and $\Delta = \sum_{j=1}^d \partial_j^2$ is the Laplacian, has been the subject of intense study due to its role in various areas of physics, such as nonlinear optics and Bose–Einstein condensates (BEC). It is well known that the NLS (1.1) possesses solutions that become singular in a finite time [37]. Of special interest is the critical NLS ($\sigma = 2/d$)

$$i\psi_t(t,\mathbf{x}) + \Delta\psi + |\psi|^{4/d}\psi = 0, \qquad \psi(0,\mathbf{x}) = \psi_0(\mathbf{x}) \in H^1(\mathbb{R}^d), \tag{1.2}$$

which models the collapse of intense laser beams that propagate in a bulk Kerr medium.

In this study, we consider the focusing biharmonic nonlinear Schrödinger equation (BNLS)

$$i\psi_t(t,\mathbf{x}) - \Delta^2\psi + |\psi|^{2\sigma}\psi = 0, \qquad \psi(0,\mathbf{x}) = \psi_0(\mathbf{x}) \in H^2(\mathbb{R}^d), \tag{1.3}$$

where Δ^2 is the biharmonic operator. This equation occurs in models of solitons in a magnetic field and in propagation of ultrashort pulses; see [11] for further details. The BNLS (1.3) is called "L^2-*critical*" or simply "critical" if $\sigma d = 4$. In this case,

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the L^2-norm ("*power*") is conserved under the BNLS dilation symmetry $\psi(t,\mathbf{x}) \mapsto L^{-2/\sigma}\psi\left(t/L^4, \mathbf{x}/L\right)$. The critical BNLS can be rewritten as

$$i\psi_t(t,\mathbf{x}) - \Delta^2\psi + |\psi|^{8/d}\psi = 0, \qquad \psi(0,\mathbf{x}) = \psi_0(\mathbf{x}) \in H^2(\mathbb{R}^d). \tag{1.4}$$

Correspondingly, the BNLS with $\sigma d < 4$ is called subcritical, and the BNLS with $\sigma d > 4$ is called supercritical. This is analogous to the NLS, where the critical case is $\sigma d = 2$.

In [5], Ben-Artzi, Koch, and Saut proved that the BNLS (1.3) is locally well-posed in H^2, when σ is in the H^2-*subcritical* regime

$$\begin{cases} 0 < \sigma < \infty, & d \le 4, \\ 0 < \sigma < \frac{4}{d-4}, & d > 4. \end{cases} \tag{1.5}$$

Local well-posedness also follows from Kenig, Ponce, and Vega [18]. Global existence and scattering of BNLS radial solutions in the H^2-critical case $\sigma = 4/(d-4)$ were studied by Miao, Xu, and Zhao [26] and by Pausader [32]. The H^2-critical defocusing BNLS was studied by Miao, Xu, and Zhao [25] and by Pausader [30, 31]. Global existence of the L^2-critical BNLS in dimension $d > 4$ was proved by Pausader and Shao [33] for general L^2 initial data in the defocusing case, and for bounded L^2 initial data in the focusing case.

The above studies focused on nonsingular solutions. In this work, we study singular solutions of the BNLS in H^2, i.e., solutions that exist in $H^2(\mathbb{R}^d)$ over some finite time interval $t \in [0, T_c)$, but for which $\lim_{t\to T_c} \|\psi\|_{H^2} = \infty$. The first study of singular BNLS solutions was done by Fibich, Ilan, and Papanicolau [11], who proved the following results for H^2 initial conditions.

THEOREM 1. *All solutions of the subcritical ($\sigma d < 4$) focusing BNLS* (1.3) *exist globally.*

THEOREM 2. *Let $\|\psi_0\|_2^2 < P_{cr}^{B}$, where $P_{cr}^{B} = \|R_B\|_2^2$, and R_B is the ground state of*

$$-\Delta^2 R_B(\mathbf{x}) - R_B + |R_B|^{8/d} R_B = 0, \qquad R_B \in H^2. \tag{1.6}$$

Then, the solution of the critical focusing BNLS (1.4) *exists globally.*

The simulations of [11] suggested that there exist singular solutions for $\sigma d = 4$ and $\sigma d > 4$. In contradistinction with NLS theory, however, there is currently no rigorous proof that solutions of the BNLS can become singular in either the critical or the supercritical case.

To the best of our knowledge, the only work, apart from [11], which considered singular solutions of the BNLS is by Chae, Hong, and Lee [9], who proved that if singular solutions of the critical BNLS exist, then they have a power (L^2) concentration property. See section 6.3 for more details.[1]

1.1. Summary of results. In this work, we consider singular solutions of the focusing critical BNLS. Our purpose is to characterize these singular solutions, i.e., their profile, blowup rate, power concentration, etc.

[1] After this manuscript was submitted, Zhu, Zhang, and Yang [42] proved the existence of the ground state and the power concentration property for nonradial data. See sections 4.1 and 6.3 for more details.

In some cases, we assume radial symmetry, i.e., that $\psi(t, \mathbf{x}) \equiv \psi(t, r)$, where $r = \|\mathbf{x}\|_2 = \sqrt{x_1^2 + \cdots + x_d^2}$. In these cases, (1.3) reduces to

$$i\psi_t(t, r) - \Delta_r^2 \psi + |\psi|^{2\sigma} \psi = 0, \qquad \psi(0, r) = \psi_0(r), \tag{1.7}$$

where

$$\Delta_r^2 = \partial_r^4 + \frac{2(d-1)}{r}\partial_r^3 + \frac{(d-1)(d-3)}{r^2}\partial_r^2 - \frac{(d-1)(d-3)}{r^3}\partial_r \tag{1.8}$$

is the radial biharmonic operator. Specifically, the critical BNLS (1.4) reduces to

$$i\psi_t(t, r) - \Delta_r^2 \psi + |\psi|^{8/d} \psi = 0, \qquad \psi(0, r) = \psi_0(r). \tag{1.9}$$

The paper is organized as follows. In section 2, we use the Noether theorem to derive conservation laws for the BNLS. We recall that in the critical NLS, the conservation law which follows from invariance of the action integral under dilation leads to the NLS "variance identity," which is the key analytic tool for proving the existence of singular solutions. In section 2.1 we use a similar procedure to derive the equivalent variance identity for the critical BNLS, and then we generalize it to the supercritical BNLS. It is not clear, however, whether the "BNLS variance" has to be positive definite. Therefore, at present it is not clear whether this identity can lead to a proof of the existence of singular BNLS solutions.

In section 3 we extend Nawa's nonradial compactness lemma [28] from H^1 to H^2. In section 4.1 we prove the existence and variational characterizations of the ground-state solutions of (1.6). In section 4.2 we use the BNLS variance identity to calculate analytically the Hamiltonian of the standing waves. The ground states of (1.6) were previously calculated numerically only in the one-dimensional case [11], since they were computed using a shooting method, which cannot be easily generalized to multi-dimensions. In section 4.3 we use the spectral renormalization method to compute the ground states of (1.6) for one, two, and three dimensions. The calculated ground states provide the first numerical estimate of the critical power for collapse $P_{\rm cr}^{\rm B} = \|R_{\rm B}\|_2^2$ in the two- and three-dimensional cases; see section 5. Direct simulations of the critical BNLS suggest that the constant $P_{\rm cr}^{\rm B}$ in Theorem 2 is optimal.

In section 6 we use rigorous analysis to study singular solutions of the critical BNLS (1.4). The blowup rate is shown to be lower-bounded by a quartic-root, i.e., $\|\Delta\psi\|_2^{-1/2} \leq C\,(T_{\rm c} - t)^{1/4}$. The corresponding bound for the critical NLS is a square root, i.e., $\left\|\nabla\psi^{\rm NLS}\right\|_2^{-1} \leq C\,(T_{\rm c} - t)^{1/2}$. We then prove that singular radial solutions of the critical BNLS converge to a self-similar profile strongly in $L^{2+2\sigma}$ for any σ in the H^2-subcritical regime (1.5). Finally, we show that singular solutions have the power (L^2-norm) concentration property, whereby the amount of power that collapses into the singularity point is at least $P_{\rm cr}^{\rm B}$. These rigorous results mirror those of the critical NLS.

Let us denote the location of the maximal amplitude of a radially symmetric solution by

$$r_{\max}(t) = \arg\max_r |\psi|.$$

Singular solutions are called "*peak-type*" when $r_{\max}(t) \equiv 0$ for $0 \leq t \leq T_c$, and "*ring-type*" when $r_{\max}(t) > 0$ for $0 \leq t < T_{\rm c}$. In section 7, we consider the asymptotic

Table 1.1
A comparison of the properties of peak-type singular solutions of the critical NLS and BNLS. These properties are analogous "up to the change $2 \to 4$."

	NLS	BNLS
L^2-critical case	$\sigma d = 2$	$\sigma d = 4$
critical power for collapse	$P_{\rm cr} = \|R\|_2^2$	$P_{\rm cr}^{\rm B} = \|R_{\rm B}\|_2^2$
asymptotic profile	$\frac{1}{L^{d/2}(t)} R\left(\frac{r}{L(t)}\right) e^{i\int_0^t \frac{ds}{L^2(s)}}$	$\frac{1}{L^{d/2}(t)} R_{\rm B}\left(\frac{r}{L(t)}\right) e^{i\int_0^t \frac{ds}{L^4(s)}}$
blowup rate bound	$\geq 1/2$	$\geq 1/4$
numerical blowup rate	slightly faster than 1/2	slightly faster than 1/4
power concentration	yes, at least $P_{\rm cr}$	yes, at least $P_{\rm cr}^{\rm B}$

profile and blowup rate of peak-type singular solutions of the critical BNLS equation (peak-type singular solutions of the supercritical BNLS and ring-type singular solutions of the critical and supercritical BNLS are studied elsewhere [2, 3, 4]). We use informal asymptotic analysis and numerical simulations to show that peak-type singular solutions of the critical BNLS collapse with the quasi–self-similar profile

$$\psi_{R_{\rm B}}(t,r) = \frac{1}{L^{d/2}(t)} R_{\rm B}\left(\frac{r}{L(t)}\right) e^{i\int_0^t \frac{1}{L^4(t')}dt'}, \qquad \lim_{t\to T_{\rm c}} L(t) = 0,$$

where $R_{\rm B}(\rho)$ is the ground state of (1.6). The blowup rate is shown to be slightly faster than the quartic-root bound. This is analogous to the critical NLS, where the blowup rate of peak-type solutions is slightly faster than the square-root bound, due to the loglog correction (the "loglog law"). It is an open question whether the correction to the BNLS blowup rate is also a loglog.

