SUPPLEMENTARY MATERIALS: BOUNDARY EFFECTS IN THE DISCRETE BASS MODEL*

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SM1. End of proof of Lemma 4.1. Substituting the expression for $S(t;M-1)$ from (4.7) into the right-hand side of (4.6), integrating, and equating the coefficients of the exponents on both sides of (4.6) gives, after some algebra,

(SM1.1)
$$S(t;M) = e^{-(p+q)t} + qe^{-(p+q)t}\sum_{k=1}^{M-2} A_{k,M-1}\int_0^t e^{-kp\tau}d\tau + qe^{-(p+q)t}B_{M-1}\int_0^t e^{-(M-1)p\tau+q\tau}d\tau$$
$$= e^{-(p+q)t} - qe^{-(p+q)t}\sum_{k=1}^{M-2} A_{k,M-1}\frac{e^{-kpt}-1}{kp} + qe^{-(p+q)t}B_{M-1}\frac{e^{-(M-1)pt+qt}-1}{q-(M-1)p}.$$

Equating the coefficients of e^{-Mpt} in (4.7) and (SM1.1) gives $B_M = \frac{q}{q-(M-1)p}B_{M-1}$. Since $B_1 = 1$, see (3.8), we get that

$$B_M = \frac{q^{M-1}}{\prod_{j=1}^{M-1}(q-jp)}. \quad \text{(SM1.2)}$$

Equating the coefficients of $e^{-(kp+q)t}$ in (4.7) and (SM1.1) gives

$$A_{k,M} = -\frac{qA_{k-1,M-1}}{(k-1)p}, \qquad k = 2,\ldots,M-1, \quad \text{(SM1.3)}$$

and

(SM1.4)
$$A_{1,M} = 1 + \frac{q}{p}\sum_{k=1}^{M-2}\frac{A_{k,M-1}}{k} - \frac{qB_{M-1}}{q-(M-1)p} = 1 + \frac{q}{p}\sum_{k=1}^{M-2}\frac{A_{k,M-1}}{k} - \frac{q^{M-1}}{\prod_{j=1}^{M-1}(q-jp)},$$

where in the second equality we used (SM1.2). By (SM1.3) and (SM1.4),

$$\begin{aligned} A_{M-k,M} &= \frac{(-q)^{M-k-1}}{(M-k-1)!p^{M-k-1}}A_{1,k+1} \\ &= \frac{(-q)^{M-k-1}}{(M-k-1)!p^{M-k-1}}\left[1 - \frac{q^k}{\prod_{j=1}^{k}(q-jp)} + \frac{q}{p}\sum_{j=1}^{k-1}\frac{A_{j,k}}{j}\right]. \end{aligned} \quad \text{(SM1.5)}$$

Using (SM1.3) again, we get that

$$A_{j,k} = \frac{p^{M-k}(M-k+j-1)!}{(-q)^{M-k}(j-1)!}A_{M-k+j,M}. \quad \text{(SM1.6)}$$

Plugging (SM1.6) into (SM1.5) yields

$$A_{M-k,M} = \frac{(-q)^{M-k-1}}{(M-k-1)!p^{M-k-1}}\left[1 - \frac{q^k}{\prod_{j=1}^{k}(q-jp)} + \frac{q}{p}\sum_{j=1}^{k-1}\frac{p^{M-k}(M-k+j-1)!}{(-q)^{M-k}j!}A_{M-k+j,M}\right].$$

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Therefore,

$$A_{M-k,M} = \frac{(-q)^{M-k-1}}{(M-k-1)!p^{M-k-1}} c_{M-k}, \tag{SM1.7}$$

where

$$c_{M-k} := 1 - \frac{q^k}{\prod_{j=1}^{k}(q-jp)} - \sum_{j=1}^{k-1} \frac{p^{M-k-1}(M-k+j-1)!}{(-q)^{M-k-1}j!} A_{M-k+j,M}. \tag{SM1.8}$$

Relation (4.1a) follows from $f = 1 - S$ and from (4.7), (SM1.2), and (SM1.7) with $(M-k) \longrightarrow k$.

