A quantitative approach to soliton instability

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Received October 29, 2010; revised December 21, 2010; accepted December 22, 2010; posted January 7, 2011 (Doc. ID 137344); published January 31, 2011

We present an approach for instabilities of solitons that is based on the spectrum of a fourth-order linearized operator. Unlike the standard approach which is based on the slope (Vakhitov–Kolokolov) condition, this approach provides the quantitative value of the instability rate and the qualitative nature of the instability dynamics. © 2011 Optical Society of America

OCIS codes: 190.0190, 190.6135.

Solitons are localized nonlinear waves that maintain their shape as they propagate. They arise in many physical fields, including nonlinear optics and Bose-Einstein condensates, where their dynamics is modeled by the nonlinear Schrödinger equation (NLS). The key question with regard to a soliton is whether it is stable. The standard approach for answering this question goes back to Vakhitov and Kolokolov (VK), who showed that a necessary condition for stability is that the curve of the soliton power as a function of the propagation constant should have a nonnegative slope [1]. Although the VK/slope condition has been used in hundreds if not thousands of studies, it has two major limitations. The first limitation is qualitative: The slope condition determines whether the soliton is susceptible to an amplitude (focusing) instability, whereby the soliton amplitude increases (decreases) as its width decreases (increases). The slope condition, however, does not determine whether the soliton is susceptible to a drift instability, whereby the soliton drifts away from potential minima [2,3]. The occurrence of a drift instability can be determined by a different condition, the *spectral condition*, which is based on the number of negative eigenvalues of the second-order linearized operator L_+. The second limitation of the slope condition is that it is not quantitative, i.e., it provides a yes/no answer to the question whether the soliton is susceptible to an amplitude instability, but it does not provide the instability rate. This quantitative information is important, e.g., in the case of an unstable soliton whose instability rate is very small, since such a soliton is "mathematically" unstable, yet it can be experimentally stable [2,4,5].

In [2] we showed that the spectral condition is also quantitative, i.e., the rate of the drift instability depends on the magnitude of the negative eigenvalues of L_+. In this study we first ask whether the slope condition is also quantitative. Indeed, since a negative (positive) slope implies an amplitude instability (stability), this suggests that a small negative slope implies a weak amplitude instability, and more generally, that the magnitude of the negative slope $|P'(\mu)|$ is a measure of the instability rate. Surprisingly, however, we show that the magnitude of the slope does not provide any information on the instability rate. Therefore, we adopt a different approach for instability, which is based on the spectrum of the fourth-order linearized operator L_-L_+. As we shall see, the soliton is unstable if and only if this operator has negative eigenvalues. The nature of the instability can be inferred from the corresponding eigenfunctions: An even eigenfunction corresponds to an amplitude instability, whereas an odd eigenfunction corresponds to a drift instability. In both cases, the instability rate is given by the square root of the negative eigenvalue.

We briefly review the linear instability analysis that leads to the slope condition. See [1,6,7] for more details. We begin with the $(1+d)$-dimensional homogeneous NLS

$$i\psi_t(\mathbf{x},t) + \Delta\psi + F(|\psi|^2)\psi = 0, \qquad \Delta \doteq \sum_{k=1}^{d} \partial_{x_k}^2, \tag{1}$$

where $t>0$, $\mathbf{x}\in\mathbb{R}^d$, and F represents a power-law, a saturable, or a photorefractive nonlinearity The NLS (1) admits the standing-wave solution $\psi(\mathbf{x},t;\mu) = R(\mathbf{x};\mu)e^{i\mu t}$, where μ is the frequency and $R(\mathbf{x};\mu)$ satisfies

$$-\mu R + \Delta R + F(R^2)R = 0. \tag{2}$$

Let $\psi(\mathbf{x},t) = [R(\mathbf{x};\mu) + \varepsilon h(\mathbf{x},t)]e^{i\mu t}$ be a perturbed soliton solution of (1), where R is the positive (ground-state) solution of (2). The $O(\varepsilon)$ linearized equation for h is

$$h_t = -i\{[-\Delta + \mu - F(R^2)]h - R^2F'(R^2)h^*\}, \tag{3}$$

where h^* is the complex conjugate of h and F' denotes the derivative of F with respect to R^2. Let

$$h(\mathbf{x},t) = [u(\mathbf{x}) + iv(\mathbf{x})]e^{\Omega t}, \tag{4}$$

where $u(\mathbf{x})$ and $v(\mathbf{x})$ are real. If the ground state is linearly stable, there are no solutions of (3) with $\Omega > 0$. In order to investigate whether there are such unstable modes, we substitute (4) into (3) and assume that $\Omega\in\mathbb{R}$. This leads to