Section 8 presents the spectral renormalization method (SRM) for calculating numerically the BNLS standing waves in multidimensions. The numerical method for solving the BNLS is based on adaptive grids and is described in [4].

1.2. Discussion. In this study, we use rigorous theory, asymptotic theory, and numerical simulations to analyze singular solutions of the critical BNLS. All the results presented in this work mirror those of the NLS "up to a change by a factor of 2"; see Table 1.1. However, several key features of NLS theory are still missing from BNLS theory. First, the variance identity for the BNLS cannot be used to prove that singular solutions exist. Second, the critical NLS is invariant under the pseudoconformal ("lens-transformation") symmetry, which can also be used to construct explicit singular solutions. At this time, it is unknown whether an analogous identity exists for the critical BNLS. Third, in critical NLS theory, the self-similar profile is known to possess a quadratic radial phase term, i.e.,

$$\psi(t,r) \sim \frac{1}{L^{d/2}(t)} R\left(\frac{r}{L(t)}\right) e^{i\tau(t) + i\frac{L_t}{4L}r^2}.$$

This term represents the focusing of the solution towards $r = 0$ and plays a key role in the rigorous and asymptotic theory of the critical NLS. At this time, we do not know the analogous radial phase term for the critical BNLS.

Finally, we note that a similar "up to a factor of 2" connection exists between singular solutions of the nonlinear heat equation (see [17]) and the biharmonic nonlinear heat equation; see [7]. For example, the L^∞-norm of singular solutions blows up as $(T_{\rm c}-t)^{-1/2}$ for the nonlinear heat equation, and as $(T_{\rm c}-t)^{-1/4}$ for the biharmonic heat equation. The "similarity up to a factor of 2," however, is not perfect. For example, the self-similar spatial variable is $r/\sqrt{(T_c - t)\log(T_c - t)}$ for the nonlinear heat equation but $r/\sqrt[4]{T_c - t}$ for the biharmonic nonlinear heat equation. Another difference between the equations is that singular solutions are *asymptotically* self-similar for the nonlinear heat equation but truly self-similar for the biharmonic heat equation. In contrast, the NLS possesses self-similar singular solutions, whereas for the BNLS it is unknown whether singular solutions are truly, or only asymptotically, self-similar.

2. Invariance. The BNLS (1.3) is the Euler–Lagrange equation of the action integral $S = \int \mathcal{L} d\mathbf{x} dt$, where $\mathcal{L}$ is the Lagrangian density

$$\mathcal{L}(\psi, \psi^*, \psi_t, \psi_t^*, \Delta\psi, \Delta\psi^*) = \frac{i}{2}(\psi_t \psi^* - \psi_t^* \psi) - |\Delta\psi|^2 + \frac{1}{1+\sigma}|\psi|^{2(\sigma+1)}. \tag{2.1}$$

Therefore, the conserved quantities of the BNLS can be found using the Noether theorem; see Appendix A. As in the standard NLS, invariance of the action integral under phase-multiplications $\psi \mapsto e^{i\delta}\psi$ implies conservation of the "power" (L^2-norm), i.e.,

$$P(t) \equiv P(0), \qquad P(t) = \|\psi(t)\|_2^2.$$

Similarly, invariance under temporal translations $t \mapsto t + \delta t$ implies conservation of the Hamiltonian

$$H(t) \equiv H(0), \qquad H[\psi(t)] = \|\Delta\psi\|_2^2 - \frac{1}{1+\sigma}\|\psi\|_{2(\sigma+1)}^{2(\sigma+1)}, \tag{2.2}$$

and invariance under spatial translations $\mathbf{x} \mapsto \mathbf{x} + \delta\mathbf{x}$ implies conservation of linear momentum, i.e.,

$$\mathbf{M}(t) \equiv \mathbf{M}(0), \qquad \mathbf{M}(t) = \int \mathrm{Im}\{\psi^* \nabla\psi\}\, d\mathbf{x}.$$

In the critical case $\sigma \cdot d = 4$, the action integral is also invariant under the dilation transformation $\psi(t, \mathbf{x}) \to \lambda^{d/2}\psi(\lambda^4 t, \lambda\mathbf{x})$. The corresponding conserved quantity is

$$J(t) \equiv J(0), \qquad J(t) = \int \mathbf{x} \cdot \mathrm{Im}\{\psi^* \nabla\psi\}\, d\mathbf{x} + 4tH. \tag{2.3}$$

2.1. A variance identity. We recall that the action integral of the critical NLS (1.2) is invariant under the dilation transformation $\psi_{\mathrm{NLS}}(t, \mathbf{x}) \mapsto \lambda^{d/2}\psi_{\mathrm{NLS}}(\lambda^2 t, \lambda\mathbf{x})$. The corresponding conserved quantity is

$$J_{\mathrm{NLS}}(t) \equiv J_{\mathrm{NLS}}(0), \qquad J_{\mathrm{NLS}} = \int \mathbf{x} \cdot \mathrm{Im}\{\psi^* \nabla\psi\}\, d\mathbf{x} - 2tH_{\mathrm{NLS}}. \tag{2.4}$$

In addition, the integral term $\int \mathbf{x} \cdot \mathrm{Im}\{\psi^* \nabla\psi\}\, d\mathbf{x}$ is the time-derivative of the variance, i.e.,

$$\frac{d}{dt}V_{\mathrm{NLS}} = 4\int \mathbf{x} \cdot \mathrm{Im}\{\psi^* \nabla\psi\}\, d\mathbf{x}, \qquad V_{\mathrm{NLS}}(t) = \int |\mathbf{x}|^2 |\psi|^2 d\mathbf{x}.$$

Therefore, it follows that

$$\frac{d^2}{dt^2}V_{\mathrm{NLS}} = 8H_{\mathrm{NLS}}.$$

In the supercritical NLS, the second derivative of the variance is not related to a conservation law. Nevertheless, direct differentiation shows that

$$\frac{d}{dt}V_{\mathrm{NLS}} = 4\int \mathbf{x}\cdot \operatorname{Im}\{\psi^*\nabla\psi\}\,d\mathbf{x}$$

and

$$\frac{d}{dt}\int \mathbf{x}\cdot \operatorname{Im}\{\psi^*\nabla\psi\}\,d\mathbf{x} = 2H_{\mathrm{NLS}} - \frac{\sigma d-2}{\sigma+1}\|\psi\|_{2(\sigma+1)}^{2(\sigma+1)}.$$

Therefore

$$\frac{d^2}{dt^2}V_{\mathrm{NLS}} = 8H_{\mathrm{NLS}} - 4\frac{\sigma d-2}{\sigma+1}\|\psi\|_{2(\sigma+1)}^{2(\sigma+1)}. \tag{2.5}$$

Since $V_{\mathrm{NLS}} \geq 0$, the variance identity (2.5) shows that solutions of the critical and supercritical NLS, whose Hamiltonian is negative, become singular in a finite time [38].

We next extend the analogy between (2.3) and (2.4) to the noncritical case. In the case of the BNLS, direct differentiation shows that

$$\frac{d}{dt}\int \mathbf{x}\cdot \operatorname{Im}\{\psi^*\nabla\psi\}\,d\mathbf{x} = 4H - \frac{\sigma d-4}{\sigma+1}\|\psi\|_{2(\sigma+1)}^{2(\sigma+1)}. \tag{2.6}$$

Therefore, if we define

$$V_{\mathrm{BNLS}}(t) = V_{\mathrm{BNLS}}(0) + \int_{s=0}^{t}\left(\int \mathbf{x}\cdot \operatorname{Im}\{\psi^*(s,\mathbf{x})\nabla\psi\}\,d\mathbf{x}\right)ds, \tag{2.7}$$

where $V_{\mathrm{BNLS}}(0)$ is a positive constant, we get the BNLS variance identity

$$\frac{d^2}{dt^2}V_{\mathrm{BNLS}} = 4H - \frac{\sigma d-4}{\sigma+1}\|\psi\|_{2(\sigma+1)}^{2(\sigma+1)}.$$

In order to use this identity to prove singularity formation, however, one must show that V_{BNLS}, as defined by (2.7), has to remain positive. Direct integration by parts gives that

$$\int \mathbf{x}\cdot\operatorname{Im}\{\psi^*\nabla\psi\}\,d\mathbf{x} = \frac{1}{4(d+2)}\int |\mathbf{x}|^4\cdot\operatorname{Im}\{\nabla\psi^*\Delta\nabla\psi\}\,d\mathbf{x} + \frac{1}{16(d+2)}\left(\int |\mathbf{x}|^4|\psi|^2 d\mathbf{x}\right)_t.$$

While the second term on the right-hand side is a temporal derivative of a positive-definite quantity, the first term on the right-hand side is not. Therefore, it remains an open question whether V_{BNLS} has to be positive.

3. Nonradial compactness lemma (critical case).

In [28, Proposition 2.1], Nawa derived a compactness lemma for H^1 functions which are not necessarily radial. The extension of this result to H^2 is as follows.