Plugging (SM1.7) into (SM1.8) yields

$$c_{M-k} = 1 - \frac{q^k}{\prod_{j=1}^{k}(q-jp)} - \sum_{j=1}^{k-1} \frac{p^{-j}(-q)^j}{j!} c_{M-k+j}.$$

Making the change of variables $\tilde{j} := k - j$ in the summation gives (4.1b).

SM2. Proof of (4.19). Following [SM1, proof of Lemma 8], we can obtain from (2.6) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\mathrm{Prob}\,(X_j(t)=0) = &-p\mathrm{Prob}\,(X_{j-1}(t)=0, X_j(t)=0, X_{j+1}(t)=0) \\ &-\left(p+\frac{q}{2}\right)\mathrm{Prob}\,(X_{j-1}(t)=0, X_j(t)=0, X_{j+1}(t)=1) \\ &-\left(p+\frac{q}{2}\right)\mathrm{Prob}\,(X_{j-1}(t)=1, X_j(t)=0, X_{j+1}(t)=0) \\ &-(p+q)\,\mathrm{Prob}\,(X_{j-1}(t)=1, X_j(t)=0, X_{j+1}(t)=1)\,. \end{aligned} \tag{SM2.1}$$

The configuration $\{X_j(t)=0, X_{j+1}(t)=0\}$ can be written as a union of two disjoint configurations:

$$\begin{aligned} &\{X_j(t)=0, X_{j+1}(t)=0\} = \\ &\qquad \{X_{j-1}(t)=0, X_j(t)=0, X_{j+1}(t)=0\} \cup \{X_{j-1}(t)=1, X_j(t)=0, X_{j+1}(t)=0\}. \end{aligned}$$

Therefore, it probability is the sum of the probabilities of the disjoint configurations:

$$\begin{aligned} \mathrm{Prob}\,(X_j(t)=0, X_{j+1}(t)=0) = &\ \mathrm{Prob}\,(X_{j-1}(t)=0, X_j(t)=0, X_{j+1}(t)=0) \\ &+\mathrm{Prob}\,(X_{j-1}(t)=1, X_j(t)=0, X_{j+1}(t)=0)\,. \end{aligned} \tag{SM2.2}$$

Similarly,

$$\begin{aligned} \mathrm{Prob}\,(X_{j-1}(t)=0, X_j(t)=0) = &\ \mathrm{Prob}\,(X_{j-1}(t)=0, X_j(t)=0, X_{j+1}(t)=0) \\ &+\mathrm{Prob}\,(X_{j-1}(t)=0, X_j(t)=0, X_{j+1}(t)=1)\,, \end{aligned} \tag{SM2.3}$$

and

$$\begin{aligned} \mathrm{Prob}\,(X_j(t)=0) = &\ \mathrm{Prob}\,(X_{j-1}(t)=0, X_j(t)=0, X_{j+1}(t)=0) \\ &+\mathrm{Prob}\,(X_{j-1}(t)=1, X_j(t)=0, X_{j+1}(t)=0) \\ &+\mathrm{Prob}\,(X_{j-1}(t)=0, X_j(t)=0, X_{j+1}(t)=1) \\ &+\mathrm{Prob}\,(X_{j-1}(t)=1, X_j(t)=0, X_{j+1}(t)=1)\,. \end{aligned} \tag{SM2.4}$$

Rearranging (SM2.2), (SM2.3) and (SM2.4), and plugging the relevant terms of these equations into (SM2.1), leads to (4.19).

A similar derivation yields the relation

(SM2.5)
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{Prob}\left(X_j(t)=0, X_{j+1}(t)=0\right) = &-(2p+q)\mathrm{Prob}\left(X_j(t)=0, X_{j+1}(t)=0\right)\\ &+\frac{q}{2}\Big[\mathrm{Prob}\left(X_{j-1}(t)=0, X_j(t)=0, X_{j+1}(t)=0\right)\\ &+\mathrm{Prob}\left(X_j(t)=0, X_{j+1}(t)=0, X_{j+2}(t)=0\right)\Big]\end{aligned}$$

which is used in the derivation of (B.9).