$$L_-u = \Omega v, \qquad L_- \doteq -\Delta + \mu - F(R^2), \tag{5a}$$

$$L_+v = -\Omega u, \qquad L_+ \doteq L_- - 2R^2F'(R^2). \tag{5b}$$

The eigenvalues of the system (5) were studied analytically in [6,7]. Alternatively, one can apply L_- to (5b), which gives the fourth-order system

$$L_- L_+ v = \lambda v, \qquad \lambda \doteq -\Omega^2. \tag{6}$$

Let $R^\perp$ denote the subspace orthogonal to R. Then L_- and L_-^{-1} are bounded and positive definite on $R^\perp$. Applying L_-^{-1} to (6) gives

$$L_+ v = \lambda L_-^{-1} v, \qquad \forall v \in R^\perp. \tag{7}$$

Let $\lambda_{\min}$ be the smallest eigenvalue of (6), with a corresponding eigenfunction $v_{\min}$. Taking the inner product of (7) with v leads to the variational characterization of $\lambda_{\min}$

$$\lambda_{\min} \doteq \inf_{v\in R^\perp, \|v\|_2=1} \frac{(v, L_+ v)}{(v, L_-^{-1} v)} = \frac{(v_{\min}, L_+ v_{\min})}{(v_{\min}, L_-^{-1} v_{\min})}. \tag{8}$$

Hence, the necessary condition for stability becomes $\lambda_{\min} \geq 0$. Since $L_-^{-1} > 0$, $(v, L_-^{-1} v) = \|L_-^{-1/2} v\|_2^2 > 0$. Therefore,

$$\operatorname{sgn}\lambda_{\min} = \operatorname{sgn}\alpha^{L_+}, \qquad \alpha^{L_+} \doteq \inf_{v\in R^\perp, \|v\|_2=1} (v, L_+ v). \tag{9}$$

Thus, the necessary condition for stability $\lambda_{\min} \geq 0$ implies that $\alpha^{L_+} \geq 0$. By [6] Lemma E.1, $(L_+^{-1} R, R) \leq 0 \Rightarrow \alpha^{L_+} \geq 0$. Differentiating (2) with respect to μ gives $L_+ Q = -R$, where $Q \doteq \partial_\mu R$. Therefore, $-(L_+^{-1} R, R) = (Q, R) = \frac{1}{2}\frac{d\|R\|_2^2}{d\mu} = \frac{1}{2}P'(\mu)$. Hence, the necessary condition for stability is satisfied when $P'(\mu) \geq 0$, which is known as the VK/slope condition. In [6,7] it was rigorously proved that the positive solitons of Eq. (1) are orbitally stable if $P'(\mu) > 0$ and unstable if $P'(\mu) < 0$.

When $P'(\mu) < 0$, it follows from (4) that the instability rate is given by the maximal positive eigenvalue of (5), which can be expressed in terms of the most negative eigenvalue of $L_- L_+$ as

$$\Omega_{\max} \doteq \sqrt{-\lambda_{\min}}. \tag{10}$$

To show analytically that the magnitude of $P'(\mu)$ is "unrelated" to the instability rate $\Omega_{\max}$, we consider the case of a power-law nonlinearity $F(|\psi|^2) = |\psi|^{2\sigma}$. In this case, the NLS is invariant under the dilation symmetry $(\mathbf{x}, t, \psi) \mapsto (\sqrt{\mu}\mathbf{x}, \mu t, \mu^{\frac{1}{2\sigma}}\psi)$. Therefore, the ground state can be written as $R(\mathbf{x}; \mu) = \mu^{\frac{1}{2\sigma}} R(\sqrt{\mu}\mathbf{x}; \mu = 1)$. Hence, the slope scales with μ as

$$P'(\mu) = \mu^{(c_{\sigma,d}-1)} c_{\sigma,d} \|R(\cdot, \mu = 1)\|_2^2, \qquad c_{\sigma,d} \doteq \frac{2 - \sigma d}{2\sigma}.$$

This implies that when $c_{\sigma,d} < 1$ the slope's magnitude $|P'(\mu)|$ decreases with μ. On the other hand, Eq. (4) and the dilation invariance $\Omega t \mapsto \Omega \mu t$ imply that $\Omega_{\max}$ increases linearly with μ for any (σ, d). In particular, since the instability rate can be large when the slope is small and vice versa, the magnitude of the slope is unrelated to the instability rate.

To better understand this surprising observation, we note that the derivation of the slope condition is based on the relations $\operatorname{sgn}P'(\mu) \geq 0 \Rightarrow \operatorname{sgn}\alpha^{L_+} \geq 0 \Rightarrow \operatorname{sgn}\lambda_{\min} \geq 0$. The magnitudes of $\lambda_{\min}$ and α^{L_+} are "related," as they are the minima of similar variational problems, Eqs. (8) and (9), respectively. In contrast, the magnitudes of $P'(\mu)$ and α^{L_+} are unrelated (see proof of Lemma E.1 in [6]).