LEMMA 3 (nonradial compactness lemma). *Let $\{f_n(x)\}$ be a bounded sequence of functions in $H^2(\mathbb{R}^d)$ such that for some constant K,*

$$\limsup_{n\to\infty}\|f_n\|_{8/d+2}^{8/d+2} \geq K > 0 \tag{3.1}$$

and

$$\limsup_{n\to\infty} H[f_n] = \limsup_{n\to\infty}\left(\|\Delta f_n\| - \frac{1}{\sigma+1}\|f_n\|_{8/d+2}^{8/d+2}\right) = 0. \tag{3.2}$$

Then, there exist

1. *a family of functions in* $H^2(\mathbb{R}^d)$*:* $\mathfrak{A} = \{f^1, f^2, \dots\}$*, and*
2. *a sequence in* $\mathbb{R}^d$*:* $\mathfrak{B} = \{y_n^1, y_n^2, \dots\}$

such that

$$\lim_{n\to\infty} \left|\sum_{k=2}^{j} y_n^k\right| = \infty, \qquad (j \geq 2) \tag{3.3}$$

and such that, for some subsequence (still denoted by f_n*), we have*

$$f_n^1 \equiv f_n(\cdot + y_n^1) \to f^1 \not\equiv 0, \tag{3.4a}$$

$$f_n^j \equiv \left(f_n^{j-1} - f^{j-1}\right)(\cdot + y_n^j) \to f^j \not\equiv 0, \qquad (j \geq 2) \tag{3.4b}$$

weakly in $H^2(\mathbb{R}^d)$ *and strongly in* $L^{2+2\sigma}(\Omega)$*, where* Ω *is a compact subset of* $\mathbb{R}^d$ *and* $0 \leq \sigma < 4/(d-4)$*. For this subsequence, we also have*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \left||f_n^j|^q - |f_n^j - f^j|^q - |f^j|^q\right| d\mathbf{x} = 0, \qquad q \in [2, 2+2\sigma), \tag{3.5}$$

where σ *is in the* H^2*-subcritical regime* (1.5)*,*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \left\{H(f_n^j) - H(f_n^j - f^j) - H(f^j)\right\} = 0, \tag{3.6}$$

and

$$-\lim_{n\to\infty} H(f_n^j - f^j) = \sum_{k=1}^{j} H(f^k). \tag{3.7}$$

Furthermore,

1. *if* $L = \#\mathfrak{A} < \infty$*, then*

$$\lim_{n\to\infty} \left\|f_n^L - f^L\right\|_{8/d+2} = 0, \tag{3.8}$$

and for any $R > 0$*,*

$$\lim_{n\to\infty} \left\{\sup_{y\in\mathbb{R}^d} \int_{|x-y|<R} \left|f_n^L(x) - f^L(x)\right|^2 dx\right\} = 0; \tag{3.9}$$

2. *if* $L = \#\mathfrak{A} = \infty$*, then*

$$\lim_{j\to\infty}\lim_{n\to\infty} \left\|f_n^j - f^j\right\|_{8/d+2} = 0, \tag{3.10}$$

and for any $R > 0$*,*

$$\lim_{j\to\infty}\lim_{n\to\infty} \left\{\sup_{y\in\mathbb{R}^d} \int_{|x-y|<R} \left|f_n^j(x) - f^j(x)\right|^2 dx\right\} = 0. \tag{3.11}$$

Proof. The proof is identical to the proof of [28, Proposition 2.1], simply replacing ∇, $W^{1,\alpha}$, and H^1 with Δ, $W^{2,\alpha}$, and H^2, respectively. ∎

4. Standing waves. The BNLS equation (1.3) admits the standing-wave solutions $\psi(t,\mathbf{x}) = \lambda^{2/\sigma} e^{i\lambda^4 t} R_{\rm B}(\lambda \mathbf{x})$, where $R_{\rm B}$ is the solution of

$$-\Delta^2 R_{\rm B}(\mathbf{x}) - R_{\rm B} + |R_{\rm B}|^{2\sigma} R_{\rm B} = 0. \tag{4.1}$$

For example, in one dimension, (4.1) is given by

$$-R_{\rm B}''''(x) - R_{\rm B} + |R_{\rm B}|^{2\sigma} R_{\rm B} = 0, \tag{4.2}$$

and in two dimensions by

$$-\left(\partial_{xx} + \partial_{yy}\right)^2 R_{\rm B}(x,y) - R_{\rm B} + |R_{\rm B}|^{2\sigma} R_{\rm B} = 0. \tag{4.3}$$

If we impose radial symmetry, (1.6) reduces to

$$-\Delta_r^2 R_{\rm B}(r) - R_{\rm B} + |R_{\rm B}|^{2\sigma} R_{\rm B} = 0, \tag{4.4a}$$

where Δ_r^2 is as given in (1.8). At $r = 0$, all the odd derivatives vanish, and so the solution of (1.8) is subject to the boundary conditions

$$R_{\rm B}'(0) = R_{\rm B}'''(0) = R_{\rm B}(\infty) = R_{\rm B}'(\infty) = 0. \tag{4.4b}$$

4.1. Variational characterizations (critical case). In [11], Fibich, Ilan, and Papanicolaou showed that if there exists a minimizer to the Gagliardo–Nirenberg functional (see Appendix B)

$$\inf_{\substack{f\in H^2(\mathbb{R}^d)\\ f\not\equiv 0}} J[f], \qquad J[f] = \frac{\|f\|_2^{8/d}\|\Delta f\|_2^2}{\|f\|_{2+8/d}^{2+8/d}}, \tag{4.5}$$

then the minimum is attained by the ground state of (1.6). This ground state, if it exists, is also the minimizer of the variational problem

$$\inf_{\substack{f\in H^2(\mathbb{R}^d)\\ f\not\equiv 0}} \left\{ \|f\|_2^2 \;\middle|\; H(f) \le 0 \right\}. \tag{4.6}$$

The existence of a minimizer to (4.5), hence of a ground-state solution to (1.6), was recently proved by Zhu, Zhang, and Yang [42]. Following Nawa [28], we use Lemma 3, the nonradial compactness lemma, to provide a different proof.

THEOREM 4. *Let*

$$m := \inf_{\substack{f\in H^2(\mathbb{R}^d)\\ f\not\equiv 0}} \left\{ \|f\|_2 \;\middle|\; H[f] \le 0 \right\}, \qquad H[f] = \|\Delta f\|_2^2 - \frac{1}{\sigma+1}\|f\|_{8/d+2}^{8/d+2}, \tag{4.7}$$

and let

$$\frac{1}{B_{\sigma=4/d,d}} := \inf_{\substack{f\in H^2(\mathbb{R}^d)\\ f\not\equiv 0}} J[f], \qquad J[f] = \frac{\|f\|_2^{8/d}\|\Delta f\|_2^2}{\|f\|_{2+8/d}^{2+8/d}}. \tag{4.8}$$

Then, there exists a function $R_{\rm B} \in H^2(\mathbb{R}^d)$ such that

$$\|R_{\rm B}\|_2 = m \tag{4.9}$$

and

$$H[R_{\mathrm{B}}] = 0. \tag{4.10}$$

Hence, R_{B} *is the minimizer of* (4.7)*. In addition,*

$$\frac{1}{B_{\sigma=4/d,d}} = J[R_{\mathrm{B}}] = \frac{1}{\sigma+1}\|R_{\mathrm{B}}\|_{8/d+2}^{8/d+2}. \tag{4.11}$$

Hence, R_{B} *is also the minimizer of* (4.8)*.*

The function R_{B} *is the ground-state solution of* (1.6)*. In particular, the ground state of* (1.6) *exists.*

Proof. The proof is identical to the proof of [28, Proposition 2.5], simply replacing ∇, H^1, C_d, and [28, Proposition 2.1] with Δ, H^2, $B_{\sigma=4/d,d}$, and Lemma 3, respectively. ∎

Finally, we note that uniqueness of the ground state is still an open problem.

4.2. Hamiltonian of $\boldsymbol{R}_{\mathrm{B}}$. Let us substitute the standing-wave solution $\psi = e^{it}R_{\mathrm{B}}(t)$ in the variance identity (2.6). Since the left-hand side vanishes, we obtain that

$$H[R_{\mathrm{B}}] = \frac{\sigma d - 4}{4(\sigma+1)}\|R_{\mathrm{B}}\|_{2(\sigma+1)}^{2(\sigma+1)}. \tag{4.12}$$

Thus, $H[R_{\mathrm{B}}] < 0$ in the subcritical BNLS, $H[R_{\mathrm{B}}] = 0$ in the critical BNLS, and $H[R_{\mathrm{B}}] > 0$ in the supercritical BNLS. Equation (4.12) can also be obtained from the BNLS "Pohozaev identities":

$$\|R_{\mathrm{B}}\|_2^2 = \frac{4-(d-4)\sigma}{4(\sigma+1)}\|R_{\mathrm{B}}\|_{2+2\sigma}^{2+2\sigma}, \qquad \|R_{\mathrm{B}}\|_2^2 = \frac{4-(d-4)\sigma}{\sigma d}\|\Delta R_{\mathrm{B}}\|_2^2.$$

In section 5, we consider the critical BNLS with the initial conditions of $\psi_0 = c{\cdot}R_{\mathrm{B}}$. Since $H[R_{\mathrm{B}}] = 0$, then

$$H[cR_{\mathrm{B}}] = c^2\|\Delta R_{\mathrm{B}}\|_2^2 - c^{2+2\sigma}\frac{1}{1+\sigma}\|R_{\mathrm{B}}\|_{2+2\sigma}^{2+2\sigma} = c^2\left(1-c^{2\sigma}\right)\|\Delta R_{\mathrm{B}}\|_2^2.$$

Therefore, in the critical case

$$\begin{cases} H[cR_{\mathrm{B}}] > 0, & 0 < c < 1, \\ H[cR_{\mathrm{B}}] = 0, & c = 1, \\ H[cR_{\mathrm{B}}] < 0, & c > 1. \end{cases} \tag{4.13}$$

4.3. Numerical calculation of $\boldsymbol{R}_{\mathrm{B}}$. In the one-dimensional case, the solution of (4.2) was computed numerically in [11] as follows. Equation (4.2) can be integrated once, yielding an explicit relation between $R_{\mathrm{B}}(0)$ and ${R_{\mathrm{B}}}''(0)$. This parameter reduction enables the usage of a one-dimensional shooting approach. Unfortunately, such a parameter reduction is not possible for the solution of (4.4) in higher dimensions. Therefore, in multidimensions we compute the ground states using the SRM. See section 8 for further details.