SM3. Proof of Lemma 4.8. We first consider node $j+k$ which is not on the circle, where $k=1,\ldots,K$. By the indifference principle, its probability to be a non-adopter by time t can be calculated from the equivalent network in Figure SM1(B). Therefore, $\mathrm{Prob}\left(X_{j+k}(t)=0\right) = S(t,p,q,M-K+k)$, or equivalently

$$\mathrm{Prob}\left(X_{j+k}(t)=1\right) = f_{\mathrm{circle}}(t;p,q,M-K+k). \tag{SM3.1}$$

For the $M-K$ nodes on the circle, it can easily be verified that edges $(j) \to (j+1)$ $\to \ldots$ $(j+K)$ are non-influentials to any of them. Therefore, by the indifference principle, the probability of such a node j to become an adopter is

$$\mathrm{Prob}\left(X_j(t)=1\right) = f_{\mathrm{circle}}(t;p,q,M-K). \tag{SM3.2}$$

Combining (3.1), (SM3.1), and (SM3.2) yields the result.

SM4. Proof of Lemma 4.9. By the indifference principle for $\Omega=\{1\}$, the networks in Figures SM2(A1) and SM2(A2) are equivalent for the one-sided case, and the networks in Figures SM2(B1) and SM2(B2) are equivalent for the two-sided case. By the strong version of the generalized dominance principle applied to networks SM2(A2) and SM2(B2), see remark after Lemma 3.5, $\mathrm{Prob}\left(X_1^{\mathrm{two-sided}}(t)=0\right) < \mathrm{Prob}\left(X_1^{\mathrm{one-sided}}(t)=0\right)$. Similarly, by the indifference principle for $\Omega=\{M\}$, the networks in Figures SM3(B1) and SM3(B2) are equivalent for the two-sided case. By the strong version of the generalized dominance principle applied to networks SM3(A) and SM3(B2), $\mathrm{Prob}\left(X_M^{\mathrm{two-sided}}(t)=0\right) > \mathrm{Prob}\left(X_M^{\mathrm{one-sided}}(t)=0\right)$.

SM5. Proof of Lemma A.1. Let $t>0$. Since $b(t)>0$, we have that $\frac{\mathrm{d}}{\mathrm{dt}}\sigma(t) + K\sigma(t) > 0$. Multiplying both sided by e^{Kt} yields $\frac{\mathrm{d}}{\mathrm{dt}}\left(e^{Kt}\sigma(t)\right) > 0$. Integrating both sides from 0 to t and rearranging leads to $\sigma(t) > e^{-Kt}\sigma(0)$. Since $\sigma(0)=0$, the result follows.

REFERENCE

[SM1] G. Fibich and R. Gibori, *Aggregate diffusion dynamics in agent-based models with a spatial structure*, Oper. Res., 58 (2010), pp. 1450–1468.