To confirm numerically that the instability rate, which is associated with the violation of the slope condition is indeed given by Eq. (10), let us consider the NLS (1) with the initial condition

$$\psi_0(\mathbf{x}) = R(\mathbf{x}; \mu) + \varepsilon u(\mathbf{x}), \qquad |\varepsilon| \ll 1, \tag{11}$$

where u is the eigenfunction of (5) that corresponds to $\Omega_{\max}$. Because (5) is invariant under the transformation $(u, v, \Omega) \mapsto (u, -v, -\Omega)$, the linearized solution of (1) with the initial conditions (11) is given by

$$\psi(\mathbf{x}, t) \cong \left[R + \frac{\varepsilon}{2}(u + iv)e^{\Omega_{\max} t} + \frac{\varepsilon}{2}(u - iv)e^{-\Omega_{\max} t}\right] e^{i\mu t}.$$

Hence, the on-axis amplitude of ψ satisfies

$$\begin{aligned} |\psi(0, t)| \cong |R(0) + \varepsilon[u(0)\cosh(\Omega_{\max} t) \\ + iv(0)\sinh(\Omega_{\max} t)]|. \end{aligned} \tag{12}$$

We solve directly the supercritical NLS (1) with $d = 1$, $F(|\psi|^2) = |\psi|^6$, and the perturbed ground-state initial condition (11) with $\mu = 1$. Recall that $R(x; \mu)$ is given explicitly by $R = (4\mu)^{\frac{1}{6}} \operatorname{sech}^{\frac{1}{3}}(3\sqrt{\mu}x)$. Therefore, $P'(\mu) < 0$, showing that the soliton is unstable. As noted, however, the slope condition approach does not provide the instability rate. In contrast, computing the spectrum of $L_- L_+$ yields the single negative eigenvalue $\lambda_{\min} \approx -8.44$, which gives an instability rate of $\Omega_{\max} \approx 2.9$; see (10). Figure 1 shows that the on-axis dynamics of the perturbed bound state agrees with the prediction (12) during the initial stage of the propagation, both for a focusing ($\epsilon > 0$) and a defocusing ($\epsilon < 0$) perturbation.

The above approach can be extended to the variable-coefficients NLS

$$i\psi_t(\mathbf{x}, t) + \Delta\psi + F(|\psi|^2)\psi - V(\mathbf{x})\psi = 0, \qquad \mathbf{x} \in \mathbb{R}^d. \tag{13}$$

The linear stability analysis is the same as in the constant-coefficient case, except that now

$$L_- \doteq -\Delta + \mu + V - F(R_V^2), \qquad L_+ \doteq L_- - 2R_V^2 F'(R_V^2),$$

where R_V is the bound state in the presence of the potential V. In the homogeneous case, the operators L_+ and $L_- L_+$ have d zero eigenvalues with corresponding eigenfunctions ∇R. The inhomogeneous potential breaks up the translation invariance, leading to a bifurcation of

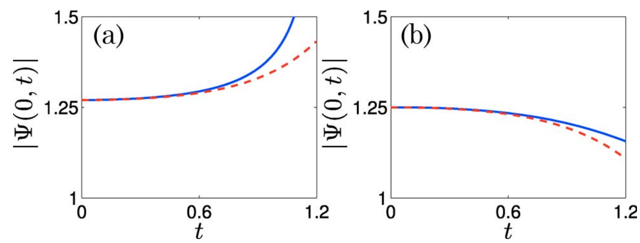

Fig. 1. (Color online) On-axis amplitude of the perturbed unstable ground state of the supercritical NLS (blue, solid line) agrees with the prediction (12) with $\Omega_{\max} \approx 2.9$ (red, dashed line) both for (a) a focusing perturbation $\varepsilon = 10^{-2}$ and (b) a defocusing perturbation $\varepsilon = -10^{-2}$.