In Figure 4.1 we display the ground states in the critical case, i.e., the solutions of (1.6). Figure 4.1A displays the ground state of (1.6) in the critical one-dimensional case, as calculated by the SRM. The solution is in excellent agreement with the

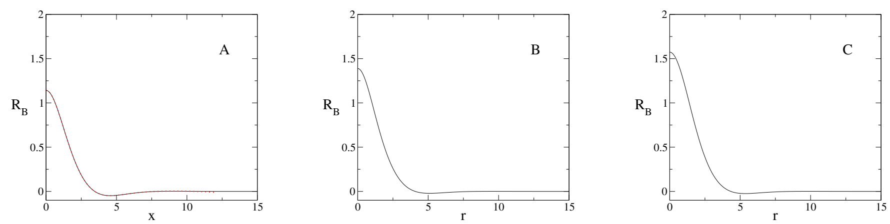

FIG. 4.1. *Numerical solution of* (1.6) *using the SRM method (solid line).* (A) $d = 1$. *Red dotted line is the solution computed in* [11] *using the shooting method. (See color version online.)* (B) $d = 2$. (A) $d = 3$.

solution computed in [11] using the shooting method. Figures 4.1B and 4.1C display the ground state in the critical two- and three-dimensional cases. We note that, while the SRM method that we use does not enforce radial symmetry, the calculated ground states for $d = 2$ and $d = 3$ are radially symmetric (data not shown). As noted in [11], the ground states of the BNLS are nonmonotonic in r and change their sign, in contradistinction with the ground states of the NLS, which are monotonically decreasing and strictly positive.

5. Critical power for collapse. Theorem 2 shows that the critical power for collapse in the critical BNLS (1.4) is $P_{\rm cr}^{\rm B} = \|R_{\rm B}\|_2^2$, when $R_{\rm B}$ is the ground state of (1.6). The computation of $R_{\rm B}$ (see section 4) allows for the numerical calculation of the critical power $P_{\rm cr}^{\rm B}$. The case $d = 1$ was found in [11] to be

$$P_{\rm cr}^{\rm B}(d=1) = \int_{x=-\infty}^{\infty} |R_{\rm B}(x)|^2 dx \approx 2.9868\,.$$

Using the calculated ground state in the two-dimensional case (see Figure 4.1B), we now calculate the critical power in the two-dimensional case, giving

$$P_{\rm cr}^{\rm B}(d=2) = \iint_{x,y=-\infty}^{\infty} |R_{\rm B}(x,y)|^2 dxdy \approx 13.143.$$

Similarly, using the calculated ground state in the three-dimensional case (see Figure 4.1C) gives

$$P_{\rm cr}^{\rm B}(d=3) = \iiint_{x,y,z=-\infty}^{\infty} |R_{\rm B}(x,y,z)|^2 dxdydz \approx 44.88.$$

We now ask whether Theorem 2 is sharp, in the sense that for any $\varepsilon > 0$, there exists an initial condition $\psi_0 \in H^2$ such that $\|\psi_0\|_2^2 \le (1+\varepsilon)P_{\rm cr}^{\rm B}$ and the corresponding solution of the critical BNLS becomes singular. As noted, at present there is no proof that solutions of the BNLS can become singular. Therefore, in particular, it is unknown whether Theorem 2 is indeed sharp. Hence, we will explore this issue numerically.

We recall that in the critical NLS, the necessary condition for collapse $\|\psi\|_2^2 \ge P_{\rm cr} := \|R\|_2^2$ is sharp, in the sense that for any $\epsilon > 0$ there exists a singular solution with $\|\psi\|_2^2 \le P_{\rm cr} + \epsilon$. Indeed, consider the one-parameter family of initial conditions $\psi_{0,\varepsilon} = (1+\varepsilon)R$ for $0 < \varepsilon$, which satisfy $\lim_{\varepsilon\to 0+} \|\psi_{0,\epsilon}\|_2^2 = P_{\rm cr}$. These initial conditions

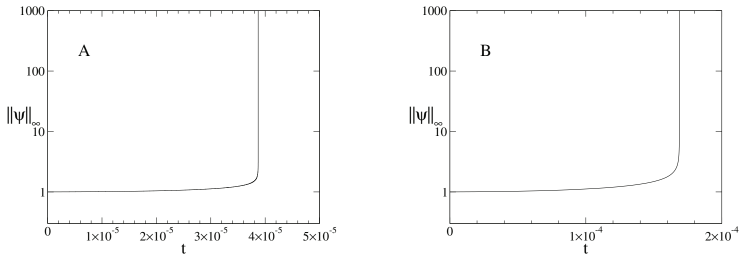

FIG. 5.1. *Collapsing solutions of the critical BNLS* (1.9) *with the perturbed ground-state initial condition* $\psi_0(\mathbf{x}) = 1.001 \cdot R_{\rm B}(r)$. (A) $d = 1$. (B) $d = 2$.

have a negative Hamiltonian and therefore become singular in a finite time [40]. In the critical BNLS, the one-parameter family of initial conditions $\psi_{0,\varepsilon} = (1+\varepsilon)R_{\rm B}$ for $0 < \varepsilon$ satisfy $\lim_{\varepsilon\to 0+} \|\psi_{0,\epsilon}\|_2^2 = P_{\rm cr}^{\rm B}$ and $H[(1+\varepsilon)R_{\rm B}] < 0$; see (4.13). Therefore, the "BNLS variance" of these solutions becomes negative at a finite time. It is not clear, however, whether this implies that these solutions have to become singular, since it is not known whether the "BNLS variance" has to be positive. Therefore, we now check numerically whether the solution of the critical BNLS with the initial condition $\psi_0 = (1+\varepsilon)R_{\rm B}(\mathbf{x})$, where $0 < \varepsilon \ll 1$, becomes singular. To do this, we solve the one- and two-dimensional critical BNLS equations with the initial condition $\psi_0 = 1.001 \cdot R_{\rm B}(\mathbf{x})$; see Figure 5.1. In both cases, the solution appears to blow up, suggesting that the value of $R_{\rm B}$ in Theorem 2 is optimal.

We also note that in the critical NLS, if the initial condition is different from the ground state, the power input required for collapse is strictly above $P_{\rm cr}$; see [10]. This is the case also in the BNLS. For example, in the one-dimensional critical BNLS with a Gaussian shaped initial condition $\psi_0 = C \cdot e^{-r^2}$, the input power required for collapse is strictly above $1.003 \cdot P_{\rm cr}^{\rm B}$; see [11]. In the two-dimensional critical BNLS with a Gaussian shaped initial condition, the input power required for collapse is strictly larger than $1.001 \cdot P_{\rm cr}^{\rm B}$; see [21].

6. Blowup rate, blowup profile, and power concentration.

6.1. Lower bound for the blowup rate.

In [8], Cazenave and Weissler proved that the blowup rate for singular solutions of the critical NLS (1.2) is not slower than a square-root, i.e., that $\left\|\nabla\psi^{\rm NLS}\right\|_2 \geq K(T_{\rm c} - t)^{-1/2}$. The analogous result for the critical BNLS is as follows.

THEOREM 5. *Let* ψ *be a solution of the critical BNLS* (1.4) *that becomes singular at* $t = T_{\rm c} < \infty$*, and let* $l(t) = \|\Delta\psi\|_2^{-1/2}$. *Then,* $\exists K = K(\|\psi_0\|_2) > 0$ *such that*

$$l(t) \leq K(T_{\rm c} - t)^{1/4}, \qquad 0 \leq t < T_{\rm c}.$$

Proof. We follow the proof given by Merle [22] for the critical NLS. For a fixed t, $0 \leq t < T_{\rm c}$, let us define $\psi_1(s,\mathbf{x}) = l^{d/2}\psi(t + s\cdot l^4,\, \mathbf{x}\cdot l)$. Then, ψ_1 is defined for $t + l^4 s < T_{\rm c} \iff s < S_{\rm c} = l^{-4}(t)\cdot(T_{\rm c} - t)$ and satisfies the BNLS equation

$$i\partial_s\psi_1 + \Delta^2\psi_1 + |\psi_1|^{8/d}\,\psi_1 = 0.$$

Since $\|\Delta\psi_1\|_2^2 = l^4 \left\|\Delta\psi(t+s\cdot l^4,\, \mathbf{x}\cdot l)\right\|_2^2$, this implies that $\lim_{s\to S_c} \|\Delta\psi_1\|_2^2 = \infty$, i.e., that $\psi_1(s)$ becomes singular as $s \to S_C$. In addition,

$$\|\Delta\psi_1(s=0,\mathbf{x})\|_2^2 = l^4 \|\Delta\psi(t,\mathbf{x})\|_2^2 = 1. \tag{6.1a}$$

From the definition of ψ_1 and power conservation it follows that

$$\|\psi_1(s=0,\mathbf{x})\|_2^2 = \|\psi(t,\mathbf{x})\|_2^2 = \|\psi_0(\mathbf{x})\|_2^2. \tag{6.1b}$$

Using (6.1a) and (6.1b) and the Cauchy–Schwarz inequality gives

$$\|\nabla\psi_1(s=0,\mathbf{x})\|_2^2 \le \|\psi_1(s=0,\mathbf{x})\|_2 \cdot \|\Delta\psi_1(s=0,\mathbf{x})\|_2 = \|\psi_0(\mathbf{x})\|_2. \tag{6.1c}$$

Together, the three formulae (6.1) imply that for any fixed $t \in [0, T_c)$,

$$\|\psi_1(s=0,\mathbf{x})\|_{H^2}^2 \le \|\psi_0\|_2^2 + \|\psi_0\|_2 + 1. \tag{6.2}$$

In other words, for each t, the initial H^2-norm of ψ_1 is bounded by a function of $\|\psi_0\|_2$. Specifically, this bound is independent of t. From the local existence theory [5], ψ_1 exists in $s \in [0, S_M(t)]$, where $S_M = S_M\left(\|\psi_1(s=0,\mathbf{x})\|_{H^2}\right)$. Therefore, it follows from (6.2) that S_M depends on $\|\psi_0\|$ but is independent of t. Since ψ_1 blows up at S_c, we have that $S_M \le S_c(t) = l^{-4}(t)\cdot(T_c - t)$, from which the result follows. ∎

6.2. Convergence to a quasi–self-similar blowup profile. In [41], Weinstein showed that the collapsing core of all singular solutions of the critical NLS approaches a self-similar profile. We now prove the analogous result for the critical BNLS.