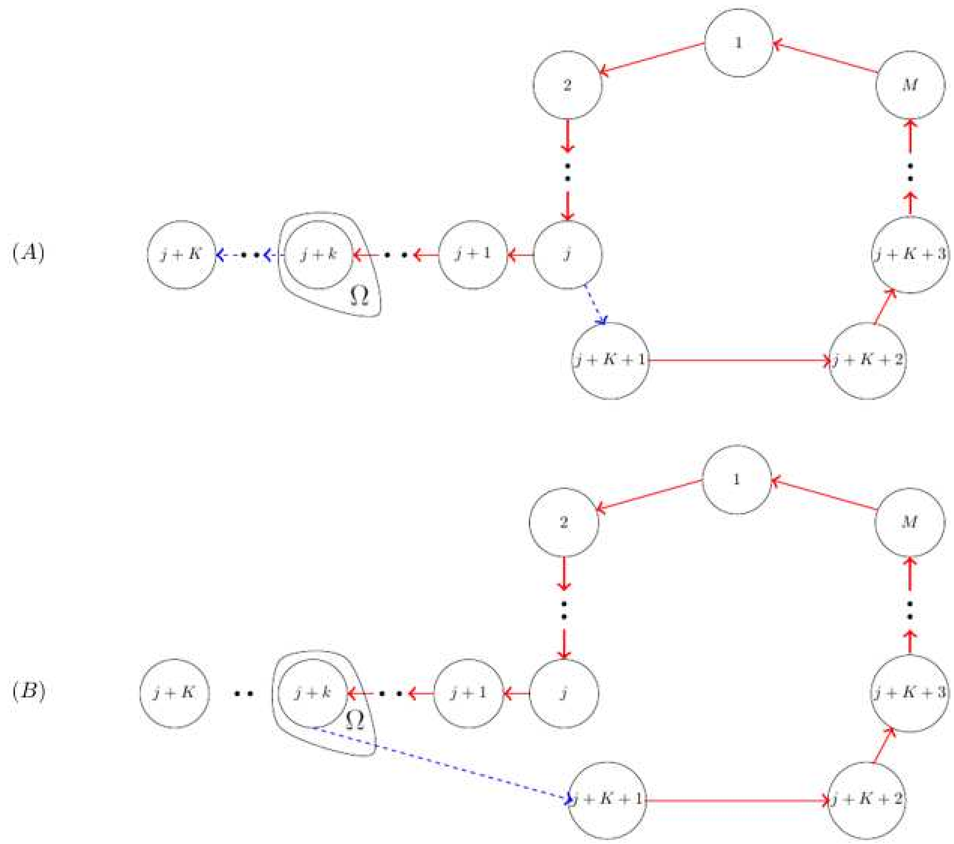

FIG. SM1. *Equivalent networks for the calculation of* $\mathrm{Prob}(X_{j+k}(t)=0)$. *Solid red and dashed blue arrows are influential and non-influential edges to* $\Omega = \{j+k\}$, *respectively.* (*A*) *One-sided circle with a one-sided ray.* (*B*) *The non-influential edges* $(j) \to (j+K+1)$ *and* $(j+k) \to (j+k+1)$ *are deleted; the non-influential edge* $(j+k) \to (j+K+1)$ *is added.*

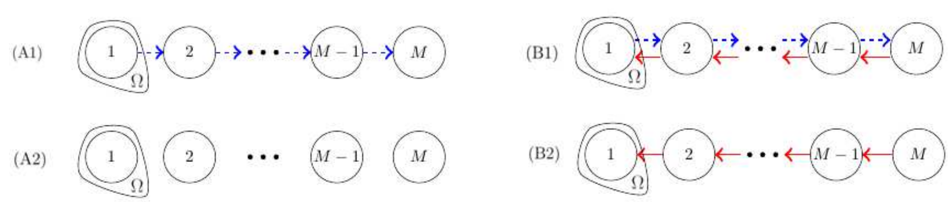

FIG. SM2. *Equivalent networks for the calculation of* $\mathrm{Prob}(X_1(t)=0)$ *for the one-sided and the two-sided cases. Solid red and dashed blue arrows correspond to influential and non-influential edges to* $\Omega = \{1\}$, *respectively.* (*A1*) *One-sided line.* (*A2*) *All non-influential edges in* (*A1*) *are deleted.* (*B1*) *Two-sided line.* (*B2*) *All non-influential edges in* (*B1*) *are deleted.*

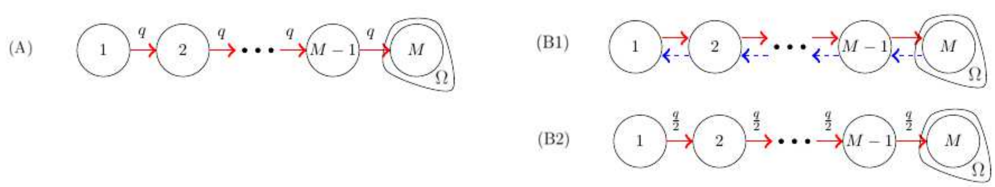

FIG. SM3. *Equivalent networks for the calculation of* $\mathrm{Prob}(X_M(t) = 0)$ *for the one-sided and the two-sided cases. Solid red and dashed blue arrows correspond to influential and non-influential edges to* $\Omega = \{M\}$*, respectively.* (A) *One-sided line (with an internal influence of* q*).* $(B1)$ *Two-sided line* $($*with an internal influence of* $\frac{q}{2}$$)$*.* $(B2)$ *All non-influential edges in* $(B1)$ *are deleted, resulting in a one-sided line with an internal influence of* $\frac{q}{2}$*.*