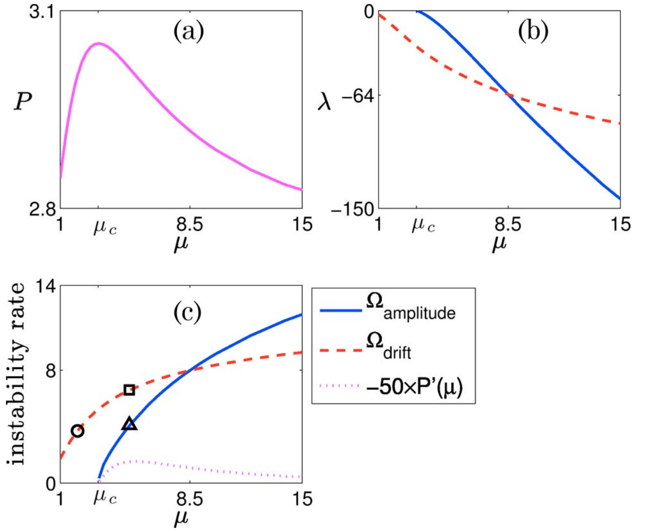

Fig. 2. (Color online) (a) Frequency-power curve for solitons of (14) centered at a lattice maximum, (b) the negative eigenvalues of L_L_+, and (C) the instability rates computed from (b) using (10) [same line types as in (b)]. The shapes in (c) are delineated for Fig. 3. Dotted curve is $100|P'(\mu)|$.

these d eigenvalues away from zero. The eigenvalues of L_L_+ that become negative are associated with a drift instability, which typically occurs when the soliton is centered at a potential maximum or at a saddle point [2,3]. Therefore, negative eigenvalues of L_L_+ can be indications of both an amplitude instability and a drift instability. Fortunately, one can easily distinguish between negative eigenvalues that induce an amplitude instability and those that induce a drift instability, as the former correspond to symmetric eigenmodes, and the latter correspond to asymmetric eigenfunctions (as they are perturbed from ∇R).

In summary, we propose the following approach for stability:

1. Compute the bound state (cf. [8,9]), and use it to compute the negative eigenvalues of L_L_+.
2. A negative eigenvalue with a symmetric eigenmode indicates an amplitude instability.
3. A negative eigenvalue with an asymmetric eigenmode indicates a drift instability.
4. In both cases, the instability rate is given by $\Omega = \sqrt{-\lambda}$.

This scheme thus completes the quantitative theory presented in [2,3]. Our approach applies to positive solitons in any dimension, any nonlinearity $F(|\psi|^2)$, as well as for any lattice configuration. To illustrate that, consider solitons of the quintic NLS with a periodic-lattice potential,

$$i\psi_t(t,x) + \psi_{xx} - 2\cos(2\pi x)\psi + |\psi|^4\psi = 0, \qquad (14)$$

that are centered at $x = 0$ (i.e., at a potential maximum). Under the standard approach, one plots the $P(\mu)$ curve. Since $P(\mu)$ has a maximum at $\mu_c \simeq 3$, see Fig. 2(a), the solitons are unstable for $\mu > \mu_c$. Alternatively, under the new approach, one computes the negative eigenvalues of L_L_+, as a function of μ, see Fig. 2(b). Because one of the negative eigenvalues exists for $0 < \mu$, these solitons are unstable for all $\mu > 0$, and not just for $\mu > \mu_c$. This negative eigenvalue corresponds to an asymmetric eigenmode [see Figs. 3(a) and 3(b)] and is thus associated with a drift instability in the x direction, away from the potential maxima. The second negative eigenvalue exists for $\mu > \mu_c$. This eigenvalue corresponds to a symmetric eigenmode [see Fig. 3(b)] and is thus associated with an amplitude instability. Plotting the instability rates given by (10) in Fig. 2(c) shows that when $\mu < 8.5$ the drift instability dominates the initial dynamics, while for $\mu > 8.5$ the amplitude instability dominates. This plot also show that, as already noted, $|P'(\mu)|$ is unrelated to the instability rate.

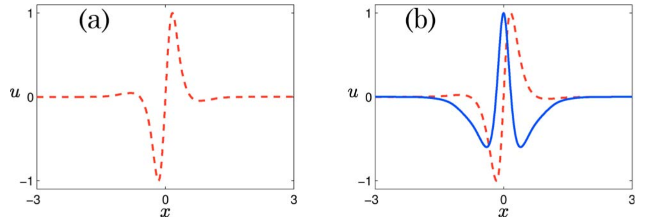

Fig. 3. (Color online) Eigenmodes for the shapes in Fig. 2(c). (a) Asymmetric drift eigenmode corresponding to $\mu = 2(\circ)$, (b) asymmetric drift eigenmode (dashed curve) and symmetric amplitude eigenmode (solid curve) corresponding to $\mu = 5$.

In conclusion, the standard VK/slope condition provides a yes/no answer to the existence of an amplitude instability. In contrast, the spectrum of L_L_+ (1) detects both amplitude and drift instabilities, (2) provides the rates of these instabilities, and (3) determines which of them is dominant. Therefore, we propose that future studies will compute the negative spectrum of L_L_+, rather than (or in addition to) the power curve $P(\mu)$. Note that the eigenvalues of the system (5) were computed in several studies [10–12]. However, to the best of our knowledge, these eigenvalues were never used to study the three aforementioned attributes but only to provide a yes/no answer to the question of stability.

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