THEOREM 6. *Let $d \ge 2$, and let $\psi(t,r)$ be a solution of the radially symmetric critical BNLS (1.4), with initial conditions $\psi_0(r) \in H^2_{\text{radial}}$, that becomes singular at $t = T_c < \infty$. Let $l(t) = \|\Delta\psi\|_2^{-1/2}$, and let*

$$S(\psi)(t,r) = l^{d/2}(t)\psi(t,\, l(t)r).$$

Then, for any sequence $t'_k \to T_c$ there is a subsequence t_k such that $S(\psi)(t_k,r) \to \Psi(r)$ strongly in L^q for all q such that[2]

$$\begin{cases} 2 < q \le \infty, & 2 \le d \le 4, \\ 2 < q < \frac{2d}{d-4}, & 4 < d. \end{cases} \tag{6.3}$$

In addition, $\|\Psi\|_2^2 \ge \|R_B\|_2^2$, where R_B is the ground state of (4.1).

Proof. Let $t_k \to T_c$ and define

$$\phi_k(r) = S(\psi)(t_k,r) = l^{d/2}(t_k)\psi(t_k, l(t_k)r).$$

From the definition of ϕ_k it follows that

$$\|\phi_k\|_2^2 = \|\psi_0\|_2^2, \qquad \|\Delta\phi_k\|_2^2 = l^4\|\Delta\psi(t_k)\|_2^2 = 1, \qquad H[\phi_k] = l^4 H[\psi(t_k)].$$

Therefore, using the Cauchy–Schwarz inequality,

$$\|\nabla\phi_k\|_2^2 \le \|\phi_k\|_2 \cdot \|\Delta\phi_k\|_2 = \|\psi_0\|_2.$$

[2] In fact, $q = 2(\sigma+1)$, where σ is in the H^2-subcritical regime (1.5).

Since $\|\phi_k\|_{H^2}$ is bounded, it follows that there exists a subsequence of ϕ_k which converges weakly in H^2 to a function $\Psi \in H^2_{\text{radial}}$. From the compactness lemma (Lemma 12; see Appendix C), it follows that $\phi_k \to \Psi$ strongly in L^q for all q given by (6.3).

Next, we prove that $H[\Psi] \le 0$. Since $\phi_k \underset{H^2}{\rightharpoonup} \Psi$, it follows that $\Delta\phi_k \underset{L^2}{\rightharpoonup} \Delta\Psi$, and so $\|\Delta\Psi\|_2 \le \lim_{k\to\infty} \|\Delta\phi_k\|_2 = 1$. Additionally, since $\phi_k \underset{L^q}{\to} \Psi$ for $q = 2 + 2\sigma$, we have that $\|\Psi\|_{2\sigma+2} = \lim_{k\to\infty} \|\phi_k\|_{2\sigma+2}$, and so

$$H[\Psi] \le \lim_{k\to\infty} H[\phi_k] = \lim_{k\to\infty} l^4 H[\psi_0] = 0.$$

In addition, since

$$0 = \lim_{k\to\infty} H[\phi_k] = \lim_{k\to\infty} \left(1 - \frac{1}{1+\sigma} \|\phi_k\|^{2(\sigma+1)}_{2(\sigma+1)}\right),$$

it follows that $\lim_{k\to\infty} \|\phi_k\|_{2(\sigma+1)} > 0$, so $\Psi \ne 0$. Therefore, Corollary 11 (see Appendix B) implies that $\|\Psi\|_2^2 \ge \|R_{\text{B}}\|_2^2$. ∎

6.3. Power concentration. Singular solutions of the critical NLS have the power concentration property, whereby the amount of power that collapses into the singularity is at least $P_{\text{cr}} = \|R\|_2^2$; see [41, 24]. Moreover, the power concentration rate is bounded by a square-root [41, 39]. In what follows, we prove the analogous results for the critical BNLS.

THEOREM 7. *Let $\psi(t, \mathbf{x})$ be an H^2 solution of the critical BNLS* (1.4) *that becomes singular at $t = T_{\text{c}} < \infty$. Then, there exists $\mathbf{y}(t) \in \mathbb{R}^d$, such that*

1. *for any monotonically decreasing $a(t) : [0, T_{\text{c}}) \to \mathbb{R}^+$ such that*

$$\lim_{t\to T_{\text{c}}} a(t) = 0 \qquad \text{and} \qquad \lim_{t\to T_{\text{c}}} \frac{(T_{\text{c}} - t)^{1/4}}{a(t)} = 0,$$

we have that

$$\liminf_{t\to T_{\text{c}}} \|\psi(t, \mathbf{x})\|^2_{L^2(|\mathbf{x}-\mathbf{y}(t)|<a(t))} \ge P^{\text{B}}_{\text{cr}};$$

2. *for any $\epsilon > 0$, $\exists K > 0$ such that*

$$\liminf_{t\to T_{\text{c}}} \|\psi(t, \mathbf{x})\|^2_{L^2(|\mathbf{x}-\mathbf{y}(t)|<K(T_{\text{c}}-t)^{1/4})} \ge (1-\epsilon) P^{\text{B}}_{\text{cr}}.$$

Proof. When ψ is radial, then $\mathbf{y}(t) \equiv 0$. If, in addition, $d \ge 2$, the proof follows directly from Theorem 6 as follows. Using the notations of Theorem 6,

$$\|\psi(t_k, r)\|^2_{L^2(r<a(t_k))} = \|\phi_k(r)\|^2_{L^2(r<a(t_k)/l(t_k))}.$$

In addition, by Theorem 5,

$$\lim_{k\to\infty} \frac{a(t_k)}{l(t_k)} = \lim_{k\to\infty} \frac{a(t_k)}{(T_{\text{c}} - t_k)^{1/4}} \frac{(T_{\text{c}} - t)^{1/4}}{l(t_k)} = \infty.$$

Therefore, for all $M > 0$,

$$\liminf_{k\to\infty} \|\phi_k(r)\|^2_{L^2(r<M)} \le \liminf_{k\to\infty} \|\phi_k(r)\|^2_{L^2(r<a(t_k)/l(t_k))}.$$

Since $\phi_k(r)_{L^2} \rightharpoonup \Psi$, it follows that $\phi_k(r)_{L^2(M)} \rightharpoonup \Psi$, and so

$$\|\Psi\|^2_{L^2(r<M)} \le \liminf_{k\to\infty} \|\phi_k(r)\|^2_{L^2(r<M)}.$$

This is true for all M, and so

$$P^{\mathrm{B}}_{\mathrm{cr}} \le \|\Psi\|^2_{L^2} \le \liminf_{k\to\infty} \|\psi(t_k,r)\|^2_{L^2(r<a(t_k))}.$$

For the second part of the theorem, since $\|\Psi\|_{L^2} \ge P^{\mathrm{B}}_{\mathrm{cr}}$, it follows that for all $\epsilon > 0$, there exist $K > 0$, such that $\|\Psi\|_{L^2(r>K)} \ge (1-\epsilon)P^{\mathrm{B}}_{\mathrm{cr}}$. Therefore, since

$$\|\psi(t_k,r)\|^2_{L^2(r<K\cdot l(t_k))} = \|\phi_k(r)\|^2_{L^2(r<K)},$$

a similar argument gives

$$\begin{aligned}(1-\epsilon)P^{\mathrm{B}}_{\mathrm{cr}} \le \|\Psi\|^2_{L^2(r<K)} &\le \liminf_{k\to\infty} \|\psi(t_k,r)\|^2_{L^2(r<K\cdot l(t_k))}\\ &\le \liminf_{k\to\infty} \|\psi(t_k,r)\|^2_{L^2(r<\tilde{K}\cdot(T_{\mathrm{c}}-t_k)^{1/4})}.\end{aligned}$$

The proof in the nonradial case follows from the nonradial compactness lemma; see Appendix D. ∎

The original version of this manuscript included the proof of Theorem 7 in the radial case. Subsequently, Zhu, Zhang, and Yang [42] provided a proof for nonradial solutions. Our proof in the nonradial case, added in the revision, is different from the one in [42]. We also note that Chae, Hong, and Lee [9] used the harmonic analysis method of Bourgain [6] to prove that singular solutions of the critical BNLS with $d \ge 2$ have the power concentration property

$$\lim_{t\to T_{\mathrm{c}}} \sup_{\mathbf{x_0}\in\mathbb{R}^d} \|\psi(t,\mathbf{x})\|^2_{L^2\left(|\mathbf{x}-\mathbf{x_0}|<(T_{\mathrm{c}}-t)^{1/4}\right)} > C,$$

where C is some positive constant. This result is stronger than Theorem 7 in that it assumes only L^2 initial data, but weaker in that it does not show that the amount of power that collapses into the singularity is at least $P^{\mathrm{B}}_{\mathrm{cr}}$.

7. Peak-type singular solutions of the critical BNLS.

In this section, we consider radially symmetric singular solutions that are "*peak-type*," i.e., for which $r_{\max}(t) \equiv 0$ for $0 \le t \le T_c$, where $r_{\max}(t) = \arg\max_r |\psi|$ is the location of the maximal amplitude.

7.1. The critical NLS—review.

The critical NLS admits singular solutions that collapse with the universal ψ_R profile, i.e., $\psi \sim \psi_R$, where

$$\psi_R(t,r) = \frac{1}{L^{1/\sigma}(t)} R(\rho) e^{i\tau + i\frac{L_t}{4L}r^2}, \qquad \tau = \int_0^t \frac{ds}{L^2(s)}, \qquad \rho = \frac{r}{L(t)}. \tag{7.1}$$

The self-similar profile R is the ground-state solution of

$$-R + \Delta R + |R|^{2\sigma} R = 0.$$

The blowup rate of $L(t)$ is given by the *loglog law* [14, 19, 20, 23]

$$L(t) \sim \left(\frac{2\pi(T_c - t)}{\log|\log(T_c - t)|}\right)^{\frac{1}{2}}, \qquad t \to T_c. \tag{7.2}$$

Since the blowup rate (7.2) is slightly faster than a square-root, $\lim_{t\to T_c} LL_t = \lim_{t\to T_c} \frac{1}{2}\left(L^2\right)_t = 0$. Therefore, the phase term $\frac{L_t}{4L}r^2 = \frac{LL_t}{8}\rho^2$ in (7.1) vanishes as $t \to T_c$. Hence, the blowup profile reduces to

$$\psi_R(t,r) = \frac{1}{L^{1/\sigma}(t)} R(\rho) e^{i\tau}, \qquad \tau = \int_0^t \frac{ds}{L^2(s)}, \qquad \rho = \frac{r}{L(t)}. \tag{7.3}$$

7.2. Informal analysis. We now look for the "corresponding" peak-type singular solutions of the critical BNLS (1.4). Theorem 6 suggests that the collapsing core of the singular solution approaches a self-similar form, i.e.,

$$\psi(t,r) \sim \psi_B(t,r), \qquad 0 \le r \le \rho_c \cdot L(t),$$

where

$$\psi_B(t,r) = \frac{1}{L^{d/2}(t)} B(\rho) e^{i\tau(t)}, \qquad \rho = \frac{r}{L}, \tag{7.4}$$

and $\rho_c = \mathcal{O}(1)$. Substituting (7.4) into (1.9) and requiring that $[\psi_t] \sim [\Delta\psi] \sim \left[|\psi|^{d/2}\psi\right]$ suggests that

$$\tau(t) = \int_{s=0}^t \frac{1}{L^4(s)} ds.$$

Let us consider the self-similar profile $B(\rho)$. In the singular region $r = \mathcal{O}(L)$ we have that

$$\Delta^2\psi \sim \Delta^2\psi_B \sim \frac{e^{i\tau}}{L^{4+d/2}}\Delta_\rho^2 B, \qquad |\psi|^{8/d}\psi \sim |\psi_B|^{8/d}\psi_B = \frac{e^{i\tau}}{L^{4+d/2}}|B|^{8/d}B,$$

and

$$\psi_t \sim (\psi_B)_t \sim \frac{e^{i\tau}}{L^{4+d/2}}\left\{iB - L_t L^3\left(\frac{d}{2}B + \rho B_\rho\right)\right\}.$$

Hence, $B(\rho)$ satisfies

$$-B(\rho) - \Delta_\rho^2 B + |B|^{8/d} = i\left(\lim_{t\to T_c} L_t L^3\right)\left(\frac{d}{2}B + \rho B_\rho\right). \tag{7.5}$$

Theorem 5 shows that that blowup rate of $L(t)$ is lower-bounded by a quartic-root. In the critical NLS the blowup rate of peak-type solutions is slightly faster than the analogous square-root rate, due to the loglog correction. Hence, we expect that the blowup rate of peak-type critical BNLS solutions is slightly faster than a quartic-root, i.e.,

$$\frac{L(t)}{\sqrt[4]{T_c - t}} \to 0. \tag{7.6}$$

In that case, $\lim_{t\to T_c} L_t L^3 = \lim_{t\to T_c} \frac{1}{4}\left(L^4\right)_t = 0$, and (7.5) reduces to the standing-wave equation (1.6). Since the ground states of (1.6) attain their maximal amplitudes at $\rho = 0$ (see section 4), they are peak-type solutions.

The above informal analysis thus leads to the following conjecture.

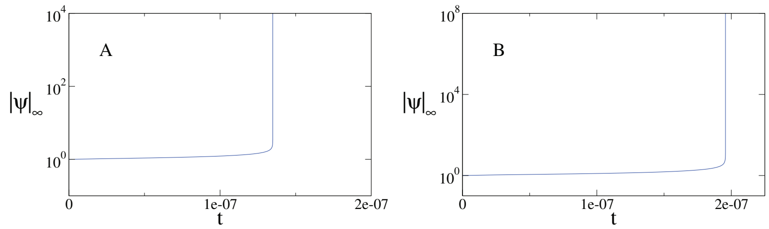

FIG. 7.1. *Maximal amplitude of peak-type singular solutions of the critical BNLS* (1.9). (A) $d = 1$. (B) $d = 2$.

CONJECTURE 8. *The critical BNLS admits peak-type singular solutions such that*

1. *the collapsing core approaches the self-similar profile*

$$\psi(t, r) \sim \psi_{R_{\mathrm{B}}}(t, r), \qquad 0 \leq r \leq \rho_c \cdot L(t), \tag{7.7a}$$

where

$$\psi_{R_{\mathrm{B}}}(t, r) = \frac{1}{L^{d/2}(t)} R_{\mathrm{B}}(\rho) e^{i\tau(t)}, \qquad \rho = \frac{r}{L}, \quad \tau(t) = \int_{s=0}^{t} \frac{1}{L^4(s)} ds, \tag{7.7b}$$

and R_{B} *is the ground state of* (1.6)*;*

2. *the blowup rate of* $L(t)$ *is slightly faster than a quartic-root, i.e.,*

$$\lim_{t \to T_{\mathrm{c}}} \frac{L(t)}{(T_{\mathrm{c}} - t)^p} = \begin{cases} 0, & p = 1/4, \\ \infty, & p > 1/4. \end{cases} \tag{7.8}$$

In section 7.3 we provide numerical evidence in support of Conjecture 8.

7.3. Simulations. The BNLS was solved numerically using an adaptive grid technique; see [4] for details. The one-dimensional critical BNLS

$$i\psi_t(t, x) - \psi_{xxxx} + |\psi|^8 \psi = 0 \tag{7.9}$$

was solved with the Gaussian initial condition $\psi_0(x) = 1.618e^{-x^2}$, whose power is $\|\psi_0\|_2^2 = 1.1 \cdot P_{\mathrm{cr}}^{\mathrm{B}}(d = 1)$. The maximal amplitude of the solution $\|\psi\|_\infty$ as a function of time is plotted in Figure 7.1A. The amplitude increases abruptly by a factor of 10^4 around $T_{\mathrm{c}} \approx 0.0499$, suggesting that the solution blows up in a finite time.

The simulation was repeated for the radially symmetric two-dimensional critical BNLS

$$i\psi_t(t, r) - \frac{1}{r^3}\psi_r + \frac{1}{r^2}\psi_{rr} - \frac{2}{r}\psi_{rrr} - \psi_{rrrr} + |\psi|^4 \psi = 0, \tag{7.10}$$

with the Gaussian initial condition $\psi_0(r) = 3.034e^{-r^2}$, whose power is $\|\psi_0\|_2^2 = 1.1 P_{\mathrm{cr}}^{\mathrm{B}}(d = 2)$. The amplitude increases abruptly by a factor of 10^8 around $T_{\mathrm{c}} \approx 0.0606$ (see Figure 7.1B), again suggesting that the solution becomes singular in a finite time.

We next consider the self-similar profile of the collapsing solutions from Figure 7.1. In order to verify that it is given by (7.7), we rescale the solutions as

$$\psi_{\mathrm{rescaled}}(t, \rho) = L^{d/2}(t)\psi(t, r = \rho \cdot L), \qquad L(t) = \|\psi\|_\infty^{-2/d}. \tag{7.11}$$

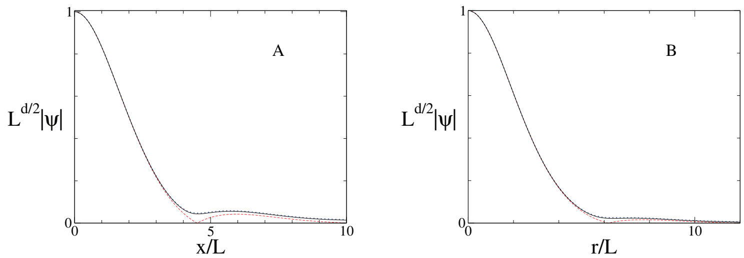

FIG. 7.2. *The solutions of Figure* 7.1*, rescaled according to* (7.11)*, at focusing levels* $L(t) = 10^{-4}$ *(blue dotted line) and* $L(t) = 10^{-8}$ *(black solid line). Red dashed line is the rescaled ground-state* $|R_{\rm B}|$*. The three curves are indistinguishable for* $0 \le r/L \le 4$*.* (A) $d = 1$. (B) $d = 2$. *(See color version online.)*

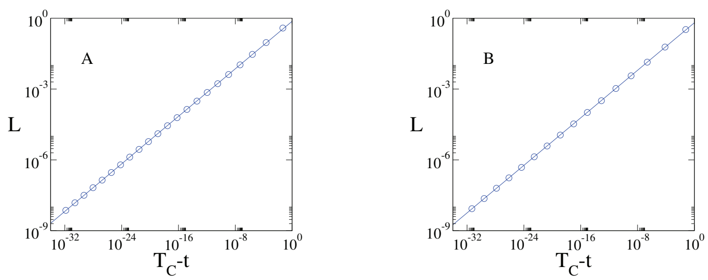

FIG. 7.3. $L(t)$ *as a function of* $(T_{\rm c} - t)$*, on a logarithmic scale, for the solutions of Figure* 7.1 *(circles).* (A) *One-dimensional case. Solid line is* $L = 0.742 \cdot (T_{\rm c} - t)^{0.2516}$*.* (B) *Two-dimensional case. Solid line is* $L = 0.641 \cdot (T_{\rm c} - t)^{0.2516}$*.*

The rescaled solutions at focusing levels of $L = 10^{-4}$ and $L = 10^{-8}$ are indistinguishable (see Figure 7.2) indicating that the collapsing core is indeed self-similar according to (7.7). As predicted, the self-similar profile is very close to the ground-state $R_{\rm B}$ in the core region $0 \le \rho \le 4$.

We next compute the blowup rate p, defined by the relation

$$L \sim \kappa(T_c - t)^p.$$

To do that, we perform a least-squares fit of $\log(L)$ with $\log(T_c - t)$ (see Figure 7.3), obtaining a value of $p \approx 0.2516$ for both $d = 1$ and $d = 2$. This value of p is slightly above $1/4$, implying that the quartic-root lower bound given by Theorem 5 is close to the actual blowup rate of peak-type singular solutions.

Next, we provide two indications that the blowup rate is faster than $1/4$. First, if the blowup rate is exactly $1/4$, then $\lim_{t \to T_{\rm c}} L^3 L_t$ should be finite and strictly negative. However, up to a focusing level of $L = 10^{-8}$, $L^3 L_t$ does not appear to converge to a negative constant, but rather to increase slowly towards 0^-; see Figure 7.4A. Second, according to the informal analysis in section 7.2, the blowup rate is faster than a quartic-root if and only if the self-similar profile $B(\rho)$ satisfies the standing-wave equation (4.1), which is indeed what we observed numerically in Figure 7.2.

Remark. In the critical NLS the blowup rate of peak-type solutions is slightly faster than the analogous square-root lower bound, due to the well-known loglog-correction; see (7.2). Figure 7.3 shows that the BNLS blowup rate is slightly faster

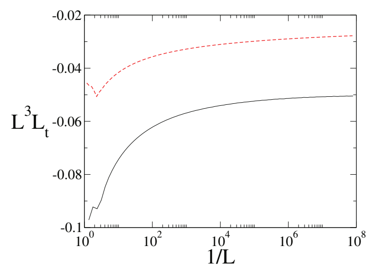

FIG. 7.4. L^3L_t *as a function of* $1/L$ *for the solution of Figure* 7.1A *(solid line) and of Figure* 7.1B *(dashed line).*

than a quartic-root, and Figure 7.4 shows that $L^3L_t \to 0$ very slowly. Together, these suggest that the blowup rate in the critical BNLS is only slightly faster than the analogous quartic-root. At present, we do not know if the blowup rate of peak-type solutions of the critical BNLS is a quartic-root with a *loglog* correction. We note, however, that the loglog correction in the critical NLS cannot be detected numerically [12] and can only be derived analytically. Therefore, we expect that the determination of the analogous correction to the 1/4 blowup rate of the critical BNLS will be done analytically and not numerically.

8. The spectral renormalization method. The SRM was introduced by Petviashvili [35], and more recently by Ablowitz and Musslimani [1], for computing ground-state solutions of NLS equations. A rigorous proof that the obtained sequence converges to the NLS ground state was given by Pelinovsky and Stepanyants [34]. See also [13, Appendix B] for an intuitive explanation of why the obtained sequence does not diverge.

Here, we describe our adaptation of the SRM to the standing-wave BNLS equation (1.6). Denoting the Fourier transform of $R(\mathbf{x})$ by $\mathcal{F}[R](\mathbf{k})$, (1.6) transforms into

$$\mathcal{F}[R](\mathbf{k}) = \tfrac{1}{|\mathbf{k}|^4+1}\mathcal{F}\left[|R|^{2\sigma}R\right], \tag{8.1}$$

leading to the fixed-point iterative scheme

$$\mathcal{F}[R_{m+1}] = \tfrac{1}{|\mathbf{k}|^4+1}\mathcal{F}\left[|R_m|^{2\sigma}R_m\right], \qquad m = 0, 1, \dots.$$

Typically this iterative scheme diverges either to ∞ or to 0. In order to avoid this problem, we renormalize the solution as follows. Multiplying (8.1) by $\mathcal{F}[R]^*$ and integrating over $\mathbf{k}$ gives the integral relation

(8.2a)

$$SL = SR, \quad \text{where} \quad SL[R] \equiv \int |\mathcal{F}[R]|^2 d\mathbf{k}, \quad SR[R] \equiv \int \frac{1}{|\mathbf{k}|^4+1}\mathcal{F}\left[|R|^{2\sigma}R\right]\mathcal{F}[R]^* d\mathbf{k}.$$

We now define $R_{m+\frac{1}{2}} = C_m R_m$ such that the integral relation (8.2a) is satisfied by $R_{m+\frac{1}{2}}$, i.e., that

$$SL\left[R_{m+1/2}\right] = C_m^2 SL[R_m] = C_m^{2\sigma+2} SR[R_m] = SR\left[R_{m+1/2}\right],$$

leading to $C_m = \left(\frac{SL[R_m]}{SR[R_m]}\right)^{\frac{1}{2\sigma}}$, and hence to

$$\left|R_{m+1/2}\right|^{2\sigma} R_{m+1/2} = \left(\frac{SL[R_m]}{SR[R_m]}\right)^{1+\frac{1}{2\sigma}} |R_m|^{2\sigma} R_m.$$

The SRM is therefore given by the iterations

$$\mathcal{F}(R_{m+1}) = \left(\frac{SL[R_m]}{SR[R_m]}\right)^{1+\frac{1}{2\sigma}} \frac{1}{|\mathbf{k}|^4+1}\mathcal{F}\left(|R_m|^{2\sigma}R_m\right), \qquad m = 1, 2 \dots. \tag{8.2b}$$

In this work, we use the SRM to solve (1.6) for the cases $d = 1, 2, 3$ without imposing radial symmetry. Alternatively, one might have solved the radial equation (4.4) using a modified Hankel-like transform instead of the Fourier transform. Our main reason for not doing so is the convenience and cost-effectiveness of using the fast Fourier transform. We also note that the fact that our nonradially symmetric method converged to a radially symmetric solution suggests that the ground state is indeed radially symmetric.

Appendix A. Application of the Noether theorem to the BNLS. The Lagrangian density of the BNLS is

$$\mathcal{L}\left(\psi, \psi^*, \psi_t, \psi_t^*, \Delta\psi, \Delta\psi^*\right) = \frac{i}{2}\left(\psi_t\psi^* - \psi_t^*\psi\right) - |\Delta\psi|^2 + \frac{1}{1+\sigma}|\psi|^{2(\sigma+1)}.$$

We cite here the Noether's theorem, as given in [37].

THEOREM 9 (Noether theorem). *If the action integral $\iint \mathcal{L}\, d\mathbf{x}dt$ is invariant under the infinitesimal transformation*

$$t \mapsto \tilde{t} = t + \delta t(\mathbf{x}, t, \psi), \quad \mathbf{x} \mapsto \tilde{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}(\mathbf{x}, t, \psi), \quad \psi \mapsto \tilde{\psi} = \psi + \delta\psi(\mathbf{x}, t, \psi),$$

then

$$\int \left[\frac{\partial\mathcal{L}}{\partial\psi_t}\left(\psi_t\delta t + \nabla\psi\cdot\delta\mathbf{x} - \delta\psi\right) + \frac{\partial\mathcal{L}}{\partial\psi_t^*}\left(\psi_t^*\delta t + \nabla\psi^*\cdot\delta\mathbf{x} - \delta\psi^*\right) - \mathcal{L}\delta t\right] d\mathbf{x} \tag{A.1}$$

is a conserved quantity.

For example, the BNLS action integral is invariant under the phase-multiplication

$$\psi(t, \mathbf{x}) \mapsto e^{i\varepsilon}\psi(t, \mathbf{x}).$$

In this case, $\delta t = 0$, $\delta\mathbf{x} = 0$, $\delta\psi = i\psi$. Therefore, Theorem 9 implies that the integral

$$\begin{aligned}
&\int \left[\frac{\partial\mathcal{L}}{\partial\psi_t}\left(\psi_t\delta t + \nabla\psi\cdot\delta\mathbf{x} - \delta\psi\right) + \frac{\partial\mathcal{L}}{\partial\psi_t^*}\left(\psi_t^*\delta t + \nabla\psi^*\cdot\delta\mathbf{x} - \delta\psi^*\right) - \mathcal{L}\delta t\right] d\mathbf{x} \\
&= \int \left[\frac{i}{2}\psi^*\left(\psi_t\cdot 0 + \nabla\psi\cdot 0 - i\psi\right) - \frac{i}{2}\psi\left(\psi_t^*\cdot 0 + \nabla\psi^*\cdot 0 + i\psi^*\right) - \mathcal{L}\cdot 0\right] d\mathbf{x} = \|\psi\|_2^2,
\end{aligned}$$

i.e., the power, is a conserved quantity. Other conservation laws can be found in a similar manner.

Appendix B. The Gagliardo–Nirenberg inequality. In the L^2-critical case $\sigma d = 4$, the appropriate Gagliardo–Nirenberg inequality in H^2 is [15, 16, 29].

LEMMA 10 (Gagliardo–Nirenberg inequality). *Let $\sigma d = 4$, and let $f \in H^2(\mathbb{R}^d)$. Then,*

$$\|f\|_{2(\sigma+1)}^{2(\sigma+1)} \le B_{\sigma,d}\|\Delta f\|_2^2\|f\|_2^{2\sigma}. \tag{B.1}$$

We note that the ground-state R_{B} of (4.1) is the minimizer of the Gagliardo–Nirenberg inequality [11], and that its L^2-norm, the critical power, satisfies

$$P_{\text{cr}}^{\text{B}} = \|R_{\text{B}}\|_2^2 = \left(\frac{\sigma+1}{B_{\sigma,d}}\right)^{1/\sigma}.$$

Hence, the Gagliardo–Nirenberg inequality implies the following corollary.

COROLLARY 11. *Let $f \in H^2$ and $\sigma d = 4$; then*

$$H[f] \geq \left[1 - \left(\|f\|_2^2 / P_{\rm cr}^{\rm B}\right)^{\sigma}\right] \cdot \|\Delta f\|_2^2,$$

so that

$$\|f\|_2^2 \leq P_{\rm cr}^{\rm B} \implies H[f] \geq 0. \tag{B.2}$$

Appendix C. Radial compactness lemma. Here we provide an extension of the compactness lemma for $H^1_{\rm radial}$ functions [36] to the case of $H^2_{\rm radial}$.

LEMMA 12 (radial compactness lemma). *Let $d \geq 2$, and let $\sigma > 0$ be in the H^2-subcritical regime* (1.5)*. Then, the embedding $H^2_{\rm radial}(\mathbb{R}^d) \to L^{2(\sigma+1)}(\mathbb{R}^d)$ is compact; i.e., every bounded sequence $u_{n'} \in H^2_{\rm radial}(\mathbb{R}^d)$ has a subsequence u_n which converges strongly in $L^{2(\sigma+1)}(\mathbb{R}^d)$.*

Proof. If $\|u_{n'}\|_{H^2} \leq M$, then the sequence $u_{n'}$ has a subsequence u_n which converges weakly to u in H^2. Since the limit of radial functions is a radial function, $u \in H^2_{\rm radial}$. In addition, since for any bounded domain Ω, the embedding $H^2(\Omega) \to L^2(\Omega)$ is compact, there is a subsequence which converges strongly to u in $L^2(\Omega)$, i.e., $\lim_{n\to\infty} \int_\Omega |u_n - u|^2 \, d\mathbf{x} = 0$. From the Gagliardo–Nirenberg inequality on the bounded domain Ω (see [15, 16, 29]),

$$\|f\|^{2(\sigma+1)}_{L^{2(\sigma+1)}(\Omega)} \leq B_{\sigma,d,\Omega} \|\Delta f\|^{\sigma d/2}_{L^2(\Omega)} \cdot \|f\|^{2(\sigma+1)-\sigma d/2}_{L^2(\Omega)} = B_{\sigma,d,\Omega} \|\Delta f\|^{\sigma d/2}_{L^2(\Omega)} \cdot \|f\|^{2(1-\sigma\frac{d-4}{4})}_{L^2(\Omega)},$$

and since $1 > \sigma\frac{d-4}{4}$ in the H^2-subcritical case, it follows that $u_n \to u$ strongly in $L^{2(\sigma+1)}(\Omega)$, so that $\lim_{n\to\infty} \int_\Omega |u_n - u|^{2(\sigma+1)} \, d\mathbf{x} = 0$.

Next, the Strauss radial lemma [36] for H^1 functions gives that for all $\rho_\epsilon > 1$ and n,

$$\int_{|x|>\rho_\epsilon} |u_n|^{2(\sigma+1)} \, d\mathbf{x} \leq \frac{C}{\rho_\epsilon^{(d-1)\sigma}},$$

so that for all ϵ there exists ρ_ϵ such that for all n, $\int_{|x|>\rho_\epsilon} |u_n|^{2(\sigma+1)} \, d\mathbf{x} \leq \epsilon$. Finally, since

$$\|u_n - u\|_{L^{2(\sigma+1)}(\mathbb{R}^d)} \leq \|u_n - u\|_{L^{2(\sigma+1)}(|\mathbf{x}|<\rho_\epsilon)} + \|u_n\|_{L^{2(\sigma+1)}(|\mathbf{x}|>\rho_\epsilon)} + \|u\|_{L^{2(\sigma+1)}(|\mathbf{x}|>\rho_\epsilon)},$$

the convergence in $\mathbb{R}^d$ is obtained. ☐

Appendix D. Proof of the concentration theorem, Theorem 7 (nonradial case). The argument in this proof is due to [27]. As in the proof of Theorem 6, let

$$f_n(\mathbf{x}) = S(\psi)(t_n, \mathbf{x}) = l^{d/2}(t_n)\psi(t_n, l(t_n)\mathbf{x}), \qquad l(t) = \|\Delta\psi\|_2^{-1/2}. \tag{D.1}$$

Since

$$\|f_n\|_2^2 = \|\psi_0\|_2^2, \qquad \|\Delta f_n\|_2^2 = 1,$$

the sequence f_n is bounded in H^2. Direct calculations give that

$$H[f_n] = l^4(t_n) H[\psi] = l^4(t_n) H[\psi_0] \longrightarrow 0, \qquad t_n \to T_{\rm c}.$$

In addition, from Hamiltonian conservation it follows that

$$\frac{1}{\sigma+1}\|f_n\|_{8/d+2}^{8/d+2} = l^4 \frac{1}{\sigma+1}\|\psi\|_{8/d+2}^{8/d+2} \longrightarrow 1.$$

Hence, f_n satisfies the conditions of the nonradial compactness lemma, Lemma 3. Therefore, by (3.7), (3.8), (3.10),

$$0 = \lim_{j\to L}\lim_{n\to\infty} \frac{1}{\sigma+1}\left\|f_n^j - f^j\right\|_{2\sigma+2}^{2\sigma+2} \geq -\lim_{j\to L}\lim_{n\to\infty} H(f_n^j - f^j) = \sum_{k=1}^{L} H(f^k).$$

Hence, there exists $1 \leq k \leq L$ such that $H(f^k) \leq 0$. Since, in addition, $f^k \not\equiv 0$, this implies that $||f^k||_2^2 \geq P_{\rm cr}^{\rm B}$. Therefore, for any $\epsilon > 0$, there exists $R > 0$ such that

$$||f^k||_{L^2(B_R)}^2 \geq P_{\rm cr}^{\rm B} - \epsilon. \tag{D.2}$$

From the nonradial compactness lemma, Lemma 3, we also have that for $1 \leq j \leq k$,

$$\lim_{n\to\infty}\int_{\mathbb{R}^d} \left||f_n^j|^2 - |f_n^j - f^j|^2 - |f^j|^2\right| d\mathbf{x} = 0.$$

Hence, for any domain $\Omega \in \mathbb{R}^d$ and any sequence $\{\mathbf{z}_n^j\}$,

$$\lim_{n\to\infty}\int_{\mathbf{x}\in\mathbf{z}_n^j+\Omega} |f_n^j|^2 - |f_n^j - f^j|^2 - |f^j|^2 d\mathbf{x} = 0.$$

By (3.4b), this inequality can be rewritten as

$$\lim_{n\to\infty}\int_{\mathbf{x}\in\mathbf{z}_n^j+\Omega} |f_n^j|^2 - |f_n^{j+1}(\cdot - \mathbf{y}_n^{j+1})|^2 - |f^j|^2 d\mathbf{x} = 0.$$

Summing this relation over $j = 1, \ldots, k$ with $\mathbf{z}_n^j = \sum_{m=j+1}^{k} \mathbf{y}_n^m$ and $\mathbf{z}_n^{k+1} = -\mathbf{y}_n^{k+1}$ gives

$$\lim_{n\to\infty}\sum_{j=1}^{k}\left[\int_{\mathbf{x}\in\mathbf{z}_n^j+\Omega} |f_n^j|^2 - \int_{\mathbf{x}\in\mathbf{z}_n^{j+1}+\Omega} |f_n^{j+1}|^2\right] = \lim_{n\to\infty}\sum_{j=1}^{k}\int_{\mathbf{x}\in\mathbf{z}_n^j+\Omega} |f^j|^2.$$

Since the left-hand side is a telescopic sum, and since $\mathbf{z}_n^k \equiv 0$,

$$\lim_{n\to\infty}\int_{\mathbf{x}\in\mathbf{z}_n^1+\Omega} |f_n^1|^2 = \lim_{n\to\infty}\int_{\mathbf{x}\in\mathbf{z}_n^{k+1}+\Omega} |f_n^{k+1}|^2 + \lim_{n\to\infty}\sum_{j=1}^{k}\int_{\mathbf{x}\in\mathbf{z}_n^j+\Omega} |f^j|^2 \geq \int_{\mathbf{x}\in\Omega} |f^k|^2.$$

Since $f_n^1(x) = f_n(\mathbf{x} - \mathbf{y}_n^1)$, from the last inequality with $\Omega = B_R$, we have that

$$\lim_{n\to\infty}\int_{\mathbf{x}\in B_R} |f_n(\mathbf{x} - \sum_{m=1}^{k}\mathbf{y}_n^m)|^2 = \lim_{n\to\infty}\int_{\mathbf{x}\in\mathbf{z}_n^1+B_R} |f_n^1|^2 \geq \int_{\mathbf{x}\in B_R} |f^k|^2.$$

Therefore, by (D.1),

$$\lim_{n\to\infty}\int_{\mathbf{x}\in B_{R\cdot l(t_n)}} |\psi(t_n, \mathbf{x} - l(t_n)\sum_{m=1}^{k}\mathbf{y}_n^m)|^2 \geq \int_{\mathbf{x}\in B_R} |f^k|^2.$$

Theorem 7, the concentration theorem, follows from this inequality, (D.2), and Theorem 5.